MATHEMATICS OF COMPUTATION, VOLUME 30, NUMBER 135
JULY 1976, PAGES 565–570

Computation of π Using Arithmetic-Geometric Mean

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Abstract. A new formula for π is derived. It is a direct consequence of Gauss' arithmetic-geometric mean, the traditional method for calculating elliptic integrals, and of Legendre's relation for elliptic integrals. The error analysis shows that its rapid convergence doubles the number of significant digits after each step. The new formula is proposed for use in a numerical computation of π, but no actual computational results are reported here.

1. Introduction. This paper announces the discovery of a new formula for π. It is based upon the arithmetic-geometric mean, a process whose rapid convergence doubles the number of significant digits at each step. The arithmetic-geometric mean, together with π as a known quantity, is the basis of Gauss' method for the calculation of elliptic integrals. But with the help of an elliptic integral relation of Legendre, Gauss' method can be turned around to express π in terms of the arithmetic-geometric mean. The resulting algorithm retains the property of doubling the number of digits at each step.

The proof of the main result (Theorem 1a) from first principles can be conducted on the elementary calculus level. The references cited here for the theorems of Landen, Gauss and Legendre have been chosen to achieve this goal, thus allowing the widest possible reader audience comprehension.

The formula presented in this paper is proposed as a method for the numerical computation of π. It has not yet been tested on a calculation of nontrivial length, although such a calculation is currently in progress [2].

2. The Arithmetic-Geometric Mean. Let a_0, b_0, c_0 be positive numbers satisfying $a_0^2 = b_0^2 + c_0^2$. Define a_n, the sequence of arithmetic means, and b_n, the sequence of geometric means, by

$$a_n = \tfrac{1}{2}(a_{n-1} + b_{n-1}), \qquad b_n = (a_{n-1} b_{n-1})^{1/2}.$$

Also, define a positive sequence c_n:

$$c_n^2 = a_n^2 - b_n^2.$$

Two relations easily follow from these definitions.

$$c_n = \tfrac{1}{2}(a_{n-1} - b_{n-1}). \tag{1}$$

$$c_n^2 = 4a_{n+1} c_{n+1}. \tag{2}$$

Received May 27, 1975; revised November 3, 1975.
AMS (MOS) subject classifications (1970). Primary 10A30, 10A40, 33A25; Secondary 41A25.
Key words and phrases. π, arithmetic-geometric mean, elliptic integral, Landen's transformation, Legendre's relation, fast Fourier transform multiplication.

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The arithmetic-geometric mean is the common limit

$$\operatorname{agm}(a_0, b_0) = \lim a_n = \lim b_n.$$

Because of the rapidity of convergence of the arithmetic-geometric mean, as exhibited by Eq. (2), the formula to be derived should be regarded as a plausible candidate for the numerical computation of π.

3. Elliptic Integrals. The complete elliptic integrals are the functions

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{-1/2}\, dt, \qquad E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{1/2}\, dt.$$

Also, if $k^2 + k'^2 = 1$, then $K'(k) = K(k')$ and $E'(k) = E(k')$ are two more elliptic integrals.

There is also a symmetric form of these integrals:

$$I(a, b) = \int_0^{\pi/2} (a^2 \cos^2 t + b^2 \sin^2 t)^{-1/2}\, dt,$$

$$J(a, b) = \int_0^{\pi/2} (a^2 \cos^2 t + b^2 \sin^2 t)^{1/2}\, dt.$$

It is clear that

$$I(a, b) = a^{-1} K'(b/a), \qquad J(a, b) = aE'(b/a).$$

4. Landen's Transformation and the Computation of Elliptic Integrals. Using the notation developed in Section 2 of this paper, these transformations are [6, Section 25.15],

(3) $$I(a_n, b_n) = I(a_{n+1}, b_{n+1}),$$

(4) $$J(a_n, b_n) = 2J(a_{n+1}, b_{n+1}) - a_n b_n I(a_{n+1}, b_{n+1}).$$

From Eq. (3) it follows that

(5) $$I(a_0, b_0) = \pi/2 \operatorname{agm}(a_0, b_0),$$

and, after a little work, Eq. (4) yields

(6) $$J(a_0, b_0) = \left(a_0^2 - \tfrac{1}{2} \sum_{j=0}^{\infty} 2^j c_j^2\right) I(a_0, b_0).$$

For $a_0 = 1, b_0 = k'$, the integrals in Eqs. (5) and (6) are equal to $K(k)$ and $E(k)$, respectively, while for $a_0 = 1, b_0 = k$, they equal $K'(k)$ and $E'(k)$. This is the well-known method of Gauss for the numerical calculation of elliptic integrals [5, pp. 78–80], [1, Section 17.6].

