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A Bound on the L_∞-Norm of L_2-Approximation by Splines in Terms of a Global Mesh Ratio

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Abstract. Let $L_k f$ denote the least-squares approximation to $f \in L_1$ by splines of order k with knot sequence $\mathbf{t} = (t_i)_1^{n+k}$. In connection with their work on Galerkin's method for solving differential equations, Douglas, Dupont and Wahlbin have shown that the norm $\|L_k\|_\infty$ of L_k as a map on L_∞ can be bounded as follows,

$$\|L_k\|_\infty \leqslant \text{const}_k M_{\mathbf{t}},$$

with $M_{\mathbf{t}}$ a global mesh ratio, given by

$$M_{\mathbf{t}} := \max_i \Delta t_i / \min \{\Delta t_i | \Delta t_i > 0\}.$$

Using their very nice idea together with some facts about B-splines, it is shown here that even

$$\|L_k\|_\infty \leqslant \text{const}_k (M_{\mathbf{t}}^{(k)})^{1/2}$$

with the smaller global mesh ratio $M_{\mathbf{t}}^{(k)}$ given by

$$M_{\mathbf{t}}^{(k)} := \max_{i,j} (t_{i+k} - t_i)/(t_{j+k} - t_j).$$

A mesh independent bound for L_2-approximation by continuous piecewise polynomials is also given.

1. Introduction. This note is an addendum to the clever paper by Douglas, Dupont and Wahlbin [2] in which these authors bound the linear map of least-squares approximation by splines of order k with knot sequence $\mathbf{t} := (t_i)$, as a map on L_∞, in terms of the particular global mesh ratio

$$M_{\mathbf{t}} := \max_i \Delta t_i / \min \{\Delta t_i | \Delta t_i > 0\}.$$

Their argument is very elegant. But their result is puzzling in one aspect: The ratio $M_{\mathbf{t}}$ is not a continuous function of t. If, e.g., $\mathbf{t}$ is uniform, hence $M_{\mathbf{t}} = 1$, and we now let $\mathbf{t} \longrightarrow \mathbf{t}^*$ by letting just one knot approach its neighbor, leaving all other knots fixed, then

$$\lim_{\mathbf{t} \to \mathbf{t}^*} M_{\mathbf{t}} = \infty, \quad \text{while } M_{\mathbf{t}^*} = 2.$$

Correspondingly, their bound goes to infinity as $\mathbf{t} \longrightarrow \mathbf{t}^*$, yet is again finite for the particular knot sequence $\mathbf{t}^*$.

This puzzling aspect is removed below. It is shown that (as asserted in a footnote to [1]) their very nice argument can be used to give a bound in terms of the smaller global mesh ratio

$$M_{\mathbf{t}}^{(k)} := \max_i (t_{i+k} - t_i)/\min_i (t_{i+k} - t_i) \tag{1}$$

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which does depend continuously on $\mathbf{t}$ in $\{\mathbf{t} \in \mathbf{R}^{n+k} \mid t_i \leqslant t_{i+1}, t_i < t_{i+k}, \text{all } i\}$.

2. Least-Squares Approximation by Splines of Order k. Let $\mathbf{t} := (t_i)_1^{n+k}$ be a nondecreasing sequence, with $t_i < t_{i+k}$, all i. A spline of order k with knot sequence $\mathbf{t}$ is, by definition, any function of the form

$$\sum_{i=1}^{n} \alpha_i N_i$$

with $\alpha \in \mathbf{R}^n$ and N_i the normalized B-spline of order k with knots $t_i, \ldots, t_{i+k}$, i.e.,

$$N_i(t) := N_{i,k,\mathbf{t}}(t) := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}.$$

In words, for each t, $N_i(t)$ is $(t_{i+k} - t_i)$ times the kth divided difference at $t_i, \ldots, t_{i+k}$ of $(s - t)_+^{k-1}$ as a function of s.

We denote the totality of all splines of order k with knot sequence $\mathbf{t}$ by $\mathbf{S}_{k,\mathbf{t}}$. More detail about $\mathbf{S}_{k,\mathbf{t}}$ is provided in [1] and its references.

