MATHEMATICS OF COMPUTATION, VOLUME 30, NUMBER 134
APRIL 1976, PAGES 263–277

On the Evaluation of Some Integrals Occurring in Scattering Problems

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Abstract. Some definite integrals which occur in transport problems through a scattering medium are studied. They are expressed in terms of such functions as the exponential integral of the first and second order, the dilogarithm, and a newly introduced and tabulated function.

1. **Introduction.** Definite integrals of the type

$$\int_0^1 dx\, Q(x) \prod_{i=1}^{q} (a_i x + b_i)^{-1} f(x), \qquad a_i \neq 0, \tag{1}$$

where $Q(x)$ is a real polynomial in x of order p $(p \leqslant q)$, and $f(x)$ is one of the functions $e^{-\gamma/x}$, $\ln|mx + n|$, $\ln|mx + n| \cdot e^{-\gamma/x}$ or the exponential integral $E_1(m/x + n)$, occur in various transport problems involving scattering (e.g. penetration of particles through a solid or scattering of the light in a planetary atmosphere). The author has met them while calculating the scattering of excited electrons in thin metal films [4].

The straightforward numerical integration in (1) can be rather difficult because of the possible logarithmic singularity and/or possible poles at $x_i = - b_i/a_i$ in the integrand. Moreover, it is sometimes desirable to possess a closed solution of the above integral. The purpose of this paper is to present such formulae which can be used as an expedient for calculations in scattering problems. To obtain clearly tractable results, we perform the whole calculation in terms of principal parts of the individual functions. It means that the expressions of the type $\ln|u| \cdot \ln|v|$ and $\ln|u| \cdot E_1(v)$ in the resulting formulae are determined up to the additive constant π^2 which must be considered in particular cases.

The integrand of the integral (1) can be written in the form (A_0, A_i are constants):

$$Q(x) \prod_{i=1}^{q} (a_i x + b_i)^{-1} f(x) = A_0 f(x) + \sum_{i=1}^{q} A_i (a_i x + b_i)^{-1} f(x). \tag{2}$$

Consequently, the evaluation of (1) reduces to the calculation of integrals of the type $\int_0^1 dx\, f(x)$, and $\int_0^1 dx\,(ax + b)^{-1} f(x)$. These integrals lead to formulae which contain besides the familiar functions also less-well-known higher functions such as *exponential integral of the first order* $E_1(x)$, *exponential integral of the second order* $E_1^{(2)}(x)$, and

Received April 15, 1975; revised August 25, 1975.
AMS (MOS) subject classifications (1970). Primary 26A63; Secondary 30A86, 33A70.
Key words and phrases. Mathematics of scattering problems, exponential integrals, dilogarithm, definite integrals.

dilogarithm $L_2(x)$. All these functions are generally defined for a complex variable, but we will use them only for real values of x.

The exponential integral of the first order $E_1(x)$ is defined on the real axis by [1], [5], [7]:

(3) $$E_1(-x) = -Ei(x), \qquad Ei(x) = \int_{-\infty}^{x} dt\, t^{-1}e^{t}, \qquad -\infty < x < \infty,$$

and the integral is interpreted in the sense of the Cauchy principal value if $x > 0$. For practical purposes we note:

(4) $$E_1(x) = \int_x^\infty dt\, t^{-1}e^{-t} = \int_1^\infty dt\, t^{-1}e^{-xt}.$$

The values of $E_1(x)$ and/or $Ei(x)$ are tabulated or they can be calculated up to the desired accuracy from various approximations [1], [7].

The exponential integral of the second order $E_1^{(2)}(x)$ has been introduced and tabulated by van de Hulst [3] (see also [5], [7]):

(5) $$E_1^{(2)}(x) = \int_x^\infty dt\, t^{-1}E_1(t) = \int_1^\infty dt\, t^{-1}\ln t\, e^{-xt}, \qquad x > 0.$$

For the dilogarithm $L_2(x)$ we take the standard form [6], [8]:

(6) $$L_2(x) = -\int_0^x dt\, t^{-1}\ln|1-t|, \qquad -\infty < x < \infty.$$

An extensive table ($\Delta x = 0.001$) and some properties of $L_2(x)$ can be found in Mitchell's paper [8].

For the sake of clarity the functions $E_1(x)$, $E_1^{(2)}(x)$, and $L_2(x)$ are shown in Fig. 1. The following identity will often be used.

(7) $$[x(ax+b)]^{-1} = (bx)^{-1} - a[b(ax+b)]^{-1}.$$

The calculation of (1) involves no unusual steps. Substitution and integration by parts are frequently applied. The integration can, in general, be performed in a closed form only for the case when the function $f(x)$ is $e^{-\gamma/x}$ or $\ln|mx+n|$. If $f(x)$ is $\ln|mx+n| \cdot e^{-\gamma/x}$ or $E_1(m/x+n)$, it is necessary to introduce a new function of three variables $\mathrm{Le}(A, B, \gamma)$ (see (16)), the evaluation of which is described in Section 6.

2. Integrals $\int_0^1 dx\, e^{-\gamma/x}$ and $\int_0^1 dx(ax+b)^{-1}e^{-\gamma/x}$. The first integral can be solved by the substitution $x = t^{-1}$ and subsequent integration by parts; using (4), we obtain

(8) $$\int_0^1 dx\, e^{-\gamma/x} = e^{-\gamma} - \gamma E_1(\gamma), \qquad \gamma > 0.$$

In the second integral of this group we substitute $x = u^{-1}$, use relation (7), substitute $t = \gamma(u + a/b)$ in the second term, and then apply (4). Thus,

(9) $$\int_0^1 dx\,(ax+b)^{-1}e^{-\gamma/x} = a^{-1}\{E_1(\gamma) - e^{\gamma a/b}E_1(\gamma + \gamma a/b)\}, \qquad \gamma > 0, \ a \neq 0.$$

(In the case $b = 0$, the right-hand side of (9) reduces to the first term because $\lim_{b\to 0} e^{\gamma a/b}E_1(\gamma + \gamma a/b) = 0$; see [5].)

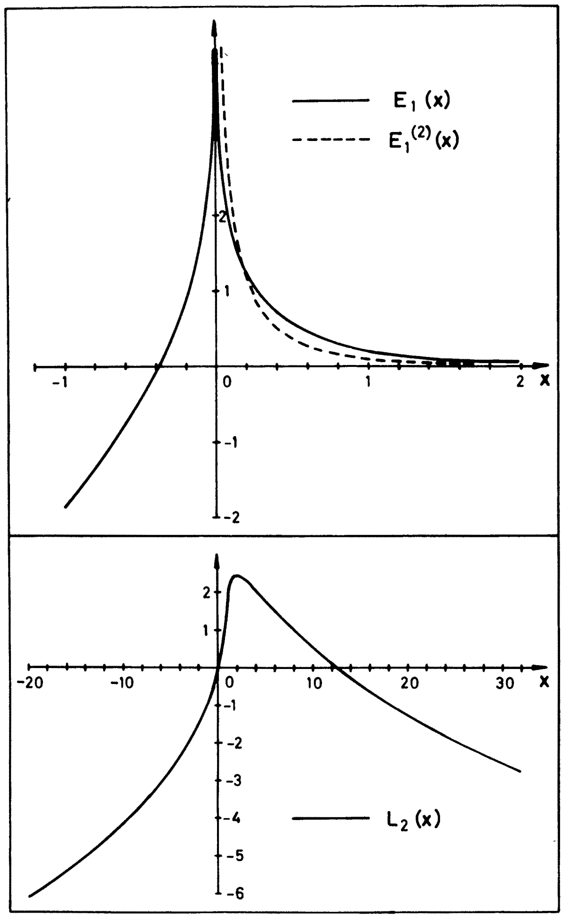

FIGURE 1. *Exponential integral of the first order $E_1(x)$ and of the second order $E_1^{(2)}(x)$ (above) and the dilogarithm $L_2(x)$ (below).*

3. **Integrals $\int_0^1 dx \ln|mx + n|$ and $\int_0^1 dx(ax + b)^{-1}\ln|mx + n|$.** For the evaluation of the first integral, we have

$$\int_0^1 dx \ln|mx + n| = m^{-1}\{(m + n)(\ln|m + n| - 1) - n(\ln|n| - 1)\}, \quad m \neq 0. \tag{10}$$

Using the indefinite integral given in [8], we obtain for the second integral,

$$\int_0^1 dx \frac{\ln|mx + n|}{ax + b} = a^{-1}\left\{\ln\left|\frac{mb - na}{a}\right|\ln\left|\frac{a + b}{b}\right| - L_2\left(\frac{m(a + b)}{mb - na}\right) + L_2\left(\frac{mb}{mb - na}\right)\right\}, \quad a \neq 0,\ b \neq 0,\ m \neq 0, \tag{11}$$

where $L_2(y)$ is the dilogarithm defined by (6).

