are addressed to a number of extremely pertinent problems confronting the practicing statistician for which tables were not previously available. However, as mentioned earlier, it is important that the terminology relating to the noncentral t tables be fully clarified so that these valuable tables can be properly understood and applied. In addition, more familiar examples are recommended to illustrate the use of the zonal polynomials, so that they will appeal to a wider class of users.

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1. THE INSTITUTE OF MATHEMATICAL STATISTICS, Editors, and H. L. HARTER & D. B. OWEN, Coeditors, *Selected Tables in Mathematical Statistics*, Vol. I, American Mathematical Society, Providence, R. I., second printing, 1973. (See *Math. Comp.*, v. 29, 1975, p. 661, RMT 32.)

32 [9].—I. O. ANGELL, *A Table of Totally Real Cubic Fields*, Royal Holloway College, Univ. of London, Surrey, England, 1975. 80 computer sheets deposited in the UMT file.

This is the table referred to in Angell's paper [1]. The 4794 nonconjugate totally real cubic fields $Q(x)$ having discriminants $D < 10^5$ are listed here in the format

$$D \quad I \quad A \quad B \quad C \quad H \quad P \quad Q \quad R \quad S \quad U \quad V \quad W \quad T.$$

Here, H is the class number and $(Px^2 + Qx + R)/S$, $(Ux^2 + Vx + W)/T$ is a fundamental pair of units. (In thirty-five fields here, one or both units have coefficients that are too large for this format and they are given in an appendix at the end of the table.) The three conjugate fields are generated by the three real roots of the polynomial

$$f(x) = x^3 - Ax^2 + Bx - C = 0 \tag{1}$$

which has index I and discriminant I^2D. The fifty-one self-conjugate (cyclic) fields included here are, of course, generated by any of the three roots.

The reader is referred to my longish review [2] of Angell's complex cubic fields for comparison with the discussion that follows. The class numbers tend to be very small here since the existence of two units implies that the regulators are relatively large. The number # of fields with class number H are as follows:

H	1	2	3	4	5	6	7	8	9
#	4184	287	268	20	19	7	7	1	1

Note the curious two-step wherein each even H has about the same population as the subsequent odd H.

The polynomial (1) follows Godwin's convention [3]; A, B and C are positive and the three roots satisfy

$$0 < x_0 < 1, \quad x_0 < x_1 < x_2, \quad 2x_1 > x_0 + x_2.$$

In the reviewer's opinion, the altered polynomial $g(x') = -f([x_2] + 1 - x')$, which has $2x'_1 < x'_0 + x'_2$ instead, is preferable. Since the polynomial coefficients are symmetric functions of the three roots, the smaller x'_1, instead of the larger x_1, implies that the coefficients of $g(x')$ will generally be smaller than those of $f(x)$ (and sometimes they will be much smaller). In Table 2 below, we follow this AG (anti-Godwin) convention.

As in [2], the index I is not always minimized here. Of the first eight cases of $I = 2$ listed, $f(x)$ for $D = 1304, 1772, 2292, 2589$ and 2920 can be easily transformed into other equations with $I = 1$. But $D = 2089$ and the cyclic $D = 31^2$ and 43^2 must have $I = 2$ since the prime 2 splits completely in these fields. This $D = 2089 = 51^2 - 2^9$, together with subsequent examples such as $4481 = 67^2 - 2^3$ and $9281 = 97^2 - 2^7$ are of a form $D = n^2 - 2^{2m+1}$ that frequently has this property; see [4, Table 2]. On the other hand, 2 is a cubic residue of 31 and 43 and therefore splits completely in those cyclic fields.

Davenport and Heilbronn [5] proved that the nonconjugate totally real cubic fields have an asymptotic density of $[12\zeta(3)]^{-1} = 0.069326$ while the empirical average density δ here is notably smaller:

TABLE 1

$D/5000$	δ	$D/5000$	δ	$D/5000$	δ	$D/5000$	δ	$D/5000$	δ
1	.0346	5	.0426	9	.0447	13	.0462	17	.0471
2	.0382	6	.0433	10	.0451	14	.0463	18	.0474
3	.0402	7	.0442	11	.0455	15	.0469	19	.0476
4	.0418	8	.0442	12	.0459	16	.0469	20	.0479

While δ is obviously increasing with D, at $D = 10^5$ it has only attained 69% of its limit. In [2], the density of the complex fields attained 76% of its limit at $|D| = 2 \cdot 10^4$. The slow convergence in [2] and even slower convergence here do not now have a good quantitative explanation but no doubt are mostly due to the delayed appearance of D having large multiplicity m. In [1], as in [2], there are D having m distinct nonconjugate fields for $m = 2, 3$, or 4, but none with $m > 4$. (For larger D, beyond these tables, there will be D with m arbitrarily large.)

