Extensions of the Mehler-Weisner and Other Results for the Hermite Function

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Abstract. The purpose of this paper is to present expansions which generalize some well-known formulae for the Hermite function. Among these are the Weisner [20] extension of Mehler's [17] bilinear relation, some recent results of Carlitz [4], and the Bateman [2] addition theorem. A bilateral generating function involving the product of the Hermite and ultraspherical polynomials is given. Finally, some general polynomial expansion theorems are derived.

I. **Introduction.** Recently we proved the expansion [5]

(1.1)
$$\sum_{n=0}^{\infty} \frac{(n+l)!\,(\alpha+\beta+l+1)_n\,t^n}{n!(\alpha+\beta+2l+2)_n}\,P_{n+l}^{(\alpha,\beta)}(x)$$
$$= \frac{(2)\,l!\,\Gamma(2l+\alpha+\beta+2)}{\Gamma(l+\alpha+1)\,\Gamma(l+\beta+1)\,t^{\alpha+\beta+l+1}}\,P_l^{(\alpha,\beta)}\left(\frac{1-\rho}{t}\right)Q_l^{(\alpha,\beta)}\left(\frac{1+\rho}{t}\right),$$

where $\rho = (1 - 2xt + t^2)^{1/2}$, $|t| < 1$, l is a nonnegative integer, and P and Q are the Jacobi polynomial and function, respectively. This type of generating function, where the degree of the polynomial is $(n + l)$, was previously not known for the Jacobi polynomial. The result (1.1) prompted us to investigate expansions of this type for bilinear generating functions.

For the Hermite polynomial, Mehler gave

(1.2) $$\sum_{n=0}^{\infty} \frac{t^n}{2^n n!}\,H_n(x)H_n(y) = (1 - t^2)^{-1/2}\exp\left[y^2 - \frac{(y - xt)^2}{1 - t^2}\right].$$

See also Askey [1, p. 16, Eq. 2.44w], [1, p. 35]. Erdélyi [7], Feldheim [8], Hardy [12], Hille [13], Sarmanov et al. [18], Watson [19], and others have worked on (1.2). A hundred years later Weisner [20], with the aid of group theoretical methods, generalized (1.2) to give

(1.3)
$$\sum_{n=0}^{\infty} \frac{t^n}{2^n n!}\,H_n(x)\,H_{n+\nu}(y)$$
$$= (1 - t^2)^{-1/2\nu - 1/2}\exp\left[\frac{y^2 t^2 - 2xyt + t^2 x^2}{t^2 - 1}\right]\cdot H_\nu\left[\frac{y - xt'}{(1 - t^2)^{1/2}}\right],$$

where $|t| < 1$, and ν is an arbitrary constant. H_ν is the Hermite function, defined as

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$$(1.4) \quad H_\nu(x) = \frac{2^\nu \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(1-\nu))} \, {}_1F_1\left[\begin{array}{c} -\frac{1}{2}\nu; \\ \frac{1}{2}; \end{array} x^2\right] + \frac{2^\nu \Gamma(-\frac{1}{2})x}{\Gamma(-\frac{1}{2}\nu)} \, {}_1F_1\left[\begin{array}{c} \frac{1}{2}(1-\nu); \\ 3/2; \end{array} x^2\right],$$

valid for all complex values of x and ν, and $\Gamma(\nu)$ is the gamma function. (1.3) is not extensively quoted in the literature and it was not known to the author when this work was done. Among other results in his paper, Carlitz [4] reproves (1.3) using series iteration and integration, but for the case ν a nonnegative integer.

We now present one generalization of (1.3).

$$(1.5) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} G_{n,s}(y, \alpha) H_{n,s}(x, \lambda) = (1 + \lambda t^s)^{\alpha-1} G_{0,s}\left[\frac{y - xt}{(1 + \lambda t^s)^{1/s}}, \alpha\right],$$

$|\lambda t^s| < 1$, where

$$(1.6) \quad G_{n,s}(y, \alpha) = \sum_{p=0}^{\infty} \frac{(-1)^p \, y^p}{p! \, \Gamma(\alpha - n/s - p/s)}$$

is valid for complex values of α, n, and y, for s an integer ≥ 2.