4. Integrals $\int_0^1 dx \ln|mx+n|e^{-\gamma/x}$ and $\int_0^1 dx(ax+b)^{-1}\ln|mx+n|e^{-\gamma/x}$. In view of the result in [8], we integrate by parts in the indefinite integral

$$\int dx \frac{\ln|mx+n|e^{-\gamma x}}{ax+b} = a^{-1}\left\{\ln\left|\frac{mb-na}{a}\right|\ln\left|\frac{m(ax+b)}{mb-na}\right| - L_2\left(\frac{m(ax+b)}{mb-na}\right)\right\}e^{-\gamma x}$$
$$\tag{12} + \gamma a^{-1}\int dx\left\{\ln\left|\frac{mb-na}{a}\right|\ln\left|\frac{m(ax+b)}{mb-na}\right| - L_2\left(\frac{m(ax+b)}{mb-na}\right)\right\}e^{-\gamma x} + \text{const.}$$

It can be seen that the solution of the integral on the right in (12) reduces to the evaluation of the integrals $\int dx\, e^{-\gamma x}$, $\int dx \ln|ax+b|e^{-\gamma x}$, and

$$\int dx\, L_2(m(ax+b)/(mb-na))e^{-\gamma x}.$$

The first one is trivial. The second integral on substituting $ax+b=y$ takes the form $\int dy \ln|y|e^{\lambda y}$ which equals $\lambda^{-1}\{\ln|y|e^{\lambda y} + E_1(-\lambda y)\}$ as given in [2]. Substituting back for y, we obtain

$$\tag{13} \int dx \ln|ax+b|e^{-\gamma x} = -\gamma^{-1}\{e^{-\gamma x}\ln|ax+b| + e^{\gamma b/a}E_1(\gamma x+\gamma b/a)\} + \text{const.}$$

Then the integral (12) reads

$$\int dx \frac{\ln|mx+n|e^{-\gamma x}}{ax+b}$$
$$\tag{14} = -a^{-1}\left\{L_2\left(\frac{m(ax+b)}{mb-na}\right)e^{-\gamma x} + e^{\gamma b/a}\ln\left|\frac{mb-na}{a}\right|E_1(\gamma x+\gamma b/a) + \gamma\int dx\, L_2\left(\frac{m(ax+b)}{mb-na}\right)e^{-\gamma x}\right\} + \text{const.}$$

In particular,

$$\int_1^\infty dx \frac{\ln|mx+n|e^{-\gamma x}}{ax+b}$$
$$\tag{15} = a^{-1}\left\{L_2\left(\frac{m(a+b)}{mb-na}\right)e^{-\gamma} + e^{\gamma b/a}\ln\left|\frac{mb-na}{a}\right|E_1(\gamma+\gamma b/a) - \gamma\,\mathrm{Le}\left(\frac{ma}{mb-na}, \frac{mb}{mb-na}, \gamma\right)\right\},$$

where we have introduced a new function $\mathrm{Le}(A, B, \gamma)$, $\gamma > 0$, defined by

$$\tag{16} \mathrm{Le}(A, B, \gamma) = \int_1^\infty dx\, L_2(Ax+B)e^{-\gamma x}.$$

The function $\mathrm{Le}(A, B, \gamma)$ is evaluated in Section 6.

We now consider the calculation of $\int_0^1 dx \ln|mx+n|e^{-\gamma/x}$. Substituting $x = t^{-1}$, we have

(17) $\int_0^1 dx \ln|mx+n|e^{-\gamma/x} = \int_1^\infty dt\, t^{-2}\ln|nt+m|e^{-\gamma t} - \int_1^\infty dt\, t^{-2}\ln t e^{-\gamma t}.$

The first integral on the right can be integrated by parts, giving the integral solved in (9) together with an integral of the type (15). The second integral splits similarly into the type (8) and the exponential integral of the second order $E_1^{(2)}(\gamma)$ defined by (5). Consequently, we write the integral (17) in the form

$$\begin{aligned}\int_0^1 dx \ln|mx+n|e^{-\gamma/x} &= \ln|m+n|e^{-\gamma} + [n/m - \gamma\ln|m| + \gamma]\, E_1(\gamma) - e^{-\gamma}\\ &\quad - n/m\, e^{\gamma m/n}E_1(\gamma+\gamma m/n) - \gamma L_2(-n/m)e^{-\gamma}\\ &\quad + \gamma E_1^{(2)}(\gamma) + \gamma^2 \mathrm{Le}(-n/m, 0, \gamma), \quad \gamma > 0, m \neq 0.\end{aligned} \tag{18}$$

In particular, if $n = 0$,

$$\int_0^1 dx \ln|mx|e^{-\gamma/x} = [\ln|m| - 1]\,[e^{-\gamma} - \gamma E_1(\gamma)] + \gamma E_1^{(2)}(\gamma), \qquad \gamma > 0, \quad m \neq 0. \tag{19}$$

The integral $\int_0^1 dx(ax+b)^{-1}\ln|mx+n|e^{-\gamma/x}$ converts by substituting $x = t^{-1}$ and applying (7) into three integrals of the type (15) and the function (5). After some calculation we obtain

$$\begin{aligned}\int_0^1 dx \frac{\ln|mx+n|e^{-\gamma/x}}{ax+b}\\ = a^{-1}\Bigg\{&\ln|m|E_1(\gamma) - e^{\gamma a/b}\ln\left|\frac{na-mb}{a}\right| E_1(\gamma+\gamma a/b)\\ &+ \left[L_2\left(-\frac{n}{m}\right) - L_2\left(\frac{n(a+b)}{na-mb}\right) + L_2\left(\frac{a+b}{a}\right)\right] e^{-\gamma} - E_1^{(2)}(\gamma)\\ &- \left[\mathrm{Le}\left(-\frac{n}{m}, 0, \gamma\right) - \mathrm{Le}\left(\frac{nb}{na-mb}, \frac{na}{na-mb}, \gamma\right) + \mathrm{Le}\left(\frac{b}{a}, 1, \gamma\right)\right]\Bigg\},\\ &\gamma > 0, a \neq 0, m \neq 0.\end{aligned} \tag{20}$$

(For $b = 0$, see the note at (9); if $m = 0$, (20) becomes (9); if $a = 0$, (20) becomes (18).)

5. Integrals $\int_0^1 dx\, E_1(m/x + n)$ and $\int_0^1 dx(ax+b)^{-1}E_1(m/x+n)$. To evaluate $\int_0^1 dx\, E_1(m/x+n)$, we proceed as follows. The substitution $x = t^{-1}$ converts this integral into $\int_1^\infty dt\, t^{-2}E_1(mt+n)$. Integrating by parts and then using the identity (7) and relation (4), we have

$$\int_0^1 dx\, E_1(m/x+n) = n^{-1}\{[m+n]\, E_1(m+n) - me^{-n}E_1(m)\}, \qquad m \neq 0, \quad n \neq 0. \tag{21}$$

(For $n = 0$, (21) becomes (8) after one integration by parts.)

In the second integral of this group we substitute $x = t^{-1}$ again and apply relation (7). In this way the problem reduces to the evaluation of $\int_1^\infty dt\, t^{-1}E_1(mt+n)$ and

$\int_1^\infty dt(bt+a)^{-1}E_1(mt+n)$. Integrating now by parts and using (15), we obtain

$$\int_0^1 dx \frac{E_1(m/x+n)}{ax+b}$$

$$(22)\quad = a^{-1}e^{-n}\left\{e^{-m}\left[L_2\left(\frac{m+n}{n}\right) - L_2\left(\frac{b(m+n)}{nb-ma}\right)\right] - e^n \ln\left|\frac{nb-ma}{n(a+b)}\right| E_1(m+n) - m\left[\mathrm{Le}\left(\frac{m}{n},1,m\right) - \mathrm{Le}\left(\frac{mb}{nb-ma},\frac{nb}{nb-ma},m\right)\right]\right\},$$

$$m > 0,\ n \neq 0,\ a \neq 0.$$

(For $n = 0$, (22) can be evaluated using (5) and (15).)

6. Evaluation of $\mathrm{Le}(A, B, \gamma) = \int_1^\infty dx\, L_2(Ax+B)e^{-\gamma x}$. In (16) of Section 4 we have introduced the function $\mathrm{Le}(A, B, \gamma)$ of three variables A, B, γ. This function converges always for $\gamma > 0$ and real A, B. In particular, if $A = 0$,

$$(23)\qquad \mathrm{Le}(0, B, \gamma) = L_2(B)\gamma^{-1}e^{-\gamma}.$$

Let $A \neq 0$, $\gamma > 0$. The substitution $Ax + B = y$ in (16) leads to the function $\mathrm{Le}^*(\alpha, \kappa)$ defined by

$$(24)\qquad \mathrm{Le}^*(\alpha,\kappa) = \begin{cases} \int_\alpha^\infty dy\, L_2(y)e^{\kappa(\alpha-y)}, & \kappa > 0, \\ \int_\alpha^{-\infty} dy\, L_2(y)e^{\kappa(\alpha-y)}, & \kappa < 0. \end{cases}$$

Then

$$(25)\qquad \mathrm{Le}(A, B, \gamma) = A^{-1}e^{-\gamma}\mathrm{Le}^*(A+B, \gamma/A).$$

The function $\mathrm{Le}^*(\alpha, \kappa)$ has been computed numerically by using the tables of the dilogarithm from Mitchell's paper [8] for the integrand and Simpson's rule for the integration.

For applications in physical problems, accuracy to three decimals seems to be sufficient. A table of the function $\mathrm{Le}^*(\alpha, \kappa)$ for $|\alpha| \leqslant 50$ $(\Delta\alpha = 1)$ and $2 \leqslant |\kappa| \leqslant 10$ $(\Delta\kappa = 1)$, $0 < |\kappa| \leqslant 1$ $(\Delta\kappa = 0.1)$ is given at the end of this paper.