While the first $m = 4$ in [2] is for $D = -3299$, $m = 4$ does not occur here until $D = 32009$. In [2], there are twenty-two D with $m = 4$ while here there are only five such D even though there are more fields and $|D|$ can be five times as large. But for $D > 10^5$, as we show below, the proportion of D having $m = 4$ increases strongly, and if this proportion has a limit as $D \longrightarrow \infty$, cf. [5, p. 406], the slowness in attaining this limit correlates with the slow convergence of δ above.

Prior to the computation of this table there were three known cases of $m = 4$ for $D < 10^5$. Two are prime [4, p. 161]:

$$32009 = 5^6 + 4 \cdot 4^6 = 179^2 - 2^5; \quad 62501 = 1^6 + 4 \cdot 5^6$$

and one is even [6, p. 540]:

$$94636 = 4 \cdot 23659 = 4\Delta(-5).$$

The table was computed because the reviewer suggested to Professor Godwin that it would be desirable to extend his earlier table [3] in order to verify that $D = 32009$ and 94636 are indeed the smallest D and smallest even D having $m = 4$, That is true; the only new cases found here are two odd, composite D related to 32009:

$$42817 = 47 \cdot 911 = 207^2 - 2^5; \quad 72329 = 151 \cdot 479 = 269^2 - 2^5.$$

But for $10^5 < D < 2 \cdot 10^5$ there are at least eight more cases and probably about 10. There are four primes: $151141 = \Delta_2(-7)$ was given in [6, Table II] and Lakein [9, Table 5] gave

$$114889 = 339^2 - 2^5; \quad 142097 = 377^2 - 2^5,$$

together with $D = 153949$ of no known series. I found that there are exactly two even D:

$$4 \cdot 43063 \quad \text{and} \quad 4 \cdot 2 \cdot 17 \cdot 1279.$$

The odd composite D were not systematically examined. Two are known: $\Delta_6(5) = 3 \cdot 17 \cdot 2999$ is due to me and $130397 = 19 \cdot 6863$ is due to Heilbronn [3, p. 109]. Probably there are at least two or three others. So the relative proportion of D having $m = 4$ about doubles in this next interval. Other $m > 1$ will also become relatively more numerous.

In Table 2 of [2], I showed that for three known series of $D < 0$, with $m = 4$, it was possible to give the four cubic polynomials a priori. For the present fields with $D > 0$ that is no longer the case. But in the *nonescalatory* cases in [6] and [7] we can give *one* polynomial, but only one, a priori. For example, for the

$$D = A^6 + 4B^6, \quad 3 \nmid B,$$

of [7],

$$x^3 - (A^2 + B^2)x + A(A^2 + 2B^2)/3 = 0$$

gives one field. This is suitable for the $D = 32009$ and 62501 above. For the Series 1 and 2 and Complementary Series 3 and 6 of [6], one can also give one polynomial. Further, these polynomials can even be put into AG form a priori. They would give one field for the examples $D = 4\Delta(-5)$, $\Delta_2(-7)$ and $\Delta_6(5)$ above.

The smallest known [8] real $Q(\sqrt{D})$ having 3-rank = 3 is $D = 44806173$. So this D gives the smallest known case of $m = 13$. In Table 2, I give its thirteen polynomials in AG form and show how thirteen primes split (shown as S) in these thirteen fields. Compare Tables 3 and 4 in [2]. The reader is invited to transform these cubics into Angell's form and to note the effects of this upon the coefficients.

