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Asymptotic Formulas Related to Free Products of Cyclic Groups

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Abstract. Asymptotic formulas for the number of subgroups of a given index of the free product of finitely many cyclic groups are given. The classical modular group Γ is discussed in detail, and a table of the number of subgroups of Γ of index n is given for $1 \leqslant n \leqslant 100$.

Formulas for the number of subgroups of a given index of free groups of finite rank have been given by M. Hall [2], and have been generalized to the case of free products of finitely many cyclic groups by I. M. S. Dey [1]. In this note we consider the asymptotic behavior of these numbers, and also give some tabular material for the case of the classical modular group.

These formulas have the common feature that the recurrence formulas associated with them have the same structure; and before considering questions of asymptotic behavior, we consider the formulas from a purely formal point of view.

Let $\alpha_0, \alpha_1, \alpha_2, \ldots, M_1, M_2, M_3, \ldots$ be sequences of real numbers such that

$$\alpha_0 = 1, \qquad \sum_{k=1}^{n} \alpha_{n-k} M_k = n\alpha_n, \qquad n \geqslant 1. \tag{1}$$

Define the formal power series $f(x)$, $g(x)$ by

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^n, \qquad g(x) = \sum_{n=1}^{\infty} M_n x^n.$$

Then (1) is equivalent to the identity

$$g(x) = xf'(x)/f(x). \tag{2}$$

Formula (2) implies that

$$\sum_{n=1}^{\infty} \frac{M_n}{n} x^n = \log f(x), \tag{3}$$

so that

$$\sum_{n=1}^{\infty} \frac{M_n}{n} x^n = \log(1 + f(x) - 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (f(x) - 1)^n. \tag{4}$$

Comparing coefficients of corresponding powers of x in (4), we find the following result, which we state as a theorem:

THEOREM 1. *The numbers M_n are given explicitly as functions of the numbers α_n by the formula*

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$$\frac{M_n}{n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} S_k, \tag{5}$$

where

$$S_k = \sum_{n_1+n_2+\cdots+n_k=n;\, n_i\geq 1} \alpha_{n_1}\alpha_{n_2}\cdots\alpha_{n_k}. \tag{6}$$

Conversely, if we rewrite (4) as

$$f(x) = \exp\left(\sum_{n=1}^{\infty} \frac{M_n}{n} x^n\right) = \prod_{n=1}^{\infty} \exp\left(\frac{M_n}{n} x^n\right),$$

we find that

$$\alpha_n = \sum_{r_1+2r_2+\cdots+nr_n=n;\, r_i\geq 0} M_1^{r_1}M_2^{r_2}\cdots M_n^{r_n}/1^{r_1}r_1!\, 2^{r_2}r_2!\cdots n^{r_n}r_n!.$$

This discussion implies an interesting formal identity, which we mention in passing. If we consider (1) as a system of equations for $M_1, M_2, \ldots, M_n$, then Cramer's rule implies the following: If A is the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & & & 1 \\ \alpha_1 & 1 & & 2\alpha_2 \\ \alpha_2 & \alpha_1 & 1 & 3\alpha_3 \\ & \cdots & & \cdots \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & n\alpha_n \end{bmatrix},$$

then

$$\det(A) = n \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} S_k, \tag{7}$$

where S_k is given by (6).

We now assume certain properties of the sequences $\alpha_0, \alpha_1, \alpha_2, \ldots, M_1, M_2, M_3, \ldots$, and use them to derive the following lemma, which will form the basis of the discussion of asymptotic properties that follows.

LEMMA 1. *Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be a sequence of positive numbers such that $\alpha_0 = 1$. Suppose that $M_1, M_2, M_3, \ldots$ is also a sequence of positive numbers, and that*

$$\sum_{k=1}^{n} \alpha_{n-k} M_k = n\alpha_n, \qquad n \geq 1.$$

Put

$$A_n = \sum_{k=1}^{n-1} \alpha_k\alpha_{n-k}/\alpha_n,$$

and assume that $A_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then $M_n \sim n\alpha_n$.