Next, let L_k denote the linear projector on $\mathbf{L}_1$ defined by the condition that $L_k f \in \mathbf{S}_{k,\mathbf{t}}$, and, for all $g \in \mathbf{S}_{k,\mathbf{t}}$, $\int (f - L_k f) g = 0$, i.e., $L_k f$ is the $\mathbf{L}_2$-approximation to f in $\mathbf{S}_{k,\mathbf{t}}$. We are interested in estimating the norm $\|L_k\|_p$ of L_k as a map on $\mathbf{L}_p$. Since

$$\|L_k\|_p = \|L_k\|_q \quad \text{for } 1/p + 1/q = 1,$$

and $\|L_k\|_2 = 1$, interpolation will given a bound on $\|L_k\|_p$ in terms of $\|L_k\|_\infty = \|L_k\|_1$, as is pointed out in [2]. It therefore suffices to consider $\|L_k\|_\infty$.

Let $L_k f = \Sigma \alpha_j N_j$. Then $\|L_k f\|_\infty \leqslant \|\alpha\|_\infty$ since $N_i \geqslant 0$, all i, and $\Sigma_j N_j \leqslant 1$, while

$$\sum_j \int N_i N_j \alpha_j = \int N_i f \leqslant [(t_{i+k} - t_i)/k] \, \|f\|_\infty, \quad \text{all } i,$$

since $N_i \geqslant 0$ and $\int N_i = (t_{i+k} - t_i)/k$. Therefore,

$$\|L_k\|_\infty \leqslant \|G^{-1}\|_\infty \tag{2}$$

with

$$G := G_\infty = E^{1/2} G_2 E^{-1/2}, \tag{3}$$

where E is a diagonal matrix,

$$E := \ulcorner k/(t_{k+1} - t_1), \ldots, k/(t_{k+n} - t_n) \lrcorner, \tag{4}$$

and G_2 is the Gramian matrix for the basis $(\overset{2}{N}_i)$ of $\mathbf{S}_{k,\mathbf{t}}$, i.e.

$$G_2 := \left(\int \overset{2}{N}_i \overset{2}{N}_j \right)_{i,j=1}^{n} \tag{5}$$

and

$$\overset{p}{N}_i := [k/(t_{i+k} - t_i)]^{1/p} N_i. \tag{6}$$

With this normalization, we are assured of the existence of a positive constant D_k depending only on k and not at all on $\mathbf{t}$ or n so that

(7) $$D_k^{-1}\|\alpha\|_p \leqslant \left\|\sum_j \alpha_j \overset{p}{N_j}\right\|_p \leqslant \|\alpha\|_p, \quad \text{all } \alpha \in \mathbf{R}^{n+k}$$

(see the theorem on p. 539 of [1]). This inequality implies that

(8) $$\|G_2^{-1}\|_\infty \leqslant \text{const}_k$$

for some const_k depending only on k as we will show below; and, on combining this with (2)–(4), we obtain the desired conclusion

(9) $$\|L_k\|_\infty \leqslant \text{const}_k (M_{\mathbf{t}}^{(k)})^{1/2}.$$

3. **A Bound for** $\|G_2^{-1}\|_\infty$. With $(\alpha_{ij})_{i,j=1}^n := G_2^{-1}$, let $f_i := \Sigma_j \alpha_{ij} \overset{2}{N_j}$. Then

$$\int f_i \overset{2}{N_j} = \delta_{ij}, \quad \text{all } j;$$

hence

$$\int \alpha_{ii} \overset{2}{N_i} f_i + \sum_{j \neq i} \alpha_{ij} \overset{2}{N_j} f_i = \alpha_{ii},$$

i.e.,

(10) $$\|f_i\|_2^2 = \alpha_{ii}.$$

Therefore, by (7),

$$D_k^{-2}\alpha_{ii}^2 \leqslant D_k^{-2} \sum_j |\alpha_{ij}|^2 \leqslant \|f_i\|^2 = \alpha_{ii},$$

hence, as $\alpha_{ii} = \|f_i\|_2^2 \neq 0$ (G_2^{-1} is invertible!), we have $\alpha_{ii} \leqslant D_k^2$; and so, $\|f_i\|_2 \leqslant D_k$ and