$$(1.7) \quad H_{n,s}(x, \lambda) = \sum_{k=0}^{[n/s]} \frac{\lambda^k x^{n-sk} n!}{k! \, (n - sk)!} = x^n \, {}_sF_0\left[\begin{array}{cc} \Delta(-n, s); & \frac{s^s(-1)^s\lambda}{x^s} \\ --; & \end{array}\right],$$

where $\Delta(-n, s) = -n/s, (-n+1)/s, \ldots, (-n+s-1)/s$. Note that the Hermite function defined in (1.4) is connected with (1.6) through the relation

$$(1.8) \quad G_{n,2}(2x, \frac{1}{2} - \frac{1}{2}\nu) = \frac{2^{-n-\nu} \exp(-x^2)}{\pi^{1/2}} H_{n+\nu}(x)$$

and with the generalized Hermite polynomial (1.7) by

$$(1.9) \quad H_{n,2}(2x, -1) = H_n(x).$$

One could consider (1.6) as a definition for a generalized Hermite function. On the other hand, (1.7) is the well-known generalized Hermite polynomial, which occurs in statistical problems. See Gupta and Jain [11], Gould and Hopper [9], and also Lukacs [15] for properties and applications. In Cohen [6], expansion problems connected with this polynomial are established. Putting $s = 2$, $\alpha = \frac{1}{2} - \frac{1}{2}\nu$, $\lambda = -1$ in (1.5), and algebraic manipulation yields Weisner's result (1.3).

(1.5) is in fact a special case of the more general expansion

$$(1.10) \quad \sum_{n_1, n_2, \ldots, n_i} G_{N,s}(y, \mu) \prod_{j=1}^{i} \frac{t_j^{n_j}}{n_j!} H_{n_j, s}(x_j, \lambda_j)$$

$$= \{1 + \sum \lambda_j t_j^s\}^{\mu-1} G_{0,s}\left[\frac{y - \sum t_j x_j}{\{1 + \sum \lambda_j t_j^s\}^{1/s}}, \mu\right]$$

for $|\sum \lambda_j t_j^s| < 1$, where $N = n_1 + n_2 + \cdots + n_i$, and

$$\sum x_j t_j = x_1 t_1 + x_2 t_2 + \cdots + x_i t_i, \qquad \sum \lambda_j t_j^s = \lambda_1 t_1^s + \lambda_2 t_2^s + \cdots + \lambda_i t_i^s.$$

In (1.10), let $s = 2$, $y = 2\eta$, $\mu = \frac{1}{2} - \frac{1}{2}\nu$, $\lambda_j = -1$, $x_j = 2\xi_j$. Then for the Hermite polynomial

$$
(1.11) \quad
\begin{aligned}
&\sum_{n_1,n_2,\ldots,n_i} 2^{-N} H_{N+\nu}(\eta) \prod_{j=1}^{i} \frac{t_j^{n_j}}{n_j!} H_{n_j}(\xi_j) \\
&= \exp\left[\eta^2 - \frac{(\eta - \sum t_j\xi_j)^2}{(1 - \sum t_j^2)}\right] \{1 - \sum t_j^2\}^{-\frac{1}{2} - \frac{1}{2}\nu} H_\nu\left[\frac{\eta - \sum t_j\xi_j}{(1 - \sum t_j^2)^{\frac{1}{2}}}\right].
\end{aligned}
$$

For $\nu = 0$, one obtains Eq. (2.3) of Carlitz [4], which is one of the main results of his paper.

Now Bateman [2, p. 890, Eq. 29] derived an addition theorem for the Hermite function

$$
(1.12) \quad (1 + t^2)^{\frac{1}{2}\nu} H_\nu\left[\frac{y + xt}{(1 + t^2)^{\frac{1}{2}}}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n(-\nu)_n \, t^n H_{\nu-n}(y) H_n(x)}{n!},
$$

where $\binom{\nu}{n} = (-1)^n(-\nu)_n/n!$ and $|t| < 1$.

For the case ν a nonnegative integer, (1.12) was first given by Kampé de Fériet [14]. (1.12) is also essentially an addition theorem for the parabolic cylinder function as

$$
H_\nu(x) = 2^{\frac{1}{2}\nu} \exp(\frac{1}{2}x^2) D_\nu(2^{\frac{1}{2}}x) \quad [16, p. 212, Eq. 5].
$$

We present a generalization of (1.12):

$$
(1.13) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} \bar{G}_{n,s}(y, \alpha) H_{n,s}(x, \lambda) = (1 - \lambda t^s)^{-\alpha} \bar{G}_{0,s}\left[\frac{y - xt}{(1 - \lambda t^s)^{1/s}}, \alpha\right]
$$

for $|\lambda t^s| < 1$, where

$$
(1.14) \quad \bar{G}_{n,s}(y, \alpha) = \sum_{p=0}^{\infty} \frac{(-1)^p y^p \Gamma(\alpha + n/s + p/s)}{p!}
$$

is valid for complex values of α, n and y, for s an integer ≥ 2. (1.14) is seen to be connected to the Hermite function through the relation

$$
(1.15) \quad \bar{G}_{n,2}(2y, \frac{1}{2}a) = 2\Gamma(n + a) H_{-a-n}(y).
$$

Putting $s = 2$, $\alpha = -\frac{1}{2}\nu$, $\lambda = -1$ in (1.13) and simplifying gives (1.12). We have, in fact, a more general expression of which (1.13) is a particular example.