Proof. Because of the positivity, we have that

$$M_n \leqslant \sum_{k=1}^{n} \alpha_{n-k} M_k = n\alpha_n, \qquad n \geqslant 1.$$

Thus

$$\sum_{k=1}^{n-1} \alpha_{n-k} M_k \leqslant \sum_{k=1}^{n-1} \alpha_{n-k} \cdot k\alpha_k = \frac{n}{2} \sum_{k=1}^{n-1} \alpha_{n-k}\alpha_k,$$

$$\sum_{k=1}^{n-1} \alpha_{n-k} M_k \leqslant \frac{n}{2} \alpha_n A_n.$$

It follows that

$$n\alpha_n = M_n + \sum_{k=1}^{n-1} \alpha_{n-k} M_k \leqslant M_n + \frac{n}{2} \alpha_n A_n, \qquad M_n \geqslant n\alpha_n \left(1 - \frac{1}{2} A_n\right).$$

Thus

$$(8) \qquad 1 - \tfrac{1}{2} A_n \leqslant M_n/n\alpha_n \leqslant 1;$$

and so, $M_n/n\alpha_n \longrightarrow 1$ as $n \longrightarrow \infty$, since $A_n \longrightarrow 0$ as $n \longrightarrow \infty$. This completes the proof.

For the case of free groups of finite rank we also require the following lemma:

LEMMA 2. *Suppose that* $s \geqslant 1$, *and put*

$$A_n = \sum_{k=1}^{n-1} \binom{n}{k}^{-s}, \qquad n \geqslant 1.$$

Then $A_n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Proof. We may assume that $n \geqslant 3$. Since $\binom{n}{k} \geqslant \binom{n}{2}$ for $2 \leqslant k \leqslant n-2$, we have

$$A_n = 2n^{-s} + \sum_{k=2}^{n-2} \binom{n}{k}^{-s} \leqslant 2n^{-s} + (n-3)\binom{n}{2}^{-s}$$
$$= 2n^{-s} + n^{-s}(n-3)\left(\frac{n-1}{2}\right)^{-s}.$$

Now $(n-1)/2 \geqslant 1$, since $n \geqslant 3$. It follows that

$$A_n \leqslant 2n^{-s} + n^{-s}(n-3)\left(\frac{n-1}{2}\right)^{-1} \leqslant 4n^{-s}.$$

Since A_n is positive, the result follows.

We now use these lemmas to obtain our first asymptotic result:

THEOREM 2. *Let* $M_r(n)$ *be the number of subgroups of index n of the free group of rank r, where* $r \geqslant 2$. *Then*

$$M_r(n) \sim n \cdot n!^{r-1}.$$

Proof. M. Hall's recurrence formula for $M_r(n)$ [2] states that

$$\sum_{k=1}^{n} (n-k)!^{r-1} M_r(k) = n \cdot n!^{r-1}, \qquad n \geqslant 1.$$

By Lemma 1 we need only show that if

$$A_n = \sum_{k=1}^{n-1} k!^{r-1}(n-k)!^{r-1}/n!^{r-1} = \sum_{k=1}^{n-1} \binom{n}{k}^{1-r},$$

then $A_n \longrightarrow 0$ as $n \longrightarrow \infty$. But this is the content of Lemma 2, since $r - 1 \geqslant 1$. In fact, inequality (8) implies that

$$(9) \qquad 1 - \frac{2}{n^{r-1}} \leqslant \frac{M_r(n)}{n \cdot n!^{r-1}} \leqslant 1, \qquad n \geqslant 3.$$

This completes the proof.

$$(5) \qquad \frac{M_n}{n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} S_k,$$

where

$$(6) \qquad S_k = \sum_{n_1+n_2+\cdots+n_k=n; n_i \geqslant 1} \alpha_{n_1}\alpha_{n_2} \cdots \alpha_{n_k}.$$

Conversely, if we rewrite (4) as

$$f(x) = \exp\left(\sum_{n=1}^{\infty} \frac{M_n}{n} x^n\right) = \prod_{n=1}^{\infty} \exp\left(\frac{M_n}{n} x^n\right),$$

we find that

$$\alpha_n = \sum_{r_1+2r_2+\cdots+nr_n=n; r_i \geqslant 0} M_1^{r_1} M_2^{r_2} \cdots M_n^{r_n}/1^{r_1}r_1!\, 2^{r_2}r_2! \cdots n^{r_n}r_n!.$$

This discussion implies an interesting formal identity, which we mention in passing. If we consider (1) as a system of equations for $M_1, M_2, \ldots, M_n$, then Cramer's rule implies the following: If A is the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & & & 1 \\ \alpha_1 & 1 & & 2\alpha_2 \\ \alpha_2 & \alpha_1 & 1 & 3\alpha_3 \\ & \cdots & & \cdots \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & n\alpha_n \end{bmatrix},$$

then

$$(7) \qquad \det(A) = n \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} S_k,$$

where S_k is given by (6).