5. Legendre's Relation. This relation is [4, Article 171], [1, Eq. 17.3.13],

(7) $$K(k)E'(k) + K'(k)E(k) - K(k)K'(k) = \pi/2.$$

Equivalently,

(8) $$a^2 I(a, b) J(a', b') + a'^2 I(a', b') J(a, b) - a^2 a'^2 I(a, b) I(a', b') = (\pi/2) a a',$$

where a, b, a', b' are subject to the restriction $(b/a)^2 + (b'/a')^2 = 1$.

6. New Expression for π. Take $a_0 = a_0' = 1$, $b_0 = k$, $b_0' = k'$. As in Section 2, define the sequences a_n, b_n, c_n, a_n', b_n', c_n'. In Eq. (8) eliminate the J integrals by use of (6), and then eliminate the I integrals by use of (5). Lo and behold, the resulting equation can be solved for π!

THEOREM 1a.

$$\pi = \frac{4\,\mathrm{agm}(1, k)\mathrm{agm}(1, k')}{1 - \sum_{j=1}^{\infty} 2^j(c_j^2 + c_j'^2)}. \tag{9}$$

The $j = 0$ term in the summation has been eliminated by use of $c_0^2 + c_0'^2 = k'^2 + k^2 = 1$. It is best to compute c_j from Eq. (1).

Theorem 1a is a one-dimensional continuum of formulae for π. This provides for an elegant and simple computational check. For example, π could be calculated starting with $k = k' = 2^{-1/2}$, and then checked using $k = 4/5$, $k' = 3/5$. The symmetric choice, $k = k'$, causes the two agm sequences to coincide, thus halving the computational burden.

THEOREM 1b.

$$\pi = \frac{4(\mathrm{agm}(1, 2^{-1/2}))^2}{1 - \sum_{j=1}^{\infty} 2^{j+1} c_j^2}.$$

7. Error Analysis. Although Theorem 1a is true for all complex values of k (except for a discrete set), the error analysis will assume real k and k'. Then $0 < k, k' < 1$. Let n square roots be taken in the process of computing agm $=$ agm$(1, k)$, and n' square roots in computing agm$' =$ agm$(1, k')$. Then no further square roots are needed to calculate the approximation

$$\pi_{nn'} = \frac{4a_{n+1}a'_{n'+1}}{1 - \sum_{j=1}^{n} 2^j c_j^2 - \sum_{j=1}^{n'} 2^j c_j'^2}. \tag{10}$$

A rough estimate shows that a_{n+1} differs from agm by c_{n+2}, and that the finite sum differs from the infinite sum by $2^{n+3}c_{n+2}$. Thus, the numerator and denominator in (10) have been truncated for compatible error contributions, and the denominator error is dominant.

To obtain rigorous error bounds, introduce the auxiliary quantity $\bar{\pi}_{nn'}$ whose denominator is taken from (10), but whose numerator is taken from (9). The first step is to establish the existence of $e_{nn'}$, $\bar{e}_{nn'}$ such that

$$0 < \pi - \bar{\pi}_{nn'} < e_{nn'}, \tag{11}$$

$$0 < \pi_{nn'} - \bar{\pi}_{nn'} < \bar{e}_{nn'}, \qquad \bar{e}_{nn'} < e_{nn'}. \tag{12}$$

These three inequalities imply that $|\pi - \pi_{nn'}| < e_{nn'}$.

The left-hand inequalities in (11) and (12) are obvious. From the general inequality $(1/x) - (1/(x + y)) < y/x^2$, valid for positive x and y, it follows that

$$\pi - \bar{\pi}_{nn'} < \frac{\pi^2}{4\ \mathrm{agm}\ \mathrm{agm}'}\left(\sum_{n+1}^{\infty} 2^j c_j^2 + \sum_{n'+1}^{\infty} 2^j c_j'^2\right).$$

This establishes (11), with error bound

$$e_{nn'} = \frac{\pi^2}{2 \text{ agm agm}'} \left(\sum_{n+2}^{\infty} 2^j a_j c_j + \sum_{n'+2}^{\infty} 2^j a'_j c'_j \right). \tag{13}$$

Proceeding to the next inequality, we first get

$$\pi_{nn'} - \bar{\pi}_{nn'} < \frac{\pi}{\text{agm agm}'} (a_{n+1} a'_{n'+1} - \text{agm agm}').$$

Substitute $a_{n+1} = \text{agm} + s$, $a'_{n'+1} = \text{agm}' + s'$, where

$$s = \sum_{n+2}^{\infty} c_j, \qquad s' = \sum_{n'+2}^{\infty} c'_j,$$

and use agm < 1, agm$' < 1$ to get

$$\pi_{nn'} - \bar{\pi}_{nn'} < \pi(s + s' + ss')/\text{agm agm}'.$$

Also, since $s < 1$, $s' < 1$, it follows that $ss' < (s + s')/2$. Thus, inequality (12) is established with error bound

$$\bar{e}_{nn'} = \frac{3}{2} \frac{\pi}{\text{agm agm}'} \left(\sum_{n+2}^{\infty} c_j + \sum_{n'+2}^{\infty} c'_j \right). \tag{14}$$

Finally, a term-by-term comparison of (13) and (14), using $2^j a_j > 1$ and $\pi > 3$, shows that $\bar{e}_{nn'} < e_{nn'}$.

