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Dyadotropic Polynomials

By Harvey Cohn*

Abstract. Polynomials which tend to represent powers of two arise in connection with certain problems of class field theory of dihedral biquadratic fields. The availability of independent units is an immediate consequence for an infinitude of parametrized cases. An exhaustive search for such types of polynomials is made by use of computer.

1. Introduction. The title refers to polynomials which have an "inclination" toward powers of two. For example, the most startling case is perhaps

$$f(x) = x^4 + x^3 - 6x^2 + 2x + 4, \tag{1.1}$$

$$\begin{gathered} f(-3) = -2, \quad f(-2) = -16, \quad f(-1) = -4, \quad f(0) = 4, \quad f(1) = 2, \\ f(2) = 8, \quad f(3) = 64. \end{gathered} \tag{1.2}$$

To be more formal, we define a monic integral polynomial of degree N as *dyadotropic* when $N + 1$ consecutive absolute values are powers of two (higher than the zero power).

From a combinatorial point of view, we can in principle assign $N + 1$ consecutive values as $(-1)^{S_j} 2^{T_j} = f(j)$ merely by taking the precautions of "finite differencing". Thus, we need only have the Nth difference equal to $N!$ and the lower order differences divisible by the corresponding factorial.

We are concerned, however, with polynomials of special relevancy to *algebraic number theory.* Thus, the example (1.1) is a defining polynomial for $k_4 = Q(41^{1/2}, \epsilon^{1/2})$, for $\epsilon = 32 + 5 \cdot 41^{1/2}$, a fundamental unit for $k_2 = Q(41^{1/2})$. This is important as a subfield of the absolute class field of $Q((-41)^{1/2})$, (see [1], [2], [9]). The fact that all these powers of two occur merely guarantees the ready availability of independent units. This phenomenon is generalized in a systematic manner which we shall describe, and it leads to a parametrized infinitude of cases.

2. Relative-Quadratic Polynomials. We focus our attention on the cases where the degree $N = 4$ and $f(x)$ is the norm of a polynomial $g(x)$ over a quadratic field as follows:

$$k_2 = Q(d_0^{1/2}), \qquad d_0 \equiv 1 \pmod 8), d_0 \text{ square-free} \neq 1, \tag{2.1}$$

$$g(x) = x^2 + \alpha x + e, \tag{2.2}$$

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(2.3) $\alpha = (a + bd_0^{1/2})/2, \quad |e| = 2^{T_0/2}, \quad (T_0 > 0), a \equiv b \equiv 1 \pmod 2,$

(2.4) $$f(x) = N_{2/1}g(x),$$

(2.5) $$f(x) = x^4 + ax^3 + (2e + (a^2 - b^2d_0)/4)x^2 + aex + e^2.$$

The dyadotropic property is achieved if the four surds

(2.6a) $$g(\pm 2)/2 = (2 + e/2 \pm (a + bd_0^{1/2})/2) = \gamma_{\pm 2},$$

(2.6b) $$g(\pm 1) = (1 + e \pm (a + bd_0^{1/2})/2) = \gamma_{\pm 1},$$

all have norm equal to a power of two. Then

(2.7) $$f(\pm 2) = 4N_{2/1}\gamma_{\pm 2}, \quad f(\pm 1) = N_{2/1}\gamma_{\pm 1}, \quad f(0) = e^2.$$

(The illustration (1.1) is given by $\alpha = (1 + 41^{1/2})/2$, $e = 2$, and it appears in Tables I and II as [2; 1, 0].)