(11) $$\left(\sum_j |\alpha_{ij}|^2\right)^{1/2} \leqslant D_k \|f_i\|_2 = D_k (\alpha_{ii})^{1/2} \leqslant D_k^2.$$

This shows that

$$\|G_2^{-1}\|_\infty = \max_i \sum_j |\alpha_{ij}| \leqslant n^{1/2} \max_i \left(\sum_j |\alpha_{ij}|^2\right)^{1/2} \leqslant n^{1/2} D_k^2$$

and so bounds $\|G_2^{-1}\|_\infty$ in terms of only k and n. From this, one obtains

$$\|G^{-1}\|_\infty \leqslant (nM_{\mathbf{t}}^{(k)})^{1/2} D_k^2,$$

a bound in terms of the desired global mesh ratio, except that the bound goes to infinity with the number of mesh points. Note that we can express $M_{\mathbf{t}}^{(k)}$ in terms of n and the local mesh ratio

$$m_{\mathbf{t}}^{(k)} := \max_{|i-j|=1} (t_{i+k} - t_i)/(t_{j+k} - t_j);$$

hence, we even have a bound on $\|G^{-1}\|_\infty$ in terms of that *local* mesh ratio but, alas, involving also n.

In order to remove this dependence on n, we use the ideas of Douglas, Dupont and Wahlbin [2] to prove the following lemma.

LEMMA 1. *There exist* const_k *and* $\lambda_k \in (0, 1)$ *independent of* n *or* $\mathbf{t}$ *so that, for all* i *and* j,

$$|\alpha_{ij}| \leqslant \text{const}_k (\lambda_k)^{|i-j|}.$$

Proof. We observed earlier that the function $f_i = \Sigma_j \alpha_{ij} \overset{2}{N_j}$ is orthogonal to $\operatorname{span}(N_j)_{j \neq i}$. Hence, for any $m > i$,

$$f_{i,m} := \sum_{m \leqslant j} \alpha_{ij} \overset{2}{N_j}$$

is orthogonal to f_i and, therefore, also orthogonal to $f_{i,m-k+1}$ since the latter function agrees with f_i on the support of $f_{i,m}$. This proves that

$$\|f_{i,m-k+1}\|_2^2 + \|-f_{i,m}\|_2^2 = \|f_{i,m-k+1} - f_{i,m}\|_2^2 \tag{12}$$

from which we conclude that

$$\left\| \sum_{m-k<j} \alpha_{ij} \overset{2}{N_j} \right\|_2^2 \leqslant \left\| \sum_{m-k<j<m} \alpha_{ij} \overset{2}{N_j} \right\|_2^2$$

or, with the inequality (7),

$$\sum_{m-k<j<m} |\alpha_{ij}|^2 \geqslant D_k^{-2} \sum_{m-k<j} |\alpha_{ij}|^2, \qquad m = i+1, i+2, \ldots . \tag{13}$$

Faced with a similar inequality, Douglas, Dupont and Wahlbin [2] make use of what amounts to the following discrete Gronwall inequality:

LEMMA 2. *If the sequence* $a_0, a_1, \ldots$ *satisfies*

$$|a_m| \geqslant c \sum_{m \leqslant j} |a_j|, \qquad m = 0, 1, 2, \ldots, \tag{14}$$

for some $c \in (0, 1)$, *then* $\lambda := 1 - c \in (0, 1)$ *and*

$$|a_m| \leqslant |a_0| \lambda^m / c, \qquad m = 0, 1, 2, \ldots . \tag{15}$$

Proof. Let $A_m := \Sigma_{m \leqslant j} |a_j|$. Then (14) reads

$$A_m - A_{m+1} \geqslant c A_m, \quad \text{all } m,$$

or, $A_{m+1} \leqslant (1-c) A_m$, all m, therefore, with $\lambda := 1 - c$,

$$A_{m+j} \leqslant \lambda^j A_m, \quad \text{all } m, j,$$

and so,

$$|a_m| = A_m - A_{m+1} \leqslant A_m \leqslant \lambda^m A_0 \leqslant |a_0| \lambda^m / c. \quad \text{Q.E.D.}$$