At this point, a needed inequality is derived.

$$a_j < a_j + b_j = 2a_{j+1},$$

$$2a_j c_{j+1} < 4a_{j+1} c_{j+1} = c_j^2 = (a_{j-1} - a_j) c_j,$$

$$a_j(c_j + 2c_{j+1}) < a_{j-1} c_j. \tag{15}$$

Consider the first summation in (13), but with the upper limit ∞ replaced by finite N. Perform the following sequence of operations, each of which increases the sum. First, replace a_N by a_{N-1}. Next, repeatedly apply (15) to the pair of highest-indexed terms in the sum. At the end, we are left with the single term $2^{n+2} a_{n+1} c_{n+2} < 2^{n+2} c_{n+2}$, which is thus an upper bound for the initial summation. Since N was arbitrary, the infinite summation also has this upper bound. Therefore,

$$e_{nn'} < \frac{2\pi^2}{\text{agm agm}'} (2^n c_{n+2} + 2^{n'} c'_{n'+2}). \tag{16}$$

An upper bound for c_{n+2} is needed now. It is convenient to use the abbreviations

$$x_n = \log c_n, \qquad g_n = \log(4a_n).$$

Equation (2) gives x_n as the solution to an inhomogeneous linear difference equation.

$$x_n = 2^n \left(x_0 - \sum_{j=1}^{n} 2^{-j} g_j \right).$$

By rearrangement,

$$x_n = 2^n \left(x_0 - g_1 + \sum_{j=1}^{n-1} 2^{-j}(g_j - g_{j+1}) \right) + g_n.$$

Using $g_j - g_{j+1} > 0$, $g_n < \log 4$, and $x_0 - g_1 = (1/2)\log(c_1/4a_1)$, we get

$$x_n < 2^{n-1}\left[\sum_{j=1}^{\infty} 2^{-j+1}\log(a_j/a_{j+1}) - \log(4a_1/c_1)\right] + \log 4. \tag{17}$$

For the purpose of an error analysis, the expression within brackets could be calculated numerically for any case of interest. However, it can be evaluated in closed form [7, p. 14] and is equal to $-\pi K'(k)/K(k) = -\pi\,\mathrm{agm}/\mathrm{agm}'$. Then

$$x_n < -\pi(\mathrm{agm}/\mathrm{agm}')2^{n-1} + \log 4.$$

Substituting this into (16) yields the final result.

THEOREM 2a.

$$|\pi - \pi_{nn'}| < \frac{8\pi^2}{\mathrm{agm}\ \mathrm{agm}'}\left[2^n \exp\left(-\pi\frac{\mathrm{agm}}{\mathrm{agm}'}2^{n+1}\right) + 2^{n'}\exp\left(-\pi\frac{\mathrm{agm}'}{\mathrm{agm}}2^{n'+1}\right)\right].$$

In the symmetric case, with $\pi_n = \pi_{nn}$, Theorem 2a simplifies to

THEOREM 2b.

$$|\pi - \pi_n| < (\pi^2 2^{n+4}/\mathrm{agm}^2)\exp(-\pi 2^{n+1}).$$

The number of valid decimal places is then

THEOREM 2c.

$$-\log_{10}|\pi - \pi_n| > (\pi/\log 10)2^{n+1} - n\log_{10}2 - 2\log_{10}(4\pi/\mathrm{agm}).$$

8. **Numerical Computation.** Raphael Finkel, Leo Guibas and Charles Simonyi are currently engaged in calculating π using the method proposed in this paper [2]. The operations of multiprecision division and square root are reduced to multiplication using a Newton's method iteration. The multiplications are then performed by the Schönhage-Strassen fast Fourier transform algorithm [10], [8, p. 274]. The computation, to be run on the Illiac IV computer, is expected to yield 33 million bits of π in an estimated run time of four hours. This run time is determined by disc input-output, and the actual computation is estimated to be only a couple of minutes. Alas, they do not plan to convert to decimal.

9. **Concluding Remarks.** The main result of this paper, Theorem 1a, directly follows from Gauss' method for calculating elliptic integrals, Eqs. (5) and (6), which was known in 1818 [3, pp. 352, 360], and from Legendre's elliptic integral relation, Eq. (7), which was known in 1811 [9, p. 61]. It is quite surprising that such an easily derived formula for π has apparently been overlooked for 155 years. The author made his discovery in December of 1973.

The series summation which was used to simplify Eq. (17) was also discovered by Gauss [3, p. 377]. An interesting consequence of this result of Gauss is that e^π can be expressed as a rapidly convergent infinite product. If $a_0 = 1$, $b_0 = 2^{-1/2}$, then

$$e^\pi = 32\prod_{j=0}^{\infty}\left(\frac{a_{j+1}}{a_j}\right)^{2^{-j+1}}.$$

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