If we let ξ denote a root of $f(x) = 0$, then it determines

(2.8) $$k_4 = Q(\xi) = Q(d_0^{1/2}, \mu^{1/2}) = Q(\mu^{1/2}),$$

(2.9) $$\mu = \alpha^2 - 4e.$$

Finally, with m integral, we have the norm relation

(2.10) $$f(m) = N_{4/1}(m - \xi) = N_{2/1}g(m).$$

We refer to such dyadotropic polynomials $f(x)$ as *normed relative-quadratic*, and we use the symbol $f(x)$ exclusively for such polynomials from now on. For convenient reference to Galois group operations on k_4 (or on ξ and its conjugates), we fix the notation as follows:

(2.11a) $$\xi = (-\alpha + \mu^{1/2})/2, \qquad \xi' = (-\alpha' + \mu'^{1/2})/2,$$

(2.11b) $$S\xi = (-\alpha - \mu^{1/2})/2, \qquad S\xi' = (-\alpha' - \mu'^{1/2})/2,$$

where the prime denotes conjugation in k_2. Thus, with $d = b^2d_0$,

(2.12) $$\alpha = (a + d^{1/2})/2, \qquad \alpha' = (a - d^{1/2})/2,$$

(2.13) $$\mu = \alpha^2 - 4e, \qquad \mu' = \alpha'^2 - 4e.$$

Of course, S leaves k_2 invariant elementwise, so $\epsilon = S\epsilon$ for ϵ the fundamental unit of k_2. It is easy to verify that

(2.14) $$N_{4/2}\xi = \xi S\xi = \xi' S\xi' = e,$$

(2.15) $$f(e/x) = f(x)e^2/x^4.$$

Furthermore, $f(1) - f(-1) = 2a(1 + e) \equiv 2 \pmod 4$. Thus, $|f(1)|$ or $|f(-1)|$ (one choice) $= 2$. We can, therefore, normalize all polynomials $f(x)$ (against the trivial symmetry $x \longleftrightarrow -x$, $a \longleftrightarrow -a$) by assuming, henceforth,

$$f(1) = N_{2/1}\gamma_1 = \pm 2 \qquad (T_1 = 1). \tag{2.16}$$

It is now clear how $f(x)$ might be computed for some given ring-discriminant d, satisfying

$$d = b^2 d_0 \equiv 1 \pmod 8. \tag{2.17}$$

We search for pairs of powers 2^t ($\geqslant 8$) and odd z (> 0) satisfying

$$z^2 - d = \pm 2^t. \tag{2.18}$$

For any d we construct a file $(z, \pm 2^t)$. For instance, the longest file arose for $d = 17$ (within the limits of our computation):

$$(z, \pm 2^t) = (5, 8), (3, -8), (1, -16), (7, 32), (9, 64), (23, 512).$$

Then we ask for quadruples $(z_i + d^{1/2})/2$, $(i = 1, \ldots, 4)$, which satisfy the descriptions of $\gamma_{\pm 2}, \gamma_{\pm 1}$ in (2.6a, b). (We "anchor" the procedure by trying each $\pm z_i$ in turn for γ_{-2} while e successively takes the values $2, -2, 4, -4, \ldots$.) As expected, the largest number of polynomials (eight) arose from the file for $d = 17$.

By virtue of (2.16), we must satisfy (2.18) for $2^t = 8$ at least once in each file. Thus $d = z^2 \pm 8$. This implies that

$$d \geqslant -7. \tag{2.19}$$

Because of this, the computer survey is made to construct the file $(z, \pm 2^t)$ by taking values of $d \equiv 1 \pmod 8$ (excluding perfect squares) within the limit $-7 \leqslant d \leqslant 75001$, with values of $2^t < 10^{20}$. The output contained d, a, "$f(x)$" (i.e., its coefficients and values for $-3 \leqslant x \leqslant 3$, see (6.4) below). The IBM 360-50 at the City College of New York was used. The total running time was about ten minutes.

3. Thoroughness of Enumeration. The computation was performed as described, with output printed according to increasing d. It was clear that some "regularity" occurred for $e = \pm 2$. This led to the conjecture that all such cases may be accounted for parametrically, indeed by the formulas presented in Table I. (This is proved in Theorem 3.5 below.) The further conjecture seemed warranted from the size of the search that the cases where $|e| > 2$ are finite in number and are only those listed in Table I. Thus, there would be none for $|e| > 4$. This would be difficult to believe "firmly" in the absence of a computer survey because of the occasional "freakish" solutions to (2.18). (Incidentally, the solutions to (2.18) are of traditional diophantine interest largely for negative d, see [5], while the fields where d has the form $z^2 - 8$ are of appeal for reasons of class number, see [8].)