In order to apply this lemma to (12), we pick $m_0 > i$ and let

$$J_m := \{ j \in \mathbf{Z} \mid m_0 + (k-1)(m-1) \leqslant j < m_0 + (k-1)m \}, \qquad m = 0, 1, \ldots .$$

Then, with

$$a_m := \sum_{j \in J_m} |\alpha_{ij}|^2, \quad \text{all } m,$$

we obtain from (12) that

$$a_m \geqslant D_k^{-2} \sum_{m \leqslant j} a_j, \qquad m = 0, 1, 2, \ldots ;$$

hence, from the lemma,

$$\max_{j \in J_m} |\alpha_{ij}| \leqslant a_m^{1/2} \leqslant D_k (1 - D_k^{-2})^{m/2} a_0^{1/2}$$

while, by (11),

$$a_0^{1/2} \leqslant \Big(\sum_j |\alpha_{ij}|^2\Big)^{1/2} \leqslant D_k^2.$$

This proves the asserted exponential decay of $|\alpha_{ij}|$ for $j > i$; but G_2 is symmetric. Q.E.D.

It follows at once that

$$\|G_2^{-1}\|_\infty \leqslant \text{const}_k\, 2/(1-\lambda_k). \tag{16}$$

In view of the discussion at the end of Section 2, we have therefore proved the following theorem.

THEOREM 1. *There exists a constant c depending only on k so that the norm* $\|L_k\|_\infty$ *of* $\mathbf{L}_2$*-approximation by splines of order k with knot sequence* $\mathbf{t}$, *as a map on* $\mathbf{L}_\infty$, *satisfies*

$$\|L_k\|_\infty \leqslant c(M_{\mathbf{t}}^{(k)})^{1/2}$$

with the global mesh ratio $M_{\mathbf{t}}^{(k)}$ *given by*

$$M_{\mathbf{t}}^{(k)} := \max_{i,j}\ (t_{i+k}-t_i)/(t_{j+k}-t_j).$$

There seems to be little hope that this argument would even support a bound in terms of $m_{\mathbf{t}}^{(k)}$, let alone a bound independent of the mesh $\mathbf{t}$.

4. A Mesh Independent Bound for $\mathbf{L}_2$-Approximation by C^0-Piecewise Polynomials. Pick $k > 1$. Let $\xi = (\xi_i)_1^r$ in (a, b) with $a =: \xi_0 < \cdots < \xi_{r+1} := b$, and let Pf be the $\mathbf{L}_2$-approximation to f by elements of $\mathbf{P}_{k,\xi} \cap C^0 := \{f \in C[a, b] \mid f|_{(\xi_i,\xi_{i+1})} \in \mathbf{P}_k\}$. Todd Dupont [3] has shown some time ago that P can be bounded as a map on $\mathbf{L}_\infty$ independently of ξ by constructing a basis for ran P for which a certain matrix related to the Gramian is strictly diagonally dominant. We take the occasion to give a proof in terms of B-splines.

If $\mathbf{t} = (t_i)_1^{n+k}$ is the nondecreasing sequence which contains a and b exactly k times and each of $\xi_1, \ldots, \xi_r$ exactly $k-1$ times (and nothing else), then

$$\mathbf{P}_{k,\xi} \cap C^0 = \mathbf{S}_{k,\mathbf{t}},$$

hence then $P = L_k$ introduced in Section 2, therefore, $\|P\| \leqslant \|G^{-1}\|$ with G given by (3)–(6) in terms of $\mathbf{t}$ as determined from ξ.

THEOREM 2. *Let* $\hat{G} := (k\int_0^1 \hat{N}_i\hat{N}_j)_{i,j=1}^k$ *be the matrix G in the special case* $r = 0$, $[a, b] = [0, 1]$. *Then, for all* ξ, $\|G^{-1}\|_\infty = \|\hat{G}^{-1}\|_\infty$. *In particular,* $\|P\| \leqslant \|\hat{G}^{-1}\|_\infty$ *for all* ξ. *Hence (T. Dupont)* $\sup_\xi \|P\| < \infty$.