We now assume certain properties of the sequences $\alpha_0, \alpha_1, \alpha_2, \ldots, M_1, M_2, M_3, \ldots$, and use them to derive the following lemma, which will form the basis of the

The importance of these numbers stems from the work of I. M. S. Dey [1] who showed that if M_n is the number of subgroups of index n of the free product $C_{p_1} * C_{p_2} * \cdots * C_{p_k}$, where p_i is either ∞ or an integer $\geqslant 2$, then M_n satisfies the recurrence (1), with

(12) $$\alpha_n = \tau_{p_1}(n)\tau_{p_2}(n) \cdots \tau_{p_k}(n)/n!\ .$$

Of course, $\tau_\infty(n) = n!$.

When p is prime, $\tau_p(n)$ is most easily calculated by the recurrence formula

$$\tau_p(n+1) = \tau_p(n) + (p-1)! \binom{n}{p-1} \tau_p(n-p+1), \qquad n \geqslant p-1,$$

with the initial conditions

$$\tau_p(0) = \tau_p(1) = \cdots = \tau_p(p-1) = 1.$$

The asymptotic behavior of $\tau_p(n)$ for p prime was determined by L. Moser and M. Wyman in [3], by means of the generating function (11). They showed that

(13) $$\tau_p(n) \sim K_p \exp\left(\frac{p-1}{p}\, n \log n - \frac{p-1}{p}\, n + n^{1/p}\right),$$

where

(14) $$K_2 = 2^{-\frac{1}{2}}e^{-\frac{1}{4}}, \qquad K_p = p^{-\frac{1}{2}}, \qquad p > 2.$$

It follows from (13), (14), and Stirling's formula that

(15) $$\tau_2(n)\tau_3(n)/n! \sim K \exp\left(\frac{n}{6} \log n - \frac{n}{6} + n^{1/2} + n^{1/3} - \frac{1}{2}\log n\right),$$

where $K = (12\pi e^{1/2})^{-1/2}$.

Of particular interest is the case $\Gamma = C_2 * C_3$, the classical modular group. We wish to show that in this case

$$M_n \sim \tau_2(n)\tau_3(n)/(n-1)!.$$

This is rather more difficult than the problem for free groups of finite rank, and the asymptotic properties of the coefficients $\tau_2(n)$, $\tau_3(n)$ come into play. In a well-defined sense, this is the most difficult case. The coefficients in the recurrence formula grow least rapidly, corresponding to the fact that Γ has the smallest hyperbolic area of all noncompact Fuchsian groups. Further comments on this point will be made later on.

The basic problem will be to show that if

$$\alpha_n = \tau_2(n)\tau_3(n)/n!,$$

then

$$A_n = \sum_{k=1}^{n-1} \alpha_k \alpha_{n-k}/\alpha_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

The discussion that follows is devoted to this end.

The asymptotic formula (15) implies that

$$A_n = O\left(\sum_{1 \leqslant k \leqslant n/2} \exp(\theta(k) + \theta(n-k) - \theta(n)) \right), \tag{16}$$

where

$$\theta(n) = \frac{n}{6} \log n + n^{1/2} + n^{1/3}.$$

(Here we have used the symmetry of the sum, and the facts that the terms in the exponent corresponding to $-n/6$ disappear, and that $n/k(n-k)$ is bounded by an absolute constant for $1 \leqslant k \leqslant n-1$.)

Consider

$$\theta(x) = \frac{x}{6} \log x + x^{1/2} + x^{1/3}.$$

Then a brief calculation shows that

$$36x^{5/3}\theta''(x) = 6x^{2/3} - 9x^{1/6} - 8,$$

and that $\theta''(x) \geqslant 0$ for $x \geqslant 6.17250\ldots$. Hence $\theta''(x) \geqslant 0$ for $x \geqslant 7$, and it follows that $\theta'(x)$ is monotone increasing for $x \geqslant 7$.