LEMMA 3.1. *The polynomials $f(x)$ satisfy*

$$e = 2: \quad f(-2) = 4f(-1), \quad f(2) = 4f(1), \quad f(0) = 4, \tag{3.2}$$

$$e = -2: \quad f(-2) = 4f(1), \quad f(2) = 4f(-1), \quad f(0) = 4, \tag{3.3}$$

$$e = -4: \quad f(-2) = f(2), \quad f(0) = 16. \tag{3.4}$$

These statements follow from (2.15).

TABLE I

e	type	a	d	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
2	$[2; u, v]$	$(-1)^v(2^u - (-1)^u)/3$	$(a+6)^2 + (-1)^{u+v}8$	$-(-1)^v 2^{u+3}$	$-(-1)^v 2^{u+1}$	4	$-(-1)^{u+v}2$	$-(-1)^{u+v}8$
-2	$[-2; u, v, w]$	$-(-1)^w(2^u + (-1)^v)$	$(a-2)^2 - (-1)^{v+w}8$	$(-1)^{u+v}8$	$-(-1)^w 2^{u+1}$	4	$(-1)^w 2$	$-(-1)^{v+w}2^{u+3}$
4	. . .	1	113	-64	-8	16	2	-32
-4	$[-4; A]$ U	5	-7	32	32	16	2	32
-4	$[-4; B]$	1	17	-16	8	16	2	-16
-4	$[-4; C]$ U	3	17	-8	16	16	-2	-8
-4	$[-4; D]$ U	1	33	-32	4	16	-2	-32
-4	$[-4; E]$ U	-3	73	-64	16	16	2	-64

Data on $k_4 = Q(\mu^{1/2})$ *where* $\mu = \alpha^2 - 4e$, $\alpha = (a + d^{1/2})/2$

The field k_4 corresponds to the polynomial $f(x) = x^4 + ax^3 + (2e + (a^2 - d)/4)x^2 + aex + e^2 = 0$. Values of $f(m) = \pm 2^{T_m}$ are shown for $-2 \leqslant m \leqslant 2$. The integer $u \geqslant 1$ except for the exclusion of $[-2; 1, 0, 1]$. "U" denotes fields with odd discriminant or unramified factors of two.

THEOREM 3.5. *When $e = \pm 2$, the only dyadotropic normed relative-quadratic polynomials are given (as in Table I) by*

$$\text{(3.6a)}\quad e = 2: \qquad a = (-1)^v(2^u - (-1)^u)/3, \qquad d = (a+6)^2 + (-1)^{u+v}8,$$

$$\text{(3.6b)}\quad e = -2: \qquad a = -(-1)^w(2^u + (-1)^v), \qquad d = (a-2)^2 - (-1)^{v+w}8.$$

When $e = -4$, there are only the five cases listed in Table I. (Conjecturally, the only remaining case is shown for $e = 4$.)

For proof, note that when $e = 2$, there is only one degree of freedom left from (2.16) and (3.2) namely in $f(-1)$, which we write as $-(-1)^v2^{u+1}$. Then by solving both of $N_{2/1}\gamma_{\pm 1} = f(\pm 1)$ simultaneously we obtain a and d. The same holds for $e = -2$, except for a further choice of sign.