In Fig. 2 the function $\mathrm{Le}^*(\alpha, \kappa)$ is represented as a surface in the three-dimensional space over the α, κ-plane. Only the part for $\kappa < 0$ is shown. It can be seen that $\mathrm{Le}^*(\alpha, \kappa)$ possesses a sharp maximum in the vicinity of the point $(\alpha \approx 7, \kappa \approx -0.3)$. For $-0.3 \lesssim \kappa < 0$ the function decreases very steeply to negative values. For positive κ the function has a similar behavior. Notice that $\lim_{\kappa\to 0} \mathrm{Le}^*(\alpha, \kappa) = -\infty$.

7. Conclusion. The evaluation of the integrals given by (1) has been reduced to tabulated functions. In some cases the above-mentioned table of $\mathrm{Le}^*(\alpha, \kappa)$ is not useable for the evaluation of $\mathrm{Le}(A, B, \gamma)$, because it is not detailed enough for interpolation in the region of very steep changes, or when $|\alpha| > 50$ and $|\kappa| > 10$. Then the function $\mathrm{Le}(A, B, \gamma)$ can be computed by a direct numerical integration in (16) which

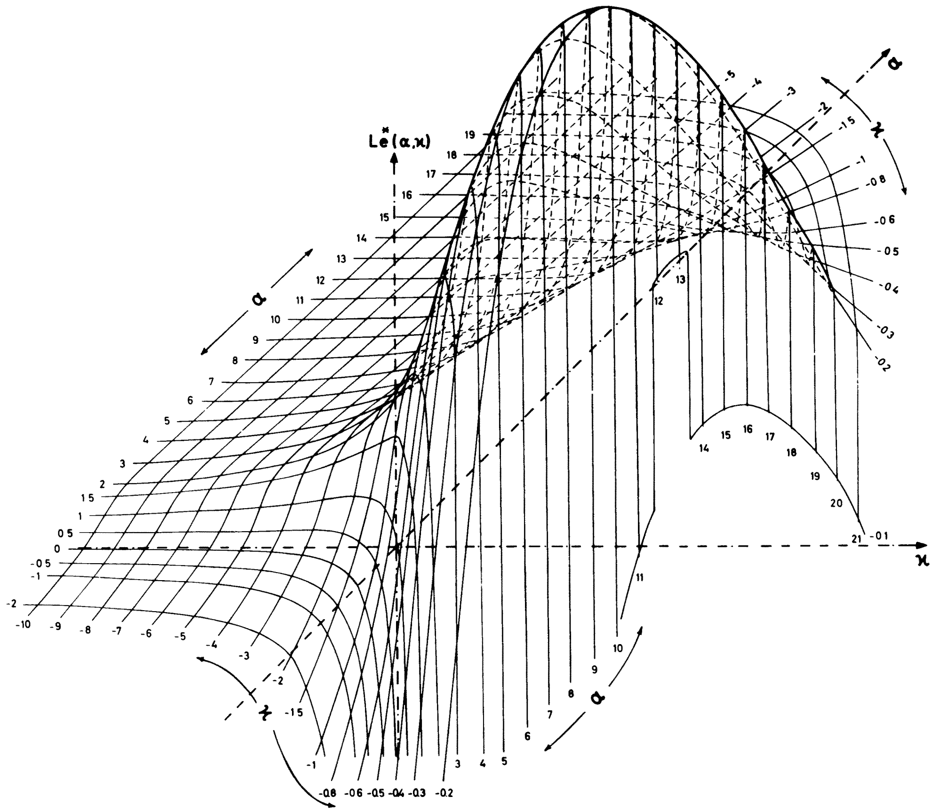

FIGURE 2. *Surface over the* α, κ*-plane described by the function* $\mathrm{Le}^*(\alpha, \kappa)$, $\kappa < 0$.

is quite simple because the singularities on the integration path in (1) have been avoided by the previous treatment.

Acknowledgement. The author wishes to thank Miss Hanna Tettenborn for computer programming and Mrs. Christine Degendorfer for preparing the manuscript.

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Le*(α, κ)

α \ κ	-10.0	-9.0	-8.0	-7.0	-6.0	-5.0	-4.0	-3.0	-2.0
0.0	-0.009	-0.012	-0.015	-0.019	-0.026	-0.037	-0.056	-0.098	-0.209
-1.0	-0.089	-0.100	-0.113	-0.131	-0.156	-0.191	-0.246	-0.345	-0.566
-2.0	-0.150	-0.167	-0.188	-0.216	-0.254	-0.309	-0.392	-0.536	-0.845
-3.0	-0.199	-0.222	-0.250	-0.287	-0.336	-0.406	-0.513	-0.695	-1.078
-4.0	-0.242	-0.269	-0.303	-0.347	-0.406	-0.490	-0.617	-0.833	-1.280
-5.0	-0.280	-0.311	-0.350	-0.400	-0.468	-0.564	-0.709	-0.955	-1.460
-6.0	-0.314	-0.348	-0.392	-0.448	-0.524	-0.631	-0.792	-1.065	-1.623
-7.0	-0.345	-0.383	-0.431	-0.492	-0.575	-0.692	-0.868	-1.166	-1.772
-8.0	-0.373	-0.414	-0.466	-0.533	-0.622	-0.748	-0.938	-1.258	-1.909
-9.0	-0.400	-0.444	-0.499	-0.570	-0.666	-0.800	-1.004	-1.345	-2.037
-10.0	-0.424	-0.471	-0.530	-0.605	-0.707	-0.849	-1.064	-1.426	-2.157
-11.0	-0.448	-0.497	-0.559	-0.638	-0.745	-0.895	-1.122	-1.502	-2.270
-12.0	-0.469	-0.521	-0.586	-0.670	-0.782	-0.939	-1.176	-1.574	-2.377
-13.0	-0.490	-0.544	-0.612	-0.699	-0.816	-0.980	-1.227	-1.642	-2.479
-14.0	-0.510	-0.566	-0.636	-0.727	-0.849	-1.019	-1.276	-1.707	-2.576
-15.0	-0.529	-0.587	-0.660	-0.754	-0.880	-1.057	-1.323	-1.769	-2,668
-16.0	-0.547	-0.607	-0.683	-0.780	-0.910	-1.093	-1.368	-1.829	-2.757
-17.0	-0.565	-0.626	-0.704	-0.805	-0.939	-1.127	-1.411	-1.886	-2.842
-18.0	-0.581	-0.645	-0.725	-0.828	-0.966	-1.160	-1.452	-1.941	-2.924
-19.0	-0.597	-0.663	-0.745	-0.851	-0.993	-1.192	-1.492	-1.993	-3.003
-20.0	-0.613	-0.680	-0.764	-0.873	-1.019	-1.223	-1.530	-2.045	-3.079
-21.0	-0.629	-0.697	-0.783	-0.895	-1.044	-1.253	-1.568	-2.094	-3.153
-22.0	-0.642	-0.713	-0.801	-0.915	-1.068	-1.282	-1.604	-2.142	-3.224
-23.0	-0.656	-0.728	-0.819	-0.935	-1.091	-1.310	-1.638	-2.188	-3.293
-24.0	-0.670	-0.743	-0.836	-0.955	-1.113	-1.337	-1.672	-2.233	-3.360
-25.0	-0.683	-0.758	-0.852	-0.973	-1.135	-1.363	-1.705	-2.277	-3.425
-26.0	-0.696	-0.772	-0.868	-0.992	-1.157	-1.389	-1.737	-2.319	-3.489
-27.0	-0.709	-0.786	-0.884	-1.009	-1.178	-1.413	-1.768	-2.360	-3.551
-28.0	-0.721	-0.800	-0.899	-1.027	-1.198	-1.438	-1.798	-2.401	-3.611
-29.0	-0.733	-0.813	-0.914	-1.044	-1.218	-1.461	-1.828	-2.440	-3.669
-30.0	-0.744	-0.826	-0.928	-1.060	-1.237	-1.484	-1.857	-2.478	-3.727
-31.0	-0.756	-0.839	-0.942	-1.076	-1.256	-1.507	-1.885	-2.516	-3.783
-32.0	-0.767	-0.851	-0.956	-1.092	-1.274	-1.529	-1.912	-2.552	-3.837
-33.0	-0.778	-0.863	-0.970	-1.108	-1.292	-1.550	-1.939	-2.588	-3.891
-34.0	-0.788	-0.875	-0.983	-1.123	-1.309	-1.572	-1.965	-2.623	-3.943
-35.0	-0.799	-0.886	-0.996	-1.137	-1.327	-1.592	-1.991	-2.657	-3.994
-36.0	-0.809	-0.897	-1.009	-1.152	-1.343	-1.612	-2.016	-2.691	-4.044
-37.0	-0.819	-0.908	-1.021	-1.166	-1.360	-1.632	-2.041	-2.724	-4.093
-38.0	-0.829	-0.919	-1.033	-1.180	-1.376	-1.651	-2.065	-2.756	-4.142
-39.0	-0.838	-0.930	-1.045	-1.194	-1.392	-1.670	-2.089	-2.788	-4.189
-40.0	-0.848	-0.940	-1.057	-1.207	-1.408	-1.689	-2.112	-2.819	-4.235
-41.0	-0.857	-0.951	-1.068	-1.220	-1.423	-1.707	-2.135	-2.849	-4.281
-42.0	-0.866	-0.961	-1.080	-1.233	-1.438	-1.726	-2.158	-2.879	-4.326
-43.0	-0.875	-0.970	-1.091	-1.246	-1.453	-1.743	-2.180	-2.908	-4.370
-44.0	-0.884	-0.980	-1.102	-1.258	-1.467	-1.761	-2.201	-2.937	-4.413
-45.0	-0.892	-0.990	-1.112	-1.270	-1.481	-1.778	-2.223	-2.966	-4.455
-46.0	-0.901	-0.999	-1.123	-1.282	-1.496	-1.795	-2.244	-2.994	-4.497
-47.0	-0.909	-1.008	-1.133	-1.294	-1.509	-1.811	-2.264	-3.021	-4.539
-48.0	-0.917	-1.018	-1.143	-1.306	-1.523	-1.828	-2.285	-3.049	-4.579
-49.0	-0.925	-1.026	-1.153	-1.317	-1.536	-1.843	-2.305	-3.075	-4.619
-50.0	-0.933	-1.035	-1.163	-1.329	-1.549	-1.859	-2.325	-3.101	-4.658