$$
(1.16) \quad
\begin{aligned}
&\sum_{n_1,n_2,\ldots,n_i} \bar{G}_{N,s}(y, \mu) \prod_{j=1}^{i} \frac{t_j^{n_j}}{n_j!} H_{n_j,s}(x_j, \lambda_j) \\
&= \{1 - \sum \lambda_j t_j^s\}^{-\mu} \bar{G}_{0,s}\left[\frac{y - \sum t_j x_j}{\{1 - \sum \lambda_j t_j^s\}^{1/s}}, \mu\right]
\end{aligned}
$$

for $|\sum \lambda_j t_j^s| < 1$, where the symbols are defined in (1.10). Letting $y = 2\eta$, $s = 2$, $\mu = $

$-\frac{1}{2}\nu$, $\lambda_j = -1$, $x_j = 2\xi_j$ in (1.16), the following expression for the Hermite polynomial presents itself.

(1.17)

$$\sum_{n_1,n_2,\ldots,n_i} (-\nu)_N \, H_{\nu-N}(\eta) \prod_{j=1}^{i} \frac{t_j^{n_j}}{n_j!} H_{n_j}(\xi_j)$$

$$= \{1 + \sum t_j^2\}^{\frac{1}{2}\nu} H_\nu \left[\frac{\eta - \sum t_j \xi_j}{\{1 + \sum t_j^2\}^{\frac{1}{2}}}\right].$$

Another extension of (1.12) is derived for the case ν a nonnegative integer. This addition theorem for the generalized Hermite polynomial takes the finite form

(1.18) $$\sum_{n=0}^{l} \frac{(-l)_n}{n!} t^n \, H_{n,s}(x,\lambda) \, H_{l-n,s}(y,\mu) = H_{l,s}(y - xt, \mu + \lambda(-t)^s).$$

$H_{n,s}(x,\lambda)$ is defined in (1.7). For $s = 2$, $\lambda = -1$, $x \to 2x$, $y \to 2y$, $\nu = l$, (1.18) reduces to (1.12), for ν a nonnegative integer. (1.18) admits a more general expression

(1.19)

$$\sum_{n_1,n_2,\ldots,n_i}^{N \leqslant l} (-l)_N \, H_{l-N,s}(y,\mu) \prod_{j=1}^{i} \frac{t_j^{n_j}}{n_j!} H_{n_j,s}(x_j,\lambda_j)$$

$$= H_{l,s}[y - \sum t_j x_j, \mu + (-1)^s \sum \lambda_j t_j^s],$$

where $N = n_1 + n_2 + \cdots + n_i$.

The next result presented is the bilateral generating function involving the Hermite and Gegenbauer polynomials.

(1.20)

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^n (\frac{1}{2}\nu + \frac{1}{2} - \frac{1}{2}n)_n \, \Gamma(\frac{1}{2} - \frac{1}{2}\nu - \frac{1}{2}n)} C_n^{\frac{1}{2}\nu + \frac{1}{2} - \frac{1}{2}n}(x) H_n(y)$$

$$= \frac{2^{-\nu}(1 + t^2)^{\frac{1}{2}\nu}}{\pi^{\frac{1}{2}}} \{1 + t^2(1 - x^2)\}^{-\frac{1}{2}\nu - \frac{1}{2}}$$

$$\cdot \exp\left[\frac{y^2 t^2 (1 - x^2)}{1 + t^2(1 - x^2)}\right] H_\nu\left[\frac{xyt}{(1 + t^2)^{\frac{1}{2}}(1 + t^2 - t^2 x^2)^{\frac{1}{2}}}\right],$$

where $|t^2(1 - x^2)| < 1$. For other bilateral relations pertaining to the Hermite polynomial, see Brafman [3].

The results in the introduction are proved in Section II. The method of proof differs from that in Carlitz [4], Hille [13], Watson [19], and others. Section III is devoted to the presentation and proof of expansions involving more generalized polynomials. These expressions appear to give new and known results for the Hermite and ultraspherical polynomials.

II. Proofs of Section I Results.

(i) *Proof of* (1.5) *and* (1.10). The general proof of (1.10) is given, which includes its one-dimensional special case (1.5).