The cases where $|e| \geqslant 4$ are handled by finite-difference conditions on $f(m) = (-1)^{S_m}2^{T_m}$, $-2 \leqslant m \leqslant 3$. For example, the fact that the fourth difference is 24 leads to

$$\text{(3.7)}\quad \begin{aligned} 6e^2 - 24 &= 4(-1)^{S_{-1}}2^{T_{-1}} + 4(-1)^{S_1}2^{T_1} \\ &\quad - (-1)^{S_{-2}}2^{T_{-2}} - (-1)^{S_2}2^{T_2}. \end{aligned}$$

Furthermore, the condition on the third difference is

$$\text{(3.8)}\quad (-1)^{S_{\pm 2}}2^{T_{\pm 2}} \equiv (-1)^{S_{\mp 1}}2^{T_{\mp 1}} \pmod 3$$

with either choice of sign. While these conditions make a polynomial dyadotropic, they do not make it normed relative-quadratic. The additional necessary condition (2.16), that $T_1 = 1$, would make (3.7) into a simpler problem of decomposing $6e^2 - 24 \pm 8$ as the sum or difference of three powers of two. With $|e| > 4$, we would be down to a small number of choices coming from the following two decompositions in some permutation:

$$\text{(3.9a)}\quad 6e^2 - 16 = 4e^2 + 2e^2 - 16 = 8e^2 - 2e^2 - 16,$$

$$\text{(3.9b)}\quad 6e^2 - 32 = 4e^2 + 2e^2 - 32 = 8e^2 - 2e^2 - 32,$$

but when $|e| = 4$, $6e^2 - 32 = 64$ has infinitely many such partitions. Nevertheless, when $e = -4$, (3.4) serves to limit (3.9a, b) to only a finite number of cases (see Table I). The remaining cases lead to equations of type (2.18) which seem unlikely to be valid "often enough". Yet, incredible exceptions such as $e = 4$ (Table I) are not completely ruled out at present.

4. Biquadratic Ramifications and Units. It is clear (see [7]) that since $d_0 \equiv d \equiv 1 \pmod 8$, then the ideal 2 factors in k_2 as

$$\text{(4.1)}\quad 2 = 2_1 2_2 \qquad (2_1 \neq 2_2).$$

We normalize the choice by taking $2_1 = (\gamma_1)$ and $2_2 = (\gamma_1')$, where

$$\text{(4.2)}\quad \gamma_1 = (1 + e) + (a + d^{1/2})/2.$$

Also, $2_1 | \gamma_{-1}$, but it is 2_2 which divides $\alpha = (a + d^{1/2})/2$ and μ $(= \alpha^2 - 4e)$. Hence,

$$\begin{cases} N_{4/2}(\xi + 1) = \gamma_{-1} = 2_1^{T-1}, \\ N_{4/2}(\xi) = e = (2_1 2_2)^{T_0/2}, \\ N_{4/2}(\xi - 1) = \gamma_1 = 2_1. \end{cases} \tag{4.3}$$

When we go to k_4, it is clear that 2_1 must split

$$2_1 = 2_{11}2_{12} \qquad (2_{11} \neq 2_{12}). \tag{4.4}$$

(For instance, if we set $2_{11} = (\xi - 1)$, then 2_{11} cannot divide ξ.)

LEMMA 4.5. *The ideal 2_2 must ramify in k_4/k_2 when $e = \pm 2$.*

For proof, note that in (4.3) with $T_0 = 2$, only one factor 2_{21} of 2_2 divides ξ while no other factor can divide $\xi \pm 1$.

When $|e| > 2$, the factor 2_2 may or may not ramify. Thus, for all cases, we have

$$(\xi + 1) = 2_{11}^{T-1}, \qquad (\xi - 1) = 2_{11}, \tag{4.6a}$$

$$(\xi) = \begin{cases} 2_{21}^{T_0/2} 2_{12}^{T_0/2} & (2_2 = 2_{21}^2 \text{ ramified}), \\ (2_{21}2_{22})^{T_0/4} 2_{12}^{T_0/2} & (2_2 = 2_{21}2_{22} \text{ unramified}). \end{cases} \tag{4.6b}$$