Now consider the inequality

$$\theta(k+1) + \theta(n-k-1) \leqslant \theta(k) + \theta(n-k). \tag{17}$$

This will hold if and only if

$$\theta(k+1) - \theta(k) \leqslant \theta(n-k) - \theta(n-k-1).$$

We have

$$\theta(k+1) - \theta(k) = \theta'(k+\sigma_1), \qquad 0 \leqslant \sigma_1 \leqslant 1,$$

$$\theta(n-k) - \theta(n-k-1) = \theta'(n-k-1+\sigma_2), \qquad 0 \leqslant \sigma_2 \leqslant 1.$$

Assume that

$$7 \leqslant k \leqslant \tfrac{1}{2}n - 1. \tag{18}$$

Then $k+1 \leqslant n-k-1$, and using the fact that $\theta'(x)$ is monotone increasing for $x \geqslant 7$, we get

$$\begin{aligned}\theta(k+1) - \theta(k) &= \theta'(k+\sigma_1) \leqslant \theta'(k+1) \leqslant \theta'(n-k-1) \\ &\leqslant \theta'(n-k-1+\sigma_2) = \theta(n-k) - \theta(n-k-1).\end{aligned}$$

It follows that (17) holds, provided that k satisfies (18). We state the consequence of this result as a lemma.

LEMMA 3. *The function $\theta(k) + \theta(n-k)$ satisfies*

$$\theta(k) + \theta(n-k) \leqslant \theta(7) + \theta(n-7), \qquad 7 \leqslant k \leqslant n/2.$$

We also remark that if k remains bounded, then

$$\theta(k) + \theta(n-k) - \theta(n) = -\frac{k}{6}\log n + O(1). \tag{19}$$

We can now prove

LEMMA 4. *Let* $A_n = \sum_{k=1}^{n-1} \alpha_k \alpha_{n-k}/\alpha_n$, *where* $\alpha_n = \tau_2(n)\tau_3(n)/n!$. *Then* $A_n = O(n^{-1/6})$, *so that* $A_n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Proof. By Lemma 3,

$$A_n = O\left(\sum_{1 \leq k \leq 6} \exp(\theta(k) + \theta(n-k) - \theta(n))\right) + O\left(\sum_{7 \leq k \leq n/2} \exp(\theta(7) + \theta(n-7) - \theta(n))\right).$$

We have

$$\sum_{1 \leq k \leq 6} \exp(\theta(k) + \theta(n-k) - \theta(n)) = \sum_{1 \leq k \leq 6} \exp\left(-\frac{k}{6}\log n + O(1)\right) = O(n^{-1/6}),$$

and

$$\sum_{7 \leq k \leq n/2} \exp(\theta(7) + \theta(n-7) - \theta(n)) = O\left(n \exp\left(-\frac{7}{6}\log n + O(1)\right)\right) = O(n^{-1/6}).$$

The result now follows.

Lemmas 1 and 4 now imply our desired result:

THEOREM 4. *Let* M_n *denote the number of subgroups of index* n *of the classical modular group* Γ. *Then*

$$M_n \sim \tau_2(n)\tau_3(n)/(n-1)! \sim K \exp\left(\frac{n}{6}\log n - \frac{n}{6} + n^{1/2} + n^{1/3} + \frac{1}{2}\log n\right),$$

where $K = (12\pi e^{1/2})^{-1/2}$.

Precisely the same discussion applies to the more general case when M_n is the number of subgroups of index n of the free product $C_{p_1} * C_{p_2} * \cdots * C_{p_k}$, with one exception. The corresponding result is that

$$M_n \sim \tau_{p_1}(n)\tau_{p_2}(n) \cdots \tau_{p_k}(n)/(n-1)!,$$

where $\tau_p(n)$ is the number of homomorphisms of C_p into S_n. The exception occurs for $C_2 * C_2$. The difficulty here is that $\tau_2(n)^2/n!$ does not grow fast enough; in fact,

$$\tau_2(n)^2/n! \sim K \exp\left(2n^{1/2} - \frac{1}{2}\log n\right),$$

where $K = (8\pi e)^{-1/2}$. The exception is quite natural in view of the fact that this is the only group of the form $C_{p_1} * C_{p_2} * \cdots * C_{p_k}$ which does not have a representation as a Fuchsian group, since it would correspond to one of genus 0, with a single parabolic generator and 2 elliptic generators of order 2; and so would have zero hyperbolic area, which is not possible.