The left-hand side of (1.10) is by definition

$$
\sum_{n_1, n_2, \ldots, n_i} \frac{(t_1 x_1)^{n_1} (t_2 x_2)^{n_2} \cdots (t_i x_i)^{n_i}}{n_1! \, n_2! \cdots n_i!} \, {}_sF_0 \left[\begin{array}{c} \Delta(-n_1, s); \\ \underline{}; \end{array} \quad \frac{(-1)^s s^s \lambda_1}{x_1^s} \right]
$$

(2.1)

$$
\cdot \, {}_sF_0 \left[\begin{array}{c} \Delta(-n_2, s); \\ \underline{}; \end{array} \quad \frac{(-1)^s s^s \lambda_2}{x_2^s} \right] \cdots \, {}_sF_0 \left[\begin{array}{c} \Delta(-n_i, s); \\ \underline{}; \end{array} \quad \frac{(-1)^s s^s \lambda_i}{x_i^s} \right]
$$

$$
\cdot \sum_{p=0}^{\infty} \frac{(-1)^p y^p}{p! \, \Gamma(\alpha - n_1/s - n_2/s - \cdots - n_i/s - p/s)} \, .
$$

Using the double series transformation

(2.2)
$$
\sum_{n=0}^{\infty} \sum_{k=0}^{[n/s]} f(n, k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(n + sk, k)
$$

for i sums

$$
n_1 \to n_1 + sk_1, n_2 \to n_2 + sk_2, \ldots, n_i \to n_i + sk_i,
$$

the gamma function relation

$$
1/\Gamma(\alpha - p/s - n_1/s - n_2/s - \cdots - n_i/s - k_1 - k_2 - \cdots - k_i)
$$

(2.3)
$$
= \frac{(-1)^{k_1 + k_2 + \cdots + k_i}(1 - \alpha + p/s + n_1/s + n_2/s + \cdots + n_i/s)_{k_1 + k_2 + \cdots + k_i}}{\Gamma(\alpha - p/s - n_1/s - n_2/s - \cdots - n_i/s)},
$$

and the expansion

$$
\sum_{k_1, k_2, \ldots, k_i} \frac{(1 - \alpha + p/s + n_1/s + n_2/s + \cdots + n_i/s)_{k_1 + k_2 + \cdots + k_i}}{k_1! \, k_2! \cdots k_i!}
$$

(2.4)

$$
\cdot (-\lambda_1 t_1^s)^{k_1} (-\lambda_2 t_2^s)^{k_2} \cdots (-\lambda_i t_i^s)^{k_i}
$$

$$
= (1 + \lambda_1 t_1^s + \lambda_2 t_2^s + \cdots + \lambda_i t_i^s)^{\alpha - 1 - n_1/s - n_2/s - \cdots - n_i/s - p/s},
$$

(2.1) simplifies to

$$
\sum_{n_1, n_2, \ldots, n_i} \sum_{p=0}^{\infty} \frac{(x_1 t_1)^{n_1} (x_2 t_2)^{n_2} \cdots (x_i t_i)^{n_i} (-1)^p y^p}{n_1! \, n_2! \cdots n_i! \, p! \, \Gamma(\alpha - p/s - n_1/s - n_2/s - \cdots - n_i/s)}
$$

(2.5)

$$
\cdot (1 + \lambda_1 t_1^s + \lambda_2 t_2^s + \cdots + \lambda_i t_i^s)^{\alpha - 1 - n_1/s - n_2/s - \cdots - n_i/s - p/s}
$$

$$
= \sum_{p=0}^{\infty} \sum_{n_1, \ldots, n_i}^{\leqslant p} \frac{(-1)^p (-p)_{n_1 + \cdots + n_i} (x_1 t_1)^{n_1} \cdots (x_i t_i)^{n_i}}{n_1! \cdots n_i! \, p! \, \Gamma(\alpha - p/s)} \, .
$$

(2.6) $y^{p-n_1-\cdots-n_i}(1 + \lambda_1 t_1^s + \cdots + \lambda_i t_i^s)^{\alpha-1-p/s}$

(2.7)
$$= \sum_{p=0}^{\infty} \frac{(-1)^p(1 + \lambda_1 t_1^s + \cdots + \lambda_i t_i^s)^{\alpha-1-p/s}(y - t_1 x_1 - \cdots - t_i x_i)^p}{p!\,\Gamma(\alpha - p/s)}.$$

Going from (2.5) to (2.6) entails the use of the series transformation, and from (2.6) to (2.7) we have employed an expansion similar to (2.4).

(2.7) is the right-hand side of (1.10) by definition.

(ii) *Proof of* (1.13) *and* (1.16). Again, it is sufficient to prove (1.16).

We follow the procedure as in the proof of (1.10), with modifications.

The left-hand side of (1.16) may be expressed as

(2.8)
$$\sum_{n_1,\ldots,n_i} \frac{(t_1 x_1)^{n_1} \cdots (t_i x_i)^{n_i}}{n_1! \cdots n_i!}\, {}_sF_0\left[\begin{matrix} \Delta(-n_1, s); & (-1)^s s^s \lambda_1 \\ —; & x_1^s \end{matrix}\right]$$

$$\cdots {}_sF_0\left[\begin{matrix} \Delta(-n_i, s); & (-1)^s s^s \lambda_i \\ —; & x_i^s \end{matrix}\right]$$

$$\cdot \sum_{p=0}^{\infty} \frac{(-1)^p y^p \Gamma(\alpha + n_1/s + \cdots + n_i/s + p/s)}{p!}.$$

Using the double series transformation (2.2), the gamma function simplification

(2.9)
$$\Gamma(\alpha + n_1/s + \cdots + n_i/s + p/s + K)$$
$$= \Gamma(\alpha + n_1/s + \cdots + n_i/s + p/s)(\alpha + n_1/s + \cdots + n_i/s + p/s)_K$$

where $K = k_1 + \cdots + k_i$, the remainder of the proof is essentially the same as in (i), giving the right-hand side of (1.16).