The conditions for ramification of 2_2 are that either μ contain a nonremovable even factor (i.e., $2_2^g \,\|\, \mu$ where g is odd), or that if $\mu_0 = \mu/2_2^g$ is odd (for g even) then (see [7]),

$$\mu_0 \not\equiv 1 \pmod 4. \tag{4.7}$$

(Since $d \equiv 1$ mod 8, the only odd square in k_2 is 1 mod 4.) Thus, we test the cases where $e = \pm 4$ and find only the following are unramified over 2:

$$[-4; A], [-4; C], [-4; D], [-4; E]. \tag{4.8}$$

In all other cases in Table I, 2_2 ramifies. In some cases, remarkably, it is the only ramified prime for k_4/k_2 (see [4]). For instance, for [2; 1, 0], $N_{2/1}$ disc $k_4/k_2 = 4$. For the case $e = 4$ and the cases [2; 2, 1], [2; 1, 1], [2; 3, 0], [−2; 4, 0, 0], $N_{2/1}$ disc $k_4/k_2 = 8$. Further anomalies occur. For instance, for [−2; 4, 0, 0] μ has a removable odd square 19_1^2 (where $19 = 19_1 19_2$ in k_2). Our interest, however, is primarily in the units.

As a consequence of (4.6b), $(\xi - 1)^{T-1}/(\xi + 1)$ is the unit ideal. Also, in either case of (4.6b), $(\xi)^2 = 2_2^{T_0/2} 2_{12}^{T_0}$, so $\xi^2(\xi - 1)^{T_0} = 2^{T_0/2} 2_1^{T_0/2}$. Thus, whether or not Table I is complete (as conjectured in Theorem 3.5), the following result is implied by the formula (4.2) for 2_1:

THEOREM 4.9. *The field k_4 has the units* Ω_1, Ω_2 defined as

$$\Omega_1 = (\xi - 1)^{T-1}/(\xi + 1), \qquad |f(-1)| = 2^{T-1}, \tag{4.10}$$

(4.11) $$\Omega_2^g = \xi^2(\xi - 1)^{T_0}/(2\gamma_1)^{T_0/2}, \qquad e^2 = 2^{T_0}, \qquad g = (2, T_0/2),$$

and $\gamma_1 = (1 + e) + (a + d^{1/2})/2$.

5. Independence of Units. The field k_4 has r independent (torsion-free) units, where r (the so-called Dirichlet rank, see [7]) depends on the four roots of $f(x) = 0$ (see (2.11a, b)) as follows:

$r = 1$ when no roots are real,
$r = 2$ when just two roots are real,
$r = 3$ when all roots are real.

We should hope to choose independent units from among Ω_1, Ω_2, and ϵ. These roots are independent, by definition, unless

(5.1) $$\Omega_1^{g_1}\Omega_2^{g_2}\epsilon^{g_0} = \pm 1$$

for some nonzero triple of integers (g_0, g_1, g_2). (Of course, ϵ^{g_0} is ignored when $d = -7$.) By the conjugation operations in (2.11) and (2.12), $S\Omega_1^{g_1}S\Omega_2^{g_2}\epsilon^{g_0} = \pm 1$, so that in any case we could eliminate the ϵ and obtain

(5.2a) $$(\Omega_1/S\Omega_1)^{g_1}(\Omega_2/S\Omega_2)^{g_2} = \pm 1,$$

(5.2b) $$(\Omega_1'/S\Omega_1')^{g_1}(\Omega_2'/S\Omega_2')^{g_2} = \pm 1.$$

If we take logarithms, we see that a "regulator-type" determinant δ vanishes (when (5.1) holds), namely,

(5.3) $$\delta = \begin{vmatrix} \log|\Omega_1/S\Omega_1| & \log|\Omega_2/S\Omega_2| \\ \log|\Omega_1'/S\Omega_1'| & \log|\Omega_2'/S\Omega_2'| \end{vmatrix}.$$

THEOREM 5.4. *For the cases in Table* I (*with a finite number of exceptions*), *the units* Ω_1, Ω_2, *and* ϵ *are an independent system.* (*Conjecturally, the only exceptions occur in Table* II, *when the Dirichlet rank* $r < 3$.)