M(N) is the number of subgroups of the classical modular group of index N

N	M(N)	N	M(N)
1	1	51	1042904230435308
2	1	52	2353258168183056
3	4	53	5018827370579404
4	8	54	10547597621517112
5	5	55	23788180556798856
6	22	56	51219574162595680
7	42	57	108420140150958464
8	40	58	244556725280402557
9	120	59	531051678812968744
10	265	60	1132058252012247936
11	286	61	2555221387154759289
12	764	62	5591154609087446054
13	1729	63	12000605074451550160
14	2198	64	27117552811153855680
15	5168	65	59749471015816115222
16	12144	66	129096123267775868166
17	17034	67	292152183906195140230
18	37702	68	647820636139303527128
19	88958	69	1408680305233863887966
20	136584	70	3193691365498309148284
21	288270	71	7123474189205298268692
22	682572	72	15585866806530333864208
23	1118996	73	35408399627074036560816
24	2306464	74	79411796926589859301294
25	5428800	75	174786866873511715532628
26	9409517	76	397988303771373972899968
27	19103988	77	897183482867489002743454
28	44701696	78	1986081323730942427050260
29	80904113	79	4533337015877143666107784
30	163344502	80	10269190024287640883669792
31	379249288	81	22858848125975541396523980
32	711598944	82	52311144432011656725753204
33	1434840718	83	119045150159430609536306820
34	3308997062	84	266408038409645923245611632
35	6391673638	85	611297865534522328810726884
36	12921383032	86	1397257675120198769089798632
37	29611074174	87	3143030617470775207826061656
38	58602591708	88	7231970266681482552134898592
39	119001063028	89	16599371653596567362869244032
40	271331133136	90	37526397797214870497477862098
41	547872065136	91	86591501222729298577623800376
42	1119204224666	92	199561184845436499408618491008
43	2541384297716	93	453310522764485671147172334106
44	5219606253184	94	1049023788372134074385428390460
45	10733985041978	95	2427029113004378647451505343600
46	24300914061436	96	5538777906538570521597333058368
47	50635071045768	97	12854913784226053720296237333900
48	104875736986272	98	29853181061514806750697886477212
49	236934212877684	99	68435931318931500012779730780508
50	499877970985660	100	159299552010504751878902805384624

To illustrate how the problem depends on the hyperbolic area, assume that each p_i is ∞ or a prime. Let the multiplicity of ∞ be $t - 1$ $(t \geqslant 1)$, and let the remaining p_i be denoted by e_i $(1 \leqslant i \leqslant s)$, so that $t + s - 1 = k$. Then if

$$\alpha_n = \tau_{p_1}(n)\tau_{p_2}(n) \cdots \tau_{p_k}(n)/n!,$$

formula (13) implies that

$$\alpha_n \sim K \exp\left(Hn \log n - Hn + \sum_{i=1}^{k} n^{1/e_i} + \left(\frac{1}{2}t - 1\right) \log n\right),$$

where K is a constant which need not be specified, and

$$H = t - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{e_i}\right).$$

Thus the growth of α_n depends upon H, and apart from a constant factor, H is just the hyperbolic area of $C_{p_1} * C_{p_2} * \cdots * C_{p_k}$.

In conclusion we append a table of M_n for $1 \leqslant n \leqslant 100$, where M_n is the number of subgroups of index n of $\Gamma = C_2 * C_3$. The table was computed in a negligible amount of time using residue arithmetic by means of the recurrence formula (1), with $\alpha_n = \tau_2(n)\tau_3(n)/n!$. The approach to 1 of the ratio $M_n/n\alpha_n$ is quite slow, and agrees well with the estimate

$$|1 - M_n/n\alpha_n| = O(n^{-1/6})$$

derived before.

A useful check on the computation is that if $n = p^e$, where p is a prime and $n > 3$, then $M_n \equiv 0 \mod p$. This is so because Γ contains no normal subgroups of index n.

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