TABLE 2

$D = 44806173$

I	A	B	$.C$	11	13	17	29	41	43	107	113	131	137	151	163	179
3	61	697	330	S	S	–	–	–	–	–	–	–	–	–	S	S
3	279	441	170	S	–	S	S	–	–	S	–	–	–	–	–	–
3	63	423	8	S	–	–	–	S	–	–	S	S	–	–	–	–
3	69	435	216	S	–	–	–	–	S	–	–	–	S	S	–	–
3	63	603	494	–	S	S	–	–	S	–	–	S	–	–	–	–
3	83	297	54	–	S	–	S	–	–	–	S	–	S	–	–	–
3	63	837	494	–	S	–	–	S	–	S	–	–	–	S	–	–
3	257	477	216	–	–	S	–	S	–	–	–	–	S	–	–	S
3	87	273	36	–	–	S	–	–	–	–	S	–	–	S	S	–
3	62	546	261	–	–	–	S	–	–	–	–	S	–	S	–	S
3	60	660	97	–	–	–	–	–	–	S	–	S	S	–	S	–
3	165	273	90	–	–	–	S	S	S	–	–	–	–	–	S	–
1	127	185	62	–	–	–	–	–	S	S	S	–	–	–	–	S

Finally, a couple of words on an erroneous first version of this table. It was instructive precisely because it was erroneous. The four class numbers for the $D = 62501$ above came out $H = 3, 3, 4, 9$. Since all H for the other four cases of $m = 4$ were divisible by 3, it did appear A) that that $H = 4$, and presumably other H, were wrong; and B) that the Gras-Callahan Theorem referred to in [2] was also valid in the real case. Georges Gras subsequently proved this B) but Frank Gerth III had already done that independently. While the thirteen H for Table 2 are not known to me, they must all be divisible by 9. The errors in A) were confirmed and corrected.

There were also errors in some units. The Artin function at argument 1 equals

$$\Phi(1) = 4RH/\sqrt{D} \tag{2}$$

where R is the regulator. Since $\Phi(1)$ is easily estimated by a determination of how all small primes split, (2) is a very powerful check on the consistency of R and H, and one can detect an error in one if the other is known. So the erroneous units were also detected and corrected. If ϵ_1 and ϵ_2 are a fundamental pair of units, then so are $\epsilon_3 = \epsilon_1^2\epsilon_2$ and $\epsilon_4 = \epsilon_1\epsilon_2$. But ϵ_3 and ϵ_2 are not a fundamental pair. Is ϵ_3 a "fundamental unit"? The moral is that it is erroneous and dangerous to speak of "a pair of fundamental units." One must say "a fundamental pair of units."

D. S.

1. I. O. ANGELL, "A table of totally real cubic fields," *Math. Comp.*, v. 30, 1976, pp. 184–187.

2. DANIEL SHANKS, UMT Review 33 of I. O. Angell, "Table of complex cubic fields," *Math. Comp.*, v. 29, 1975, pp. 661–665.

3. H. J. GODWIN & P. A. SAMET, "A table of real cubic fields," *J. London Math. Soc.*, v. 34, 1959, pp. 108–110.

4. DANIEL SHANKS, "On Gauss's class number problems," *Math. Comp.*, v. 23, 1969, pp. 151–163.

5. H. DAVENPORT & H. HEILBRONN, On the density of discriminants of cubic fields. II," *Proc. Roy. Soc. London Ser. A*, v. 322, 1971, pp. 405–420.

6. DANIEL SHANKS, "New types of quadratic fields having three invariants divisible by 3," *J. Number Theory*, v. 4, 1972, pp. 537–556.

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9. RICHARD B. LAKEIN, "Computation of the ideal class group of certain complex quartic fields. II," *Math. Comp.*, v. 29, 1975, pp. 137–144.

33 [12.15]. –R. J. ORD-SMITH & J. STEPHENSON, *Computer Simulation of Continuous Systems*, Cambridge Univ. Press, New York & London, 1975, vi + 327 pp., 23 cm. Price $9.95.

The authors have presented a good introduction to analog and hybrid computation techniques. The book is written so that students without an electronic background can follow the material. In the first chapter, for example, the operation of analog and logic components is adequately presented without detailed electronic circuitry. A more detailed description of the analog components is covered in the Appendix for those who are interested. Another favorable point is the variety of good, basic problems given at the end of several chapters.

The method of implementing a differential equation on the analog computer and the method of amplitude scaling presented in Chapter 2 are not the most convenient techniques for large scale systems. The change of variables suggested is neither necessary nor desirable when simulating a large system. However, the techniques set forth