(iii) *Proof of* (1.18) *and* (1.19). It suffices to prove (1.19).

The left-hand side of (1.19) can be put in the form

(2.10)
$$\sum_{n_1,\ldots,n_i}^{N \leqslant l} \frac{(-l)_N (t_1 x_1)^{n_1} \cdots (t_i x_i)^{n_i} y^{l-N}}{n_1! \cdots n_i!}\, {}_sF_0\left[\begin{matrix} \Delta(-n_1, s); & (-1)^s s^s \lambda_1 \\ —; & x_1^s \end{matrix}\right]$$

$$\cdots {}_sF_0\left[\begin{matrix} \Delta(-n_i, s); & (-1)^s s^s \lambda_i \\ —; & x_i^s \end{matrix}\right] {}_sF_0\left[\begin{matrix} \Delta(-l+N)/s; & (-1)^s \mu \\ —; & y^s \end{matrix}\right],$$

where $N = n_1 + \cdots + n_i$. With the aid of the double series transformation (2.2) for i sums, $n_i \rightarrow n_i + sk_i$ and the expansion

(2.11)
$$\sum_{n_1,\ldots,n_i} \frac{(-l + sk_1 + \cdots + sk_i + sp)_N}{n_1! \cdots n_i!} \left(\frac{t_1 x_1}{y}\right)^{n_1} \cdots \left(\frac{t_i x_i}{y}\right)^{n_i}$$

$$= \left\{1 - \frac{t_1 x_1}{y} - \cdots - \frac{t_i x_i}{y}\right\}^{l-sk_1-\cdots-sk_i-sp},$$

(2.10) reduces to

$$(2.12) \quad \sum_{k_1,\ldots,k_i,p}^{sk_1+\cdots+sk_i+sp\leqslant l} \frac{(-l)_{sk_1+\cdots+sk_i+sp}(t_1)^{sk_1}\cdots(t_i)^{sk_i}(-1)^{sp}\mu^p\lambda_1^{k_1}\cdots\lambda_i^{k_i}}{k_1!\cdots k_i!\, p!}$$

$$\cdot(y-t_1x_1-\cdots-t_ix_i)^{l-sk_1-\cdots-sk_i-sp}.$$

With the aid of the series iteration

$$(2.13) \quad \sum_{k_1,\ldots,k_i,p} f(p,k_1,\ldots,k_i) = \sum_{p=0}^{\infty}\sum_{k_1,\ldots,k_i}^{K\leqslant p} f(p-K,k_1,\ldots,k_i)$$

where $K = k_1+\cdots+k_i$, and

$$(2.14) \quad \sum_{k_1,\ldots,k_i}^{K\leqslant p} \frac{\mu^{p-K}(-p)_K(-1)^{(s+1)K}t_1^{sk_1}\cdots t_i^{sk_i}\lambda_1^{k_1}\cdots\lambda_i^{k_i}}{k_1!\cdots k_i!}$$

$$= \{\mu+\lambda_1(-t_1)^s+\cdots+\lambda_i(-t_i)^s\}^p,$$

(2.12) simplifies to give

$$(2.15) \quad (y-t_1x_1-\cdots-t_ix_i)^l\sum_{p=0}^{[l/s]}\frac{(y-t_1x_1-\cdots-t_ix_i)^{-sp}(-1)^{sp}(-l)_{sp}}{p!}$$

$$\cdot\{\mu+\lambda_1(-t_1)^s+\cdots+\lambda_i(-t_i)^s\}^p.$$

(2.15) is the right-hand side of (1.19).

(iv) *Proof of* (1.20). The left-hand side of (1.20) is expressed as

$$(2.16) \quad \sum_{n=0}^{\infty}\frac{(-xt)^n}{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}n)}\,{}_2F_1\!\left[\begin{array}{c}-\tfrac{1}{2}n,-\tfrac{1}{2}n+\tfrac{1}{2};\quad\dfrac{1}{x^2}\\[4pt]\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}n;\end{array}\right]\sum_{k=0}^{[n/2]}\frac{2^{n-2k}y^{n-2k}(-1)^k}{k!\,(n-2k)!}$$

$$(2.17) \quad = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{(-2xyt)^n(-1)^k(1-1/x^2)^{\tfrac{1}{2}n-\tfrac{1}{2}\nu}}{k!\,2^k(n-2k)!\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}n)y^{2k}}$$