Proof. Let $\xi_{-1} = a$, $\xi_{r+2} = b$. Then, for $m = 0, \ldots, r+1$, $N_{m(k-1)+1}$ has its support on the two intervals (ξ_{m-1}, ξ_{m+1}) of ξ. All other N_i have their support in just one interval. Correspondingly, the matrix G is almost block diagonal, with $r+1$ $k \times k$ blocks overlapping in just one row and column. For $k = 4$ (the cubic case) and $r = 2$ this looks like

$$\begin{array}{cccccccccc}
x & x & x & x & & & & & & \\
x & x & x & x & & & & & & \\
x & x & x & x & & & & & & \\
x & x & x & x & x & x & x & & & \\
 & & & x & x & x & x & & & \\
 & & & x & x & x & x & & & \\
 & & & x & x & x & x & x & x & x \\
 & & & & & & x & x & x & x \\
 & & & & & & x & x & x & x \\
 & & & & & & x & x & x & x
\end{array}$$

Since the linear change of the independent variable taking $[\xi_m, \xi_{m+1}]$ to $[0, 1]$ carries

$$N_{m(k-1)+i} \text{ on } [\xi_m, \xi_{m+1}] \text{ to } \hat{N}_i \text{ on } [0, 1], \qquad i = 1, \ldots, k,$$

we have

$$(17) \quad G_{m(k-1)+i,m(k-1)+j} = \begin{Bmatrix} (\Delta\xi_m/(\xi_{m+1} - \xi_{m-1}))\hat{G}_{1j}, \ i = 1 \\ \hat{G}_{ij}, \ i = 2, \ldots, k-1 \\ (\Delta\xi_m/(\xi_{m+2} - \xi_m))\hat{G}_{kj}, \ i = k \end{Bmatrix}, \quad j = 1, \ldots, k,$$

for $m = 0, \ldots, r$. This says that each of the $r + 1$ blocks of G is essentially equal to $\hat{G}$.

G is totally positive by [1]. Its inverse is therefore a checkerboard matrix, hence (see [1, p. 541])

$$(18) \qquad \text{if } \mathbf{y} \text{ is such that } \sum_j G_{ij}(-)^{i+j}y_j = 1, \quad \text{all } i, \text{ then } \|G^{-1}\|_\infty = \|\mathbf{y}\|_\infty.$$

But such a $\mathbf{y}$ is easily constructed. Take $\mathbf{x} = (x_1, \ldots, x_k)$ so that

$$(19) \qquad \sum_j \hat{G}_{ij}(-)^{i+j}x_j = 1, \quad \text{all } i,$$

and extend $\mathbf{x}$ to a $(k-1)$-periodic function $\mathbf{y} = (y_i)_1^n$ on all of $(1, \ldots, n)$. This is possible since $x_k = x_1$ by symmetry. Then, for $i = m(k-1) + I$, we have from (17) and (19) that

$$\sum_j G_{ij}(-1)^{i+j}y_j = \sum_{j=1}^{k} \hat{G}_{Ij}(-)^{I+j}x_j = 1, \qquad I = 2, \ldots, k-1; m = 0, \ldots, r,$$

and also

$$\sum_j G_{ij}(-)^{i+j}y_j = (\Delta\xi_{m-1}/(\xi_{m+1} - \xi_{m-1}))\sum_j \hat{G}_{kj}(-)^{k+j}x_j$$
$$+ (\Delta\xi_m/(\xi_{m+1} - \xi_{m-1}))\sum_j \hat{G}_{1j}(-)^{1+j}x_j = 1$$

for $I = 1; m = 0, \ldots, r + 1$.

This proves with (18) that

$$\|G^{-1}\|_\infty = \|y\|_\infty = \|x\|_\infty = \|\hat{G}^{-1}\|_\infty. \quad \text{Q.E.D.}$$

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