Le*(α, κ)

α \ κ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
0.0	-0.745	-0.900	-1.111	-1.407	-1.845	-2.533	-3.715	-6.041	-11.809	-35.733
-1.0	-1.392	-1.606	-1.887	-2.271	-2.819	-3.654	-5.039	-7.669	-13.956	-39.040
-2.0	-1.909	-2.172	-2.513	-2.971	-3.615	-4.576	-6.139	-9.039	-15.794	-41.951
-3.0	-2.348	-2.654	-3.047	-3.571	-4.299	-5.372	-7.095	-10.239	-17.422	-44.586
-4.0	-2.732	-3.076	-3.517	-4.093	-4.903	-6.079	-7.947	-11.315	-18.896	-47.008
-5.0	-3.076	-3.455	-3.938	-4.574	-5.447	-6.717	-8.719	-12.296	-20.250	-49.259
-6.0	-3.389	-3.800	-4.322	-5.007	-5.945	-7.301	-9.428	-13.200	-21.505	-51.370
-7.0	-3.676	-4.117	-4.675	-5.406	-6.404	-7.341	-10.085	-14.040	-22.677	-53.361
-8.0	-3.943	-4.410	-5.003	-5.777	-6.830	-8.345	-10.698	-14.826	-23.780	-55.248
-9.0	-4.191	-4.685	-5.309	-6.124	-7.230	-8.816	-11.274	-15.567	-24.821	-57.044
-10.0	-4.425	-4.943	-5.598	-6.450	-7.607	-9.261	-11.817	-16.266	-25.810	-58.760
-11.0	-4.645	-5.186	-5.870	-6.759	-7.962	-9.682	-12.333	-16.931	-26.751	-60.403
-12.0	-4.854	-5.417	-6.128	-7.051	-8.300	-10.082	-12.823	-17.565	-27.651	-61.982
-13.0	-5.053	-5.637	-6.373	-7.330	-8.622	-10.463	-13.290	-18.170	-28.513	-63.501
-14.0	-5.242	-5.847	-6.608	-7.596	-8.930	-10.828	-13.738	-18.750	-29.341	-64.967
-15.0	-5.424	-6.047	-6.832	-7.851	-9.225	-11.177	-14.167	-19.308	-30.137	-66.383
-16.0	-5.598	-6.240	-7.048	-8.096	-9.508	-11.513	-14.580	-19.844	-30.905	-67.754
-17.0	-5.765	-6.425	-7.255	-8.331	-9.780	-11.836	-14.978	-20.361	-31.646	-69.082
-18.0	-5.926	-6.603	-7.454	-8.557	-10.042	-12.147	-15.361	-20.860	-32.363	-70.370
-19.0	-6.081	-6.774	-7.647	-8.776	-10.295	-12.448	-15.732	-21.343	-33.058	-71.623
-20.0	-6.231	-6.940	-7.833	-8.987	-10.540	-12.739	-16.090	-21.811	-33.732	-72.840
-21.0	-6.376	-7.101	-8.013	-9.192	-10.777	-13.021	-16.437	-22.264	-34.386	-74.026
-22.0	-6.516	-7.257	-8.187	-9.390	-11.007	-13.294	-16.774	-22.704	-35.021	-75.182
-23.0	-6.652	-7.408	-8.356	-9.582	-11.230	-13.559	-17.102	-23.132	-35.640	-76.309
-24.0	-6.785	-7.554	-8.520	-9.769	-11.446	-13.817	-17.420	-23.549	-36.242	-77.409
-25.0	-6.913	-7.697	-8.680	-9.951	-11.657	-14.068	-17.730	-23.954	-36.830	-78.484
-26.0	-7.039	-7.835	-8.836	-10.128	-11.863	-14.312	-18.032	-24.349	-37.403	-79.535
-27.0	-7.160	-7.970	-8.987	-10.300	-12.062	-14.550	-18.326	-24.735	-37.962	-80.563
-28.0	-7.279	-8.102	-9.135	-10.468	-12.258	-14.783	-18.613	-25.111	-38.509	-81.570
-29.0	-7.395	-8.231	-9.279	-10.632	-12.448	-15.009	-18.894	-25.479	-39.043	-82.555
-30.0	-7.508	-8.356	-9.419	-10.793	-12.634	-15.231	-19.168	-25.838	-39.566	-83.521
-31.0	-7.619	-8.479	-9.557	-10.949	-12.816	-15.448	-19.436	-26.190	-40.078	-84.468
-32.0	-7.727	-8.599	-9.691	-11.102	-12.993	-15.659	-19.698	-26.534	-40.579	-85.397
-33.0	-7.833	-8.716	-9.823	-11.252	-13.167	-15.867	-19.955	-26.871	-41.070	-86.309
-34.0	-7.937	-8.831	-9.952	-11.399	-13.338	-16.069	-20.206	-27.202	-41.552	-87.204
-35.0	-8.038	-8.943	-10.078	-11.542	-13.504	-16.268	-20.452	-27.525	-42.024	-88.083
-36.0	-8.137	-9.053	-10.201	-11.683	-13.668	-16.463	-20.694	-27.843	-42.488	-88.947
-37.0	-8.235	-9.161	-10.323	-11.821	-13.828	-16.655	-20.931	-28.154	-42.943	-89.797
-38.0	-8.330	-9.267	-10.442	-11.957	-13.985	-16.842	-21.164	-28.460	-43.390	-90.632
-39.0	-8.424	-9.371	-10.558	-12.089	-14.140	-17.026	-21.392	-28.761	-43.830	-91.454
-40.0	-8.516	-9.473	-10.673	-12.220	-14.292	-17.207	-21.617	-29.056	-44.262	-92.263
-41.0	-8.606	-9.573	-10.785	-12.348	-14.440	-17.385	-21.837	-29.346	-44.686	-93.059
-42.0	-8.695	-9.672	-10.896	-12.474	-14.587	-17.560	-22.054	-29.631	-45.104	-93.843
-43.0	-8.782	-9.769	-11.004	-12.598	-14.731	-17.732	-22.267	-29.912	-45.515	-94.615
-44.0	-8.868	-9.864	-11.111	-12.720	-14.872	-17.901	-22.476	-30.188	-45.920	-95.376
-45.0	-8.953	-9.957	-11.216	-12.839	-15.012	-18.067	-22.683	-30.460	-46.318	-96.125
-46.0	-9.036	-10.050	-11.320	-12.957	-15.149	-18.230	-22.886	-30.727	-46.711	-96.865
-47.0	-9.117	-10.140	-11.422	-13.073	-15.283	-18.391	-23.085	-30.991	-47.097	-97.593
-48.0	-9.198	-10.230	-11.522	-13.188	-15.416	-18.550	-23.283	-31.250	-47.478	-98.313
-49.0	-9.277	-10.317	-11.620	-13.300	-15.547	-18.706	-23.476	-31.506	-47.854	-99.022
-50.0	-9.355	-10.404	-11.718	-13.411	-15.676	-18.861	-23.668	-31.759	-48.224	-99.722

Le*(α, κ)