$$\cdot\,{}_2F_1\!\left[\begin{array}{c}-\tfrac{1}{2}\nu,-\tfrac{1}{2}\nu+\tfrac{1}{2};\quad\dfrac{1}{x^2}\\[4pt]\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}n;\end{array}\right].$$

$$(2.18) \quad \sum_{n,p,k=0}^{\infty}\frac{[-2yt(x^2-1)^{\tfrac{1}{2}}]^n[(x^2-1)t^2]^k[(1-1/x^2)]^{-\tfrac{1}{2}\nu}(-\tfrac{1}{2}\nu)_p(\tfrac{1}{2}-\tfrac{1}{2}\nu)_p(\tfrac{1}{2}\nu+\tfrac{1}{2}+\tfrac{1}{2}n-p)_K}{n!\,p!\,k!\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}n+p)x^{2p}}.$$

In (2.16) we have used the representation (3.20) for the ultraspherical polynomial. (2.16) to (2.17) implies the Kummer transformation. (2.18) results from (2.17) by using the double series transformation (2.2).

Now

$$(2.19) \quad \sum_{k=0}^{\infty}\frac{(\tfrac{1}{2}\nu+\tfrac{1}{2}+\tfrac{1}{2}n-p)_k}{k!}\{t^2(x^2-1)\}^k = [1+t^2(1-x^2)]^{-\tfrac{1}{2}\nu-\tfrac{1}{2}-\tfrac{1}{2}n+p}$$

Combining (2.18) and (2.19) and using once again the Kummer transformation gives

$$\sum_{n=0}^{\infty} \sum_{p=0}^{[n/2]} \frac{(-2xyt)^n [1 + t^2(1 - x^2)]^{-\frac{1}{2}\nu - \frac{1}{2} - \frac{1}{2}n}(1 + t^2)^{\frac{1}{2}\nu - \frac{1}{2}n}}{2^{2p} p! \, (n - 2p)! \, \Gamma(\frac{1}{2} - \frac{1}{2}\nu - \frac{1}{2}n + p)}$$

(2.20)
$$\left[\frac{1 + t^2(1 - x^2)}{x^2} \right]^p$$

(2.21)
$$= \sum_{n=0}^{\infty} \frac{(-2xyt)^n [1 + t^2(1 - x^2)]^{-\frac{1}{2}\nu - \frac{1}{2} - \frac{1}{2}n}(1 + t^2)^{-\frac{1}{2}n + \frac{1}{2}\nu}}{n! \, \Gamma(\frac{1}{2} - \frac{1}{2}\nu - \frac{1}{2}n)}$$

$$\sum_{p=0}^{\infty} \frac{[y^2 t^2/(1 + t^2)]^p}{p!} \, .$$

From (2.20) to (2.21) we have used the double series transformation. In (2.21) the series over p is seen to be the exponential, and using the definition of the Hermite function gives the required right-hand side of (1.20).

III. **Generalized Polynomial Expansions.** This section deals with general expansion theorems. Some special cases are

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(n + l)!}{(2\nu)_n} \, y^n C_n^\nu(x) \, C_{n+l}^\mu(xy)$

(3.1)
$$= \frac{(-1)^{\frac{1}{2}l} l! \, (\mu)_{\frac{1}{2}l}}{(\frac{1}{2}l)! \, (1 - x^2 y^2)^{\mu + \frac{1}{2}l}} \, {}_2F_1 \left[\begin{matrix} \frac{1}{2}l + \mu, \, \frac{1}{2}l + \frac{1}{2}; & \dfrac{y^2(1 - x^2)}{(1 - x^2 y^2)} \\ \nu + \frac{1}{2}; & \end{matrix} \right] \quad \text{for } l \text{ even,}$$

$$= 0 \quad \text{for } l \text{ odd,}$$

and $|y| < 1$, where $C_m^\lambda(x)$ is the Gegenbauer polynomial. For special values, the right-hand side of (3.1) may be expressed as well-known functions, including the Jacobi function. It also reduces to simple expressions for $2\nu = l$ or $2\nu - 2\mu + 1 = l$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n + l)!}{2^n n!} y^n H_n(x) \, C_{n+l}^\mu(xy)$

(3.2)
$$\cong (\mu)_{\frac{1}{2}l}(1 - x^2 y^2)^{-\mu - \frac{1}{2}l} \, {}_2F_0 \left[\begin{matrix} \frac{1}{2}l + \frac{1}{2}, \, \frac{1}{2}l + \mu; & \dfrac{y^2}{1 - x^2 y^2} \\ ---; & \end{matrix} \right] \quad \text{for } l \text{ even,}$$

$$= 0 \quad \text{for } l \text{ odd,}$$

where the right-hand side is a divergent series. See Luke [16, Vol. I, Chapter IV, Section (4.7), and Chapter VI, Sections (6.2.7) and (6.2.11)] for a fine exposition of the asymptotic expansion and significance of the ${}_2F_0$ divergent series.

