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Triples of Sixth Powers With Equal Sums

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Abstract. The diophantine equation $x^6 + y^6 + z^6 = u^6 + v^6 + w^6$ is shown to have a two-parameter solution which is homogeneous of degree four. The solution also satisfies $x^2 + y^2 + z^2 = u^2 + v^2 + w^2$; and in addition, $3x + y + z = 3u + v + w$.

The diophantine equation

$$x^6 + y^6 + z^6 = u^6 + v^6 + w^6 \tag{1}$$

is a particular instance of the much-studied problem of finding equal sums of like powers of integers, surveyed by Lander, Parkin and Selfridge in 1967 [4]. The smallest nontrivial solution was published by Subba Rao in 1934 [5], namely, $3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6$. Early editions of Hardy and Wright [3] referred to this result as "an isolated curiosity". However, Lander, Parkin and Selfridge [4] discovered that (1) has ten primitive solutions in the range up to 2.5×10^{14}, and that all but one of these also satisfy

$$x^2 + y^2 + z^2 = u^2 + v^2 + w^2. \tag{2}$$

In [1] it was shown that there are infinitely many primitive solutions to (1), each also satisfying (2) and

$$v = y - z, \qquad w = y + z. \tag{3}$$

Subsequently, in [2] the complete solution to (1), (2) and (3) was obtained in terms of an infinite cyclic group of rational points on a cubic curve. (Regrettably, the solution $5P$ appeared in [2] with transcription errors in the values of x and w; it should read $x = 165809277507$, $y = 151561337462$, $z = 23038103009$, $u = 63175337782$, $v = 128523234453$ and $w = 174599440471$.)

The principal aim of this paper is to exhibit the following explicit solution to (1) in terms of parameters m, n:

$$\begin{aligned} x &= 2m^4 + 4m^3n - 5m^2n^2 - 12mn^3 - 9n^4, \\ y &= 3m^4 + 9m^3n + 18m^2n^2 + 21mn^3 + 9n^4, \\ z &= -m^4 - 10m^3n - 17m^2n^2 - 12mn^3, \\ u &= m^4 - 3m^3n - 14m^2n^2 - 15mn^3 - 9n^4, \\ v &= 3m^4 + 8m^3n + 9m^2n^2, \\ w &= 2m^4 + 12m^3n + 19m^2n^2 + 18mn^3 + 9n^4. \end{aligned} \tag{4}$$

This solution also satisfies (2); and in addition,

$$3x + y + z = 3u + v + w. \tag{5}$$

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TABLE 1

m	n	m'	n'	x	y	z	u	v	w	d	d'
0	1	1	−1	−1	1	0	−1	0	1	9	4
1	0	3	−1	2	3	−1	1	3	2	1	36
1	1	3	−2	−1	3	−2	−2	1	3	20	45
2	−1	3	1	−1	3	4	1	4	−3	5	180
1	−2	3	−5	−74	33	47	−73	23	54	1	36
1	2	9	−7	−50	81	−37	−65	11	78	5	180
2	1	9	−5	11	243	−188	−103	148	249	1	36
1	−3	3	−4	−23	15	10	−22	3	19	20	45
1	3	6	−5	−271	372	−127	−317	27	356	4	9
2	−3	3	−7	−65	15	52	−67	36	37	5	180
4	−1	9	−1	43	81	32	55	80	−3	5	180
1	−4	9	−11	−326	243	107	−311	23	282	5	180
3	2	5	−3	−26	225	−169	−121	111	230	9	4
5	−1	6	−1	169	276	65	179	275	36	4	9
5	1	9	−4	389	891	−590	−46	775	831	4	9
4	−3	3	5	−409	93	512	−293	528	−271	1	36
5	−2	9	1	86	729	655	431	775	−426	1	36
6	1	7	−3	71	147	−92	1	132	133	45	20
7	−1	9	−2	163	243	14	142	245	75	20	45

Of the ten smallest primitive solutions to (1), listed in [4], all but the sixth satisfy (2). Only the second satisfies (3), while (4) gives rise to all except the seventh (and, of course, the sixth).

It should be noted that a particular choice of m and n does not necessarily yield a *primitive* solution in (4), even if m and n are coprime. Indeed, suppose $(m, n) = 1$ and $d = (x, y, z, u, v, w)$. It is not difficult to prove that (i) $2|d$ just if $m \equiv n \pmod 2$, and then $2^2 \| d$; (ii) $3|d$ just if $m \equiv 0 \pmod 3$, and then $3^2 \| d$; and (iii) $5|d$ just if $m \equiv n$ or $2m \equiv n \pmod 5$, and then $5^1 \| d$. Moreover, suppose $p|d$ for some prime $p > 5$. Clearly, $p \nmid mn$, so $v \equiv 0 \pmod p$ yields $3m^2 + 8mn + 9n^2 \equiv 0 \pmod p$. With $y - v \equiv 0 \pmod p$ this leads to $10(m + 3n) \equiv 0 \pmod p$, and finally with $z \equiv 0 \pmod p$ this yelds $72n^4 \equiv 0 \pmod p$, which is impossible. Hence, d has no prime factor greater than 5.

Consider the transformation to (4) which results from replacing m, n by m', n' satisfying

$$m' : n' = -3(m + n) : (m + 3n). \tag{6}$$

If x, y, z, u, v, w is the solution corresponding to m, n and x', y', z', u', v', w' is the solution corresponding to m', n', then

$$x' : y' : z' : u' : v' : w' = u : v : w : x : y : z. \tag{7}$$

It follows that any particular primitive solution to (1), (2) and (5) obtained from (4) actually arises from two distinct ratios $m:n$.

Next, we remark that any solution to (1), (2) and (5) has an interesting geometrical interpretation. The points (x, y, z) and (u, v, w) in E^3 are lattice points which simultaneously lie on a sphere $X^2 + Y^2 + Z^2 = a$, a concentric closed surface $X^6 + Y^6 + Z^6 = b$, and a double cone with vertex at the origin and axis in the direction $3:1:1$. It is intriguing to speculate that the solutions might turn out to have some physical interpretation.

Finally, in Table 1 are listed all primitive solutions obtained from (4) with the property that $\max\{|x|, |y|, |z|\} < 10^3$. As remarked earlier, these include all but two of the numerical examples given in Table IX of [4].

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