α \ κ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	9.399	6.955	5.140	3.920	3.041	2.428	1.976	1.634	1.369	1.161
-1.0	8.083	5.286	3.414	2.232	1.475	0.974	0.634	0.397	0.229	0.109
-2.0	6.223	3.284	1.530	0.539	-0.024	-0.347	-0.532	-0.635	-0.689	-0.713
-3.0	4.013	1.145	-0.348	-1.050	-1.365	-1.484	-1.504	-1.475	-1.422	-1.360
-4.0	1.573	-1.102	-2.132	-2.495	-2.541	-2.454	-2.318	-2.169	-2.023	-1.888
-5.0	-2.986	-3.231	-3.802	-3.796	-3.571	-3.289	-3.010	-2.756	-2.531	-2.334
-6.0	-5.485	-5.299	-5.349	-4.962	-4.477	-4.015	-3.611	-3.265	-2.973	-2.724
-7.0	-8.056	-7.286	-6.775	-6.011	-5.281	-4.657	-4.141	-3.716	-3.365	-3.071
-8.0	-10.665	-9.183	-8.087	-6.959	-6.002	-5.232	-4.618	-4.123	-3.719	-3.385
-9.0	-13.287	-10.984	-9.297	-7.820	-6.655	-5.754	-5.051	-4.494	-4.044	-3.674
-10.0	-15.902	-12.691	-10.414	-8.608	-7.252	-6.232	-5.450	-4.836	-4.344	-3.941
-11.0	-18.494	-14.305	-11.448	-9.334	-7.802	-6.674	-5.820	-5.154	-4.623	-4.190
-12.0	-21.057	-15.831	-12.409	-10.006	-8.313	-7.086	-6.165	-5.452	-4.885	-4.424
-13.0	-23.580	-17.274	-13.306	-10.632	-8.792	-7.473	-6.490	-5.733	-5.132	-4.644
-14.0	-26.056	-18.639	-14.146	-11.219	-9.241	-7.838	-6.797	-5.998	-5.366	-4.853
-15.0	-28.480	-19.933	-14.935	-11.772	-9.666	-8.183	-7.089	-6.250	-5.588	-5.052
-16.0	-30.850	-21.159	-15.679	-12.295	-10.068	-8.511	-7.366	-6.490	-5.800	-5.242
-17.0	-33.162	-22.323	-16.382	-12.791	-10.452	-8.824	-7.630	-6.720	-6.002	-5.423
-18.0	-35.417	-23.431	-17.050	-13.263	-10.818	-9.123	-7.884	-6.939	-6.196	-5.597
-19.0	-37.611	-24.485	-17.686	-13.715	-11.168	-9.410	-8.127	-7.151	-6.383	-5.764
-20.0	-39.748	-25.492	-18.293	-14.147	-11.505	-9.686	-8.361	-7.354	-6.562	-5.924
-21.0	-41.825	-26.453	-18.875	-14.562	-11.828	-9.952	-8.586	-7.549	-6.735	-6.079
-22.0	-43.845	-27.374	-19.433	-14.962	-12.141	-10.208	-8.804	-7.739	-6.903	-6.229
-23.0	-45.809	-28.257	-19.969	-15.347	-12.442	-10.456	-9.014	-7.921	-7.064	-6.374
-24.0	-47.717	-29.106	-20.485	-15.719	-12.733	-10.696	-9.218	-8.099	-7.221	-6.515
-25.0	-49.571	-29.921	-20.984	-16.078	-13.015	-10.928	-9.416	-8.270	-7.373	-6.651
-26.0	-51.373	-30.708	-21.466	-16.426	-13.289	-11.153	-9.607	-8.437	-7.521	-6.784
-27.0	-53.124	-31.466	-21.932	-16.764	-13.555	-11.372	-9.794	-8.599	-7.664	-6.912
-28.0	-54.825	-32.199	-22.384	-17.092	-13.813	-11.585	-9.975	-8.757	-7.803	-7.037
-29.0	-56.479	-32.908	-22.823	-17.411	-14.064	-11.792	-10.151	-8.910	-7.939	-7.159
-30.0	-58.088	-33.595	-23.249	-17.722	-14.308	-11.994	-10.323	-9.060	-8.072	-7.278
-31.0	-59.652	-34.261	-23.664	-18.024	-14.546	-12.190	-10.490	-9.206	-8.201	-7.394
-32.0	-61.173	-34.907	-24.067	-18.319	-14.779	-12.382	-10.654	-9.348	-8.327	-7.507
-33.0	-62.654	-35.536	-24.461	-18.607	-15.006	-12.570	-10.813	-9.487	-8.450	-7.618
-34.0	-64.096	-36.147	-24.844	-18.887	-15.227	-12.753	-10.969	-9.623	-8.571	-7.726
-35.0	-65.500	-36.742	-25.219	-19.162	-15.444	-12.932	-11.122	-9.756	-8.689	-7.832
-36.0	-66.868	-37.322	-25.585	-19.430	-15.656	-13.107	-11.271	-9.886	-8.804	-7.935
-37.0	-68.200	-37.838	-25.942	-19.692	-15.863	-13.279	-11.417	-10.013	-8.917	-8.037
-38.0	-69.500	-38.440	-26.292	-19.949	-16.066	-13.447	-11.561	-10.138	-9.027	-8.136
-39.0	-70.768	-38.980	-26.635	-20.201	-16.265	-13.611	-11.701	-10.261	-9.136	-8.233
-40.0	-72.005	-39.507	-26.970	-20.447	-16.460	-13.773	-11.839	-10.381	-9.242	-8.329
-41.0	-73.212	-40.023	-27.299	-20.689	-16.651	-13.931	-11.974	-10.498	-9.347	-8.423
-42.0	-74.392	-40.528	-27.621	-20.926	-16.839	-14.086	-12.106	-10.614	-9.449	-8.515
-43.0	-75.544	-41.023	-27.937	-21.159	-17.023	-14.239	-12.236	-10.728	-9.550	-8.605
-44.0	-76.671	-41.507	-28.247	-21.387	-17.204	-14.389	-12.364	-10.839	-9.649	-8.694
-45.0	-77.772	-41.983	-28.552	-21.612	-17.382	-14.536	-12.490	-10.949	-9.746	-8.781
-46.0	-78.850	-42.449	-28.851	-21.832	-17.556	-14.681	-12.613	-11.056	-9.841	-8.867
-47.0	-79.905	-42.906	-29.144	-22.049	-17.728	-14.823	-12.735	-11.162	-9.935	-8.951
-48.0	-80.938	-43.355	-29.433	-22.262	-17.897	-14.963	-12.854	-11.266	-10.028	-9.034
-49.0	-81.949	-43.797	-29.717	-22.472	-18.064	-15.101	-12.972	-11.369	-10.119	-9.116
-50.0	-82.940	-44.230	-29.996	-22.678	-18.227	-15.236	-13.088	-11.470	-10.208	-9.196

Le*(α, κ)

α \ κ	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
0.0	0.330	0.141	0.075	0.046	0.031	0.022	0.017	0.013	0.010
-1.0	-0.199	-0.187	-0.159	-0.135	-0.117	-0.103	-0.092	-0.083	-0.076
-2.0	-0.562	-0.413	-0.323	-0.265	-0.224	-0.194	-0.171	-0.153	-0.139
-3.0	-0.843	-0.592	-0.455	-0.369	-0.310	-0.268	-0.236	-0.210	-0.190
-4.0	-1.077	-0.743	-0.567	-0.453	-0.384	-0.331	-0.291	-0.259	-0.234
-5.0	-1.280	-0.875	-0.664	-0.535	-0.448	-0.386	-0.339	-0.302	-0.273
-6.0	-1.460	-0.993	-0.752	-0.605	-0.506	-0.435	-0.382	-0.340	-0.307
-7.0	-1.622	-1.099	-0.831	-0.668	-0.559	-0.480	-0.421	-0.375	-0.339
-8.0	-1.771	-1.197	-0.904	-0.726	-0.607	-0.522	-0.457	-0.408	-0.368
-9.0	-1.909	-1.288	-0.972	-0.780	-0.652	-0.560	-0.491	-0.437	-0.394
-10.0	-2.037	-1.372	-1.034	-0.830	-0.693	-0.596	-0.522	-0.465	-0.420
-11.0	-2.157	-1.451	-1.094	-0.877	-0.733	-0.629	-0.552	-0.491	-0.443
-12.0	-2.270	-1.526	-1.149	-0.922	-0.770	-0.661	-0.579	-0.516	-0.465
-13.0	-2.377	-1.597	-1.202	-0.964	-0.805	-0.691	-0.605	-0.539	-0.486
-14.0	-2.479	-1.664	-1.252	-1.004	-0.838	-0.719	-0.630	-0.561	-0.506
-15.0	-2.576	-1.728	-1.300	-1.042	-0.870	-0.747	-0.654	-0.583	-0.525
-16.0	-2.668	-1.789	-1.346	-1.079	-0.900	-0.773	-0.677	-0.603	-0.544
-17.0	-2.757	-1.848	-1.389	-1.114	-0.929	-0.798	-0.699	-0.622	-0.561
-18.0	-2.842	-1.904	-1.432	-1.147	-0.957	-0.822	-0.720	-0.641	-0.578
-19.0	-2.924	-1.958	-1.472	-1.180	-0.984	-0.845	-0.740	-0.659	-0.594
-20.0	-3.002	-2.010	-1.511	-1.211	-1.010	-0.867	-0.760	-0.676	-0.610
-21.0	-3.079	-2.061	-1.549	-1.241	-1.035	-0.888	-0.778	-0.693	-0.625
-22.0	-3.152	-2.110	-1.585	-1.270	-1.060	-0.909	-0.797	-0.709	-0.639
-23.0	-3.223	-2.157	-1.621	-1.298	-1.083	-0.929	-0.814	-0.725	-0.654
-24.0	-3.293	-2.203	-1.655	-1.326	-1.106	-0.949	-0.831	-0.740	-0.667
-25.0	-3.360	-2.247	-1.688	-1.352	-1.128	-0.968	-0.848	-0.755	-0.681
-26.0	-3.425	-2.291	-1.721	-1.378	-1.150	-0.986	-0.864	-0.769	-0.694
-27.0	-3.488	-2.333	-1.752	-1.403	-1.171	-1.004	-0.880	-0.783	-0.706
-28.0	-3.550	-2.374	-1.783	-1.428	-1.191	-1.022	-0.895	-0.797	-0.718
-29.0	-3.610	-2.414	-1.813	-1.452	-1.211	-1.039	-0.910	-0.810	-0.730
-30.0	-3.669	-2.452	-1.842	-1.475	-1.230	-1.056	-0.925	-0.823	-0.742
-31.0	-3.726	-2.490	-1.870	-1.498	-1.249	-1.072	-0.939	-0.836	-0.753
-32.0	-3.782	-2.528	-1.898	-1.520	-1.268	-1.088	-0.953	-0.848	-0.765
-33.0	-3.837	-2.564	-1.925	-1.542	-1.286	-1.103	-0.966	-0.860	-0.776
-34.0	-3.890	-2.599	-1.952	-1.563	-1.303	-1.118	-0.980	-0.872	-0.786
-35.0	-3.942	-2.634	-1.978	-1.584	-1.321	-1.133	-0.993	-0.883	-0.797
-36.0	-3.994	-2.668	-2.003	-1.604	-1.338	-1.148	-1.005	-0.895	-0.807
-37.0	-4.044	-2.701	-2.028	-1.624	-1.354	-1.162	-1.018	-0.906	-0.817
-38.0	-4.093	-2.734	-2.053	-1.643	-1.371	-1.176	-1.030	-0.917	-0.827
-39.0	-4.141	-2.766	-2.077	-1.663	-1.387	-1.190	-1.042	-0.927	-0.836
-40.0	-4.188	-2.798	-2.100	-1.682	-1.402	-1.203	-1.054	-0.938	-0.846
-41.0	-4.235	-2.828	-2.123	-1.700	-1.418	-1.216	-1.065	-0.948	-0.855
-42.0	-4.280	-2.859	-2.146	-1.718	-1.433	-1.229	-1.077	-0.958	-0.864
-43.0	-4.325	-2.888	-2.168	-1.736	-1.448	-1.242	-1.088	-0.968	-0.873
-44.0	-4.369	-2.918	-2.190	-1.754	-1.462	-1.254	-1.099	-0.978	-0.882
-45.0	-4.412	-2.946	-2.212	-1.771	-1.477	-1.267	-1.109	-0.987	-0.890
-46.0	-4.455	-2.975	-2.233	-1.788	-1.491	-1.279	-1.120	-0.997	-0.899
-47.0	-4.497	-3.003	-2.254	-1.804	-1.505	-1.291	-1.130	-1.006	-0.907
-48.0	-4.538	-3.030	-2.274	-1.821	-1.518	-1.302	-1.141	-1.015	-0.915
-49.0	-4.579	-3.057	-2.295	-1.837	-1.532	-1.314	-1.151	-1.024	-0.923
-50.0	-4.618	-3.083	-2.314	-1.853	-1.545	-1.325	-1.161	-1.033	-0.931