(3.3) (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(xy)^n}{(l)_n} C_n^{\frac{1}{2}l}(x) H_{n+l}(y) = \frac{(-1)^{\frac{1}{2}l} \, l!}{(\frac{1}{2}l)!} \exp\left(\frac{y^2}{x^2} \right) \quad \text{for } l \text{ even,}$

$$= 0 \quad \text{for } l \text{ odd,}$$

where $|y| < 1$.

We now proceed with the proof of some general theorems.

For $\{a_k\}$ and $\{b_k\}$ arbitrary complex sequences, μ an arbitrary complex number, and l a nonnegative integer,

$$(3.4) \quad \sum_{n=0}^{\infty} \frac{\xi^n}{n!} H_{n+l}(\xi) A_n(\eta) = \frac{(-1)^{\frac{1}{2}l} \, l!}{(\frac{1}{2}l)!} \theta(\xi, \eta) \quad \text{for } l \text{ even,}$$

$$= 0 \qquad\qquad \text{for } l \text{ odd,}$$

where $|\xi| < 1$, and

$$(3.5) \quad A_n(\eta) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \eta^{2k}(-n)_{2k}\{a_k\}}{2^{2k}k! \, (\frac{1}{2}l + \frac{1}{2})_k},$$

$$(3.6) \quad \theta(\xi, \eta) = \exp(\xi^2) \sum_{k=0}^{\infty} \frac{(\xi^2\eta^2)^k\{a_k\}}{k!}.$$

$$(3.7) \quad \sum_{n=0}^{\infty} \frac{(\frac{1}{2}l)_n \xi^n}{(l)_n n!} D_n(\eta) B_{n+l}(\xi) = \frac{(-1)^{\frac{1}{2}l} l!}{(\frac{1}{2}l)!} \Phi(\xi, \eta) \quad \text{for } l \text{ even,}$$

$$= 0 \qquad\qquad \text{for } l \text{ odd,}$$

where $|\xi| < 1$, and

$$(3.8) \quad B_{n+l}(\xi) = \sum_{k=0}^{[\frac{1}{2}(n+l)]} \frac{[-(n+l)]_{2k}(-1)^k\{a_{\frac{1}{2}l+n-k}\}\xi^{n+l-2k}}{k! \, 2^{2k-n-l}},$$

$$(3.9) \quad D_n(\eta) = \sum_{p=0}^{[n/2]} \frac{(-n)_{2p}\{b_p\}\eta^{2p}}{2^{2p-n}p! \, (1 - \frac{1}{2}l - n)_p},$$

$$(3.10) \quad \Phi(\xi, \eta) = \sum_{k=0}^{\infty} \frac{\{a_k\}\{b_k\}\eta^{2k}\xi^{2k}}{k!},$$

$$(3.11) \quad \sum_{n=0}^{\infty} \frac{(n+l)! \, \xi^n}{n!} C_{n+l}^{\mu}(\xi) A_n(\eta) = \frac{(-1)^{\frac{1}{2}l} \, l!}{(\frac{1}{2}l)!} \Psi(\xi, \eta) \quad \text{for } l \text{ even,}$$

$$= 0 \qquad\qquad \text{for } l \text{ odd,}$$

where $|\xi| < 1$, and

$$(3.12) \quad \Psi(\xi, \eta) = (1 - \xi^2)^{-\mu - \frac{1}{2}l} \sum_{k=0}^{\infty} \frac{(\mu)_{\frac{1}{2}l+k}\{a_k\}}{k!} \left\{\frac{\eta^2\xi^2}{1-\xi^2}\right\}^k,$$

$A_n(\eta)$ is defined by (3.5) and $C_{n+l}^{\mu}(\xi)$ is the Gegenbauer polynomial.

(3.6), (3.9), and (3.12) are convergent; and $A_n(\eta)$, $B_{n+l}(\eta)$, $D_n(\eta)$ are assumed to exist.

Proof of (3.4). Putting $t = y/x$ in (1.3),

$$(3.13) \quad \sum_{n=0}^{\infty} \frac{(y/x)^n}{n!2^n} H_n(x)H_{n+l}(y) = \frac{\exp(y^2)}{(1 - y^2/x^2)^{\frac{1}{2}l + \frac{1}{2}}} H_l(0).$$

Note that $H_l(0) = \epsilon$ where

$$\epsilon = \frac{(-1)^{\frac{1}{2}l}\, l!}{(\frac{1}{2}l)!} \quad \text{if } l \text{ is even} \quad \text{and} \quad \epsilon = 0 \quad \text{if } l \text{ is odd.}$$