Le*(α, κ)

α \ κ	-10.0	-9.0	-8.0	-7.0	-6.0	-5.0	-4.0	-3.0	-2.0
50.0	-0.440	-0.488	-0.548	-0.625	-0.729	-0.874	-1.091	-1.452	-2.171
49.0	-0.432	-0.479	-0.538	-0.614	-0.716	-0.858	-1.071	-1.426	-2.132
48.0	-0.424	-0.470	-0.528	-0.603	-0.702	-0.842	-1.051	-1.399	-2.092
47.0	-0.416	-0.461	-0.518	-0.591	-0.689	-0.826	-1.031	-1.372	-2.051
46.0	-0.407	-0.452	-0.507	-0.579	-0.675	-0.809	-1.010	-1.344	-2.010
45.0	-0.399	-0.442	-0.497	-0.567	-0.661	-0.792	-0.989	-1.316	-1.968
44.0	-0.390	-0.433	-0.486	-0.555	-0.647	-0.775	-0.968	-1.288	-1.925
43.0	-0.382	-0.423	-0.475	-0.543	-0.632	-0.758	-0.946	-1.259	-1.881
42.0	-0.373	-0.414	-0.464	-0.530	-0.618	-0.741	-0.924	-1.230	-1.837
41.0	-0.364	-0.404	-0.453	-0.517	-0.603	-0.723	-0.902	-1.200	-1.792
40.0	-0.355	-0.393	-0.442	-0.504	-0.588	-0.704	-0.879	-1.169	-1.746
39.0	-0.345	-0.383	-0.430	-0.491	-0.572	-0.686	-0.856	-1.139	-1.700
38.0	-0.336	-0.373	-0.418	-0.478	-0.556	-0.667	-0.832	-1.107	-1.652
37.0	-0.326	-0.362	-0.406	-0.464	-0.540	-0.649	-0.808	-1.075	-1.604
36.0	-0.316	-0.351	-0.394	-0.450	-0.524	-0.628	-0.784	-1.042	-1.554
35.0	-0.306	-0.340	-0.382	-0.436	-0.507	-0.608	-0.759	-1.008	-1.504
34.0	-0.296	-0.328	-0.369	-0.421	-0.490	-0.587	-0.733	-0.974	-1.453
33.0	-0.286	-0.317	-0.356	-0.406	-0.473	-0.567	-0.707	-0.939	-1.400
32.0	-0.275	-0.305	-0.342	-0.391	-0.455	-0.545	-0.680	-0.904	-1.346
31.0	-0.264	-0.293	-0.329	-0.375	-0.437	-0.523	-0.653	-0.867	-1.292
30.0	-0.253	-0.280	-0.315	-0.359	-0.418	-0.501	-0.625	-0.830	-1.235
29.0	-0.241	-0.268	-0.301	-0.343	-0.399	-0.478	-0.596	-0.792	-1.178
28.0	-0.230	-0.255	-0.286	-0.326	-0.380	-0.455	-0.567	-0.753	-1.119
27.0	-0.218	-0.241	-0.271	-0.309	-0.360	-0.431	-0.537	-0.713	-1.059
26.0	-0.205	-0.228	-0.256	-0.292	-0.340	-0.406	-0.506	-0.672	-0.997
25.0	-0.193	-0.214	-0.240	-0.274	-0.318	-0.381	-0.475	-0.629	-0.933
24.0	-0.180	-0.199	-0.224	-0.255	-0.297	-0.355	-0.442	-0.586	-0.868
23.0	-0.166	-0.184	-0.207	-0.236	-0.275	-0.329	-0.409	-0.541	-0.801
22.0	-0.153	-0.169	-0.190	-0.216	-0.252	-0.301	-0.375	-0.496	-0.732
21.0	-0.139	-0.154	-0.172	-0.196	-0.228	-0.273	-0.339	-0.448	-0.660
20.0	-0.124	-0.137	-0.154	-0.175	-0.204	-0.244	-0.303	-0.399	-0.587
19.0	-0.109	-0.121	-0.135	-0.154	-0.179	-0.214	-0.265	-0.349	-0.510
18.0	-0.093	-0.103	-0.116	-0.132	-0.153	-0.182	-0.226	-0.297	-0.432
17.0	-0.077	-0.085	-0.096	-0.109	-0.126	-0.150	-0.186	-0.243	-0.350
16.0	-0.060	-0.067	-0.075	-0.085	-0.098	-0.117	-0.144	-0.187	-0.265
15.0	-0.043	-0.047	-0.053	-0.060	-0.069	-0.082	-0.100	-0.128	-0.177
14.0	-0.025	-0.027	-0.030	-0.034	-0.039	-0.046	-0.055	-0.068	-0.086
13.0	-0.006	-0.006	-0.007	-0.007	-0.008	-0.008	-0.007	-0.005	0.010
12.0	0.014	0.016	0.018	0.021	0.025	0.032	0.042	0.062	0.110
11.0	0.034	0.038	0.044	0.050	0.060	0.073	0.094	0.131	0.215
10.0	0.056	0.062	0.071	0.081	0.096	0.115	0.148	0.204	0.325
9.0	0.079	0.088	0.099	0.114	0.134	0.162	0.205	0.280	0.440
8.0	0.103	0.114	0.129	0.143	0.173	0.210	0.265	0.360	0.562
7.0	0.128	0.142	0.160	0.184	0.215	0.260	0.328	0.445	0.690
6.0	0.154	0.172	0.193	0.222	0.259	0.313	0.395	0.534	0.823
5.0	0.182	0.202	0.228	0.261	0.305	0.368	0.463	0.625	0.960
4.0	0.210	0.233	0.262	0.300	0.351	0.423	0.532	0.716	1.090
3.0	0.235	0.261	0.294	0.336	0.393	0.472	0.592	0.791	1.178
2.0	0.247	0.274	0.308	0.351	0.408	0.486	0.600	0.778	1.076
1.0	0.136	0.148	0.162	0.179	0.201	0.229	0.264	0.310	0.357
0.0	-0.009	-0.012	-0.015	-0.019	-0.026	-0.037	-0.056	-0.098	-0.209

Le*(α, κ)

α \ κ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
50.0	-4.303	-4.771	-5.353	-6.097	-7.081	-8.443	-10.449	-13.693	-19.787	-34.804
49.0	-4.223	-4.682	-5.254	-5.983	-6.947	-8.281	-10.245	-13.417	-19.359	-33.898
48.0	-4.142	-4.592	-5.152	-5.867	-6.811	-8.116	-10.037	-13.136	-18.923	-32.978
47.0	-4.060	-4.501	-5.049	-5.748	-6.673	-7.949	-9.827	-12.851	-18.481	-32.047
46.0	-3.977	-4.408	-4.944	-5.628	-6.532	-7.779	-9.612	-12.561	-18.031	-31.103
45.0	-3.892	-4.314	-4.838	-5.506	-6.389	-7.607	-9.395	-12.266	-17.572	-30.147
44.0	-3.806	-4.218	-4.729	-5.382	-6.243	-7.431	-9.173	-11.966	-17.106	-29.178
43.0	-3.718	-4.120	-4.619	-5.256	-6.096	-7.253	-9.948	-11.660	-16.630	-28.198
42.0	-3.629	-4.021	-4.507	-5.128	-5.945	-7.071	-8.718	-11.348	-16.145	-27.206
41.0	-3.538	-3.920	-4.393	-4.997	-5.792	-6.886	-8.484	-11.031	-15.653	-26.203
40.0	-3.446	-3.817	-4.277	-4.864	-5.636	-6.697	-8.246	-10.707	-15.148	-25.189
39.0	-3.352	-3.712	-4.159	-4.723	-5.477	-6.505	-8.003	-10.377	-14.634	-24.164
38.0	-3.255	-3.605	-4.038	-4.590	-5.314	-6.309	-7.756	-10.041	-14.109	-23.129
37.0	-3.159	-3.496	-3.915	-4.449	-5.149	-6.109	-7.503	-9.697	-13.573	-22.086
36.0	-3.058	-3.385	-3.790	-4.305	-4.980	-5.905	-7.245	-9.346	-13.025	-21.035
35.0	-2.956	-3.271	-3.662	-4.158	-4.808	-5.697	-6.981	-8.987	-12.465	-19.977
34.0	-2.852	-3.156	-3.531	-4.008	-4.632	-5.484	-6.712	-8.620	-11.893	-18.915
33.0	-2.746	-3.037	-3.398	-3.855	-4.452	-5.266	-6.437	-8.244	-11.308	-17.848
32.0	-2.637	-2.916	-3.261	-3.698	-4.268	-5.044	-6.155	-7.860	-10.710	-16.780
31.0	-2.526	-2.793	-3.122	-3.538	-4.080	-4.816	-5.866	-7.466	-10.098	-15.715
30.0	-2.413	-2.666	-2.979	-3.374	-3.888	-4.583	-5.570	-7.062	-9.473	-14.650
29.0	-2.296	-2.536	-2.832	-3.205	-3.690	-4.344	-5.267	-6.648	-8.832	-13.592
28.0	-2.177	-2.403	-2.682	-3.033	-3.488	-4.099	-4.956	-6.223	-8.178	-12.542
27.0	-2.055	-2.267	-2.523	-2.856	-3.280	-3.847	-4.636	-5.786	-7.508	-11.506
26.0	-1.929	-2.127	-2.370	-2.675	-3.067	-3.588	-4.308	-5.337	-6.824	-10.488
25.0	-1.800	-1.983	-2.208	-2.488	-2.847	-3.322	-3.970	-4.875	-6.125	-9.493
24.0	-1.668	-1.836	-2.041	-2.296	-2.622	-3.048	-3.622	-4.400	-5.412	-8.527
23.0	-1.531	-1.683	-1.869	-2.098	-2.389	-2.766	-3.263	-3.910	-4.686	-7.597
22.0	-1.390	-1.527	-1.692	-1.895	-2.149	-2.475	-2.892	-3.407	-3.948	-6.710
21.0	-1.245	-1.365	-1.509	-1.684	-1.902	-2.174	-2.510	-2.888	-3.200	-5.876
20.0	-1.096	-1.198	-1.320	-1.467	-1.646	-1.863	-2.114	-2.354	-2.444	-5.104
19.0	-0.941	-1.025	-1.124	-1.242	-1.381	-1.541	-1.705	-1.804	-1.683	-4.405
18.0	-0.780	-0.846	-0.922	-1.009	-1.106	-1.207	-1.282	-1.240	-0.922	-3.793
17.0	-0.614	-0.660	-0.712	-0.767	-0.821	-0.861	-0.844	-0.662	-0.168	-3.282
16.0	-0.441	-0.467	-0.493	-0.515	-0.525	-0.501	-0.391	-0.072	0.598	-2.889
15.0	-0.261	-0.266	-0.265	-0.253	-0.216	-0.127	0.077	0.529	1.323	-2.632
14.0	-0.074	-0.057	-0.029	0.020	0.105	0.261	0.559	1.133	2.012	-2.534
13.0	0.122	0.162	0.219	0.305	0.440	0.664	1.054	1.736	2.652	0.024
12.0	0.327	0.391	0.478	0.603	0.789	1.080	1.557	2.327	3.222	0.009
11.0	0.542	0.630	0.749	0.914	1.151	1.508	2.061	2.892	3.696	-3.446
10.0	0.767	0.882	1.032	1.237	1.525	1.942	2.557	3.410	4.044	-4.257
9.0	1.003	1.145	1.328	1.572	1.906	2.374	3.027	3.854	4.227	-5.384
8.0	1.250	1.418	1.632	1.912	2.285	2.786	3.445	4.184	4.194	-6.873
7.0	1.505	1.697	1.933	2.246	2.642	3.150	3.769	4.346	3.884	-8.774
6.0	1.758	1.969	2.228	2.548	2.943	3.420	3.939	4.267	3.223	-11.144
5.0	1.989	2.206	2.464	2.770	3.126	3.517	3.863	3.848	2.119	-14.045
4.0	2.142	2.341	2.566	2.817	3.080	3.318	3.410	2.960	0.465	-17.539
3.0	2.039	2.231	2.376	2.516	2.624	2.638	2.400	1.444	-2.183	-21.689
2.0	1.557	1.593	1.610	1.587	1.486	1.228	0.621	-0.887	-5.334	-26.505
1.0	0.213	0.134	0.015	-0.171	-0.470	-0.978	-1.916	-3.984	-9.060	-31.706
0.0	-0.745	-0.900	-1.111	-1.407	-1.845	-2.533	-3.715	-6.041	-11.809	-35.733