Using

$$H_n(x) = \frac{(2x)^n}{n!} \,{}_2F_0[-\tfrac{1}{2}n, -\tfrac{1}{2}n + \tfrac{1}{2}; \;-\!\!-\!\!-; -1/x^2],$$

and a corresponding expression for $H_{n+l}(y)$, putting $x = 1/z^{\frac{1}{2}}$, and expanding

$$(1 - y^2/x^2)^{-\frac{1}{2}l-\frac{1}{2}} = {}_1F_0[\tfrac{1}{2}l + \tfrac{1}{2}; \;-\!\!-\!\!-; zy^2]$$

reduces (3.13) to

$$\sum_{n=0}^{\infty} \frac{2^{n+l}y^{2n+l}}{n!} \,{}_2F_0[-\tfrac{1}{2}n, -\tfrac{1}{2}n + \tfrac{1}{2}; \;-\!\!-\!\!-; -z]$$

(3.14)
$${}_2F_0[-\tfrac{1}{2}(n + l), -\tfrac{1}{2}(n + l) + \tfrac{1}{2}; \;-\!\!-\!\!-; -1/y^2]$$

$$= \epsilon \exp(y^2) \,{}_1F_0[\tfrac{1}{2}l + \tfrac{1}{2}; \;-\!\!-\!\!-; zy^2].$$

We take the inverse Laplace of (3.14) with respect to z to give

$$\sum_{n=0}^{\infty} \frac{2^{n+l}y^{2n+l}}{n!} \,{}_2F_0[-\tfrac{1}{2}(n + l), -\tfrac{1}{2}(n + l) + \tfrac{1}{2}; \;-\!\!-\!\!-; -1/y^2]$$

(3.15)
$${}_2F_1\begin{bmatrix} -\tfrac{1}{2}n, -\tfrac{1}{2}n + \tfrac{1}{2}; \\ \tfrac{1}{2}l + \tfrac{1}{2}; \end{bmatrix} -z$$

(3.16)
$$= \epsilon \exp\{y^2(1 + z)\},$$

(3.17)
$$= \epsilon \exp(y^2) \sum_{k=0}^{\infty} \frac{(zy^2)^k}{k!}.$$

Taking Laplace and inverse Laplace transforms of (3.15) and (3.17) with respect to z, and changing variables to ξ and η, results in Eq. (3.4).

Proof of (3.7). Now (3.16) may also be expanded in the form

(3.18)
$$\epsilon \sum_{k=0}^{\infty} \frac{(1 + z)^k\, y^{2k}}{k!}.$$

Taking transforms of (3.15) and (3.18) with respect to y and z, and changing variables gives (3.7). The representations for the Gegenbauer polynomial

(3.19)
$$C_m^{\lambda}(x) = \frac{(2\lambda)_m\, x^m}{m!} \,{}_2F_1\begin{bmatrix} -\tfrac{1}{2}m, -\tfrac{1}{2}m + \tfrac{1}{2}; & \dfrac{x^2 - 1}{x^2} \\ \lambda + \tfrac{1}{2}; & \end{bmatrix}$$

(3.20)
$$= \frac{(\lambda)_m\, (2x)^m}{m!} \,{}_2F_1\begin{bmatrix} -\tfrac{1}{2}m, -\tfrac{1}{2}m + \tfrac{1}{2}; & 1/x^2 \\ 1 - \lambda - m; & \end{bmatrix}$$

are used [10, p. 1030].

Proof of (3.11). Taking Laplace transforms of (3.15) and (3.16) with respect to y, one has

$$(3.21) \quad \sum_{n=0}^{\infty} \frac{2^{n+l}(\mu + \tfrac{1}{2}l)_{n+\tfrac{1}{2}l}\, y^{2n+l}}{n!}\, {}_2F_1\left[\begin{array}{c} -\tfrac{1}{2}(n+l),\ -\tfrac{1}{2}(n+l)+\tfrac{1}{2}; \\ 1-\mu-n-l; \end{array} 1/y^2\right]$$

$$\cdot\, {}_2F_1\left[\begin{array}{c} -\tfrac{1}{2}n,\ -\tfrac{1}{2}n+\tfrac{1}{2}; \\ -\tfrac{1}{2}l+\tfrac{1}{2}; \end{array} -z\right]$$

$$(3.22) \quad = \epsilon\,\{1 - y^2(1+z)\}^{-\mu-\tfrac{1}{2}l}$$

$$(3.23) \quad = \epsilon\,\{1 - y^2\}^{-\mu-\tfrac{1}{2}l} \sum_{k=0}^{\infty} \frac{(zy^2)^k(\mu+\tfrac{1}{2}l)_k}{k!}\,.$$

Taking transforms of (3.21) and (3.23) with respect to z and changing variables gives (3.11). The representation (3.20) is employed.

The special case (3.1) may be deduced from (3.11) by applying the representation (3.19) to the polynomial $A_n(\eta)$. (3.2) is derived from (3.11), while (3.3) is derived from (3.4).

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