$Le^*(\alpha, \kappa)$

α \ κ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
50.0	-50.713	-23.732	-15.435	-11.426	-9.068	-7.515	-6.417	-5.598	-4.964	-4.459
49.0	-50.019	-23.365	-15.185	-11.237	-8.915	-7.388	-6.307	-5.501	-4.878	-4.382
48.0	-49.316	-22.993	-14.932	-11.045	-8.761	-7.258	-6.195	-5.404	-4.791	-4.304
47.0	-48.604	-22.616	-14.675	-10.850	-8.604	-7.127	-6.082	-5.304	-4.703	-4.224
46.0	-47.882	-22.233	-14.414	-10.652	-8.444	-6.993	-5.968	-5.204	-4.613	-4.143
45.0	-47.150	-21.845	-14.149	-10.451	-8.282	-6.858	-5.851	-5.102	-4.522	-4.061
44.0	-46.407	-21.451	-13.881	-10.247	-8.113	-6.720	-5.732	-4.998	-4.430	-3.977
43.0	-45.653	-21.050	-13.608	-10.040	-7.951	-6.580	-5.612	-4.892	-4.335	-3.893
42.0	-44.889	-20.644	-13.330	-9.829	-7.781	-6.433	-5.490	-4.785	-4.240	-3.806
41.0	-44.113	-20.231	-13.048	-9.615	-7.609	-6.293	-5.365	-4.675	-4.143	-3.719
40.0	-43.326	-19.811	-12.762	-9.398	-7.433	-6.146	-5.239	-4.564	-4.044	-3.630
39.0	-42.526	-19.385	-12.470	-9.176	-7.254	-5.997	-5.110	-4.451	-3.943	-3.539
38.0	-41.713	-18.950	-12.173	-8.951	-7.073	-5.844	-4.979	-4.336	-3.840	-3.446
37.0	-40.888	-18.509	-11.872	-8.721	-6.888	-5.689	-4.845	-4.219	-3.736	-3.352
36.0	-40.049	-18.060	-11.564	-8.488	-6.699	-5.531	-4.709	-4.100	-3.630	-3.256
35.0	-39.195	-17.603	-11.251	-8.249	-6.507	-5.370	-4.570	-3.978	-3.521	-3.158
34.0	-38.327	-17.137	-10.932	-8.007	-6.311	-5.206	-4.429	-3.854	-3.410	-3.058
33.0	-37.444	-16.662	-10.607	-7.759	-6.111	-5.038	-4.285	-3.727	-3.297	-2.956
32.0	-36.544	-16.179	-10.275	-7.506	-5.907	-4.867	-4.138	-3.598	-3.182	-2.853
31.0	-35.629	-15.686	-9.937	-7.249	-5.699	-4.692	-3.987	-3.466	-3.064	-2.746
30.0	-34.695	-15.183	-9.591	-6.985	-5.486	-4.514	-3.833	-3.331	-2.944	-2.638
29.0	-33.744	-14.669	-9.238	-6.716	-5.269	-4.331	-3.676	-3.192	-2.821	-2.527
28.0	-32.774	-14.144	-8.877	-6.441	-5.046	-4.145	-3.515	-3.051	-2.695	-2.413
27.0	-31.784	-13.607	-8.508	-6.159	-4.818	-3.954	-3.351	-2.907	-2.566	-2.297
26.0	-30.774	-13.058	-8.130	-5.871	-4.585	-3.758	-3.182	-2.758	-2.434	-2.177
25.0	-29.742	-12.497	-7.743	-5.576	-4.346	-3.557	-3.009	-2.606	-2.298	-2.055
24.0	-28.687	-11.922	-7.346	-5.273	-4.101	-3.351	-2.832	-2.450	-2.159	-1.929
23.0	-27.608	-11.332	-6.939	-4.962	-3.850	-3.140	-2.649	-2.290	-2.016	-1.801
22.0	-26.503	-10.728	-6.522	-4.643	-3.591	-2.923	-2.462	-2.126	-1.869	-1.668
21.0	-25.373	-10.108	-6.093	-4.314	-3.325	-2.699	-2.269	-1.956	-1.718	-1.531
20.0	-24.214	-9.471	-5.652	-3.977	-3.052	-2.469	-2.071	-1.782	-1.562	-1.391
19.0	-23.025	-8.815	-5.198	-3.629	-2.770	-2.232	-1.866	-1.602	-1.402	-1.246
18.0	-21.804	-8.141	-4.730	-3.270	-2.479	-1.987	-1.655	-1.416	-1.236	-1.096
17.0	-20.549	-7.446	-4.247	-2.900	-2.178	-1.734	-1.436	-1.223	-1.064	-0.941
16.0	-19.260	-6.729	-3.748	-2.517	-1.867	-1.473	-1.210	-1.025	-0.887	-0.781
15.0	-17.931	-5.988	-3.232	-2.120	-1.545	-1.201	-0.976	-0.818	-0.703	-0.614
14.0	-16.561	-5.222	-2.697	-1.709	-1.211	-0.920	-0.733	-0.604	-0.511	-0.441
13.0	-15.150	-4.429	-2.143	-1.283	-0.864	-0.627	-0.480	-0.382	-0.312	-0.262
12.0	-13.689	-3.606	-1.567	-0.839	-0.503	-0.323	-0.217	-0.150	-0.105	-0.074
11.0	-12.176	-2.751	-0.967	-0.376	-0.126	0.005	0.062	0.094	0.112	0.122
10.0	-10.608	-1.861	-0.342	0.175	0.287	0.333	0.348	0.347	0.338	0.327
9.0	-8.980	0.089	0.511	0.658	0.691	0.678	0.648	0.612	0.576	0.542
8.0	-7.287	0.874	1.143	1.172	1.119	1.042	0.964	0.891	0.826	0.767
7.0	1.008	1.736	1.822	1.718	1.570	1.425	1.296	1.184	1.088	1.005
6.0	2.226	2.675	2.546	2.295	2.045	1.829	1.646	1.493	1.364	1.254
5.0	3.583	3.686	3.315	2.903	2.545	2.252	2.013	1.816	1.653	1.516
4.0	5.074	4.765	4.123	3.538	3.066	2.692	2.394	2.152	1.954	1.788
3.0	6.681	5.894	4.955	4.187	3.595	3.137	2.778	2.489	2.254	2.058
2.0	8.342	7.014	5.753	4.799	4.033	3.541	3.120	2.786	2.514	2.290
1.0	9.722	7.808	6.229	5.086	4.257	3.640	3.168	2.797	2.499	2.254
0.0	9.399	6.955	5.140	3.900	3.041	2.428	1.976	1.634	1.369	1.161

Le*(α, κ)

α \ κ	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
50.0	-2.211	-1.470	-1.101	-0.880	-0.733	-0.628	-0.550	-0.490	-0.441
49.0	-2.172	-1.443	-1.081	-0.864	-0.720	-0.617	-0.540	-0.481	-0.433
48.0	-2.132	-1.417	-1.061	-0.849	-0.707	-0.606	-0.530	-0.472	-0.425
47.0	-2.092	-1.390	-1.041	-0.832	-0.693	-0.594	-0.520	-0.463	-0.417
46.0	-2.051	-1.363	-1.021	-0.816	-0.680	-0.583	-0.510	-0.454	-0.409
45.0	-2.010	-1.335	-1.000	-0.799	-0.666	-0.571	-0.500	-0.444	-0.401
44.0	-1.968	-1.307	-0.979	-0.782	-0.652	-0.559	-0.489	-0.435	-0.392
43.0	-1.925	-1.279	-0.957	-0.765	-0.637	-0.546	-0.478	-0.425	-0.383
42.0	-1.882	-1.250	-0.936	-0.748	-0.623	-0.534	-0.467	-0.416	-0.375
41.0	-1.837	-1.220	-0.913	-0.730	-0.608	-0.521	-0.456	-0.406	-0.366
40.0	-1.793	-1.190	-0.891	-0.712	-0.593	-0.508	-0.445	-0.396	-0.357
39.0	-1.747	-1.159	-0.868	-0.693	-0.578	-0.495	-0.433	-0.385	-0.347
38.0	-1.700	-1.128	-0.844	-0.675	-0.562	-0.481	-0.421	-0.375	-0.338
37.0	-1.653	-1.096	-0.820	-0.655	-0.546	-0.468	-0.409	-0.364	-0.328
36.0	-1.604	-1.064	-0.796	-0.636	-0.530	-0.454	-0.397	-0.353	-0.318
35.0	-1.555	-1.031	-0.771	-0.616	-0.513	-0.440	-0.385	-0.342	-0.308
34.0	-1.504	-0.997	-0.746	-0.596	-0.496	-0.425	-0.372	-0.331	-0.298
33.0	-1.453	-0.963	-0.720	-0.575	-0.479	-0.410	-0.359	-0.319	-0.288
32.0	-1.400	-0.928	-0.694	-0.554	-0.461	-0.395	-0.346	-0.308	-0.277
31.0	-1.347	-0.892	-0.667	-0.532	-0.443	-0.380	-0.332	-0.296	-0.266
30.0	-1.292	-0.855	-0.639	-0.510	-0.425	-0.364	-0.318	-0.283	-0.255
29.0	-1.236	-0.817	-0.611	-0.488	-0.406	-0.348	-0.304	-0.271	-0.244
28.0	-1.178	-0.779	-0.582	-0.464	-0.387	-0.331	-0.290	-0.258	-0.232
27.0	-1.119	-0.740	-0.552	-0.441	-0.367	-0.314	-0.275	-0.244	-0.220
26.0	-1.059	-0.699	-0.522	-0.416	-0.346	-0.297	-0.260	-0.231	-0.208
25.0	-0.997	-0.658	-0.491	-0.391	-0.326	-0.279	-0.244	-0.217	-0.195
24.0	-0.933	-0.615	-0.459	-0.366	-0.304	-0.260	-0.228	-0.203	-0.182
23.0	-0.868	-0.571	-0.426	-0.339	-0.282	-0.242	-0.211	-0.188	-0.169
22.0	-0.801	-0.526	-0.392	-0.312	-0.260	-0.222	-0.194	-0.173	-0.155
21.0	-0.732	-0.480	-0.357	-0.284	-0.236	-0.202	-0.177	-0.157	-0.141
20.0	-0.660	-0.432	-0.321	-0.256	-0.212	-0.182	-0.159	-0.141	-0.127
19.0	-0.587	-0.383	-0.284	-0.226	-0.187	-0.160	-0.140	-0.124	-0.112
18.0	-0.511	-0.332	-0.246	-0.195	-0.162	-0.138	-0.121	-0.107	-0.096
17.0	-0.432	-0.279	-0.206	-0.163	-0.135	-0.115	-0.101	-0.089	-0.080
16.0	-0.350	-0.224	-0.165	-0.130	-0.108	-0.092	-0.080	-0.071	-0.064
15.0	-0.265	-0.168	-0.122	-0.096	-0.079	-0.067	-0.058	-0.052	-0.046
14.0	-0.177	-0.108	-0.078	-0.060	-0.049	-0.042	-0.036	-0.032	-0.028
13.0	-0.086	-0.047	-0.031	-0.023	-0.018	-0.015	-0.013	-0.011	-0.010
12.0	0.010	0.017	0.017	0.016	0.014	0.013	0.012	0.011	0.010
11.0	0.110	0.084	0.068	0.056	0.048	0.042	0.037	0.033	0.030
10.0	0.215	0.155	0.121	0.099	0.083	0.072	0.064	0.057	0.052
9.0	0.324	0.229	0.176	0.143	0.121	0.104	0.092	0.082	0.074
8.0	0.440	0.306	0.235	0.190	0.160	0.138	0.121	0.108	0.098
7.0	0.562	0.388	0.296	0.240	0.201	0.173	0.152	0.136	0.123
6.0	0.690	0.474	0.361	0.292	0.245	0.211	0.185	0.165	0.149
5.0	0.824	0.564	0.429	0.346	0.290	0.250	0.219	0.195	0.177
4.0	0.962	0.656	0.498	0.401	0.336	0.289	0.254	0.226	0.204
3.0	1.096	0.746	0.565	0.454	0.380	0.327	0.287	0.256	0.231
2.0	1.201	0.811	0.612	0.491	0.410	0.352	0.308	0.275	0.248
1.0	1.104	0.713	0.521	0.408	0.334	0.282	0.243	0.214	0.191
0.0	0.330	0.141	0.075	0.046	0.031	0.022	0.017	0.013	0.010