The proof consists of the verification that for $|e| = 2$, the value of δ becomes infinite with the order of magnitude u^3 (or $(\log d)^3$). Details are omitted since this is a straightforward calculation based on the asymptotic estimates for the closeness of the four roots of $f(x)$ (see (2.11a, b)) to $0, 1, -1$, and ∞ as $u \longrightarrow \infty$.

For Table II, the computer tested the dependence for the cases of Table I where $e = \pm 4$ and for the cases of $e = 2$ where $u \leqslant 6$, and those of $e = -2$ where $u \leqslant 4$. (This range includes all cases where $r < 3$.) For any nonzero $\Omega \in k_4$, the real four-vector

(5.5) $$(\log|\Omega|, \log|S\Omega|, \log|\Omega'|, \log|S\Omega'|)$$

can be easily computed by double-precision complex arithmetic, particularly when Ω is a factored polynomial in ξ. This was done for $\Omega = \Omega_1$ and $\Omega = \Omega_2^* = \xi^{2/g}(\xi - 1)^2$. (Here Ω_2^* proves more convenient than Ω_2, whose denominator cancels in forming $\Omega/S\Omega$.) Whenever $\delta = 0$ "numerically," we can tell the unit relations by inspection. They consist only of cases where $r < 3$.

TABLE II

r	[e, "type"]	a	d	A	D	ϵ	unit $= \pm 1$
1	[2; 4, 1]	-5	-7	-7	56	$\cdots$	$\Omega_1\Omega_2^{-1}$
1	[-2; 1, 1, 1]	1	-7	13	44	$\cdots$	$\Omega_1\Omega_2^{-1}$
1	[-2; 2, 1, 1]	3	-7	17	88	$\cdots$	$\Omega_1\Omega_2^{-1}$
1	[-4; A]	5	-7	41	29	$\cdots$	$\Omega_1\Omega_2^{-2}$
1	[2; 2, 1]	-1	17	-7	8	$4 + 17^{1/2}$	$\Omega_1^2\epsilon^{-1}$; Ω_2
2	[2; 3, 1]	-3	17	-3	36	$4 + 17^{1/2}$	$\Omega_1\Omega_2^{-1}\epsilon^{-1}$
2	[2; 1, 1]	-1	33	1	8	$23 + 4 \cdot 33^{1/2}$	$\Omega_1^2\Omega_2^{-2}\epsilon$
2	[2; 5, 1]	-11	33	61	68	$23 + 4 \cdot 33^{1/2}$	$\Omega_1^2\Omega_2^{-1}\epsilon^{-1}$
2	[2; 1, 0]	1	41	5	4	$32 + 5 \cdot 41^{1/2}$	$\Omega_1^2\epsilon^{-1}$
2	[2; 3, 0]	3	73	25	8	$1068 + 125 \cdot 73^{1/2}$	$\Omega_1^2\epsilon^{-1}$

Interrelations of units Ω_1, Ω_2, ϵ

For the field $k_4 = Q(\mu^{1/2})$, $\mu = (A + ad^{1/2})/2$, the Dirichlet rank r and the quadratic fundamental unit (for $k_2 = Q(d^{1/2})$) are shown (see [6]). Since μ is not assumed square-free, $D = N_{2/1}$ disc k_4/k_2 is also shown.

6. Concluding Remarks. Some insight into the different nature of the units Ω_1 and Ω_2 is given by the norm operation

(6.1) $$N_{4/2}\Omega_i = \Omega_i S\Omega_i = \epsilon_i \qquad (i = 1, 2),$$

where ϵ_i is a unit in k_2. By using relations (2.11)–(2.14) we find $|\Omega_2 S\Omega_2| = 1$, but more remarkably

(6.2) $$\Omega_1 S\Omega_1 = \pm\, \gamma_1^{T-1}/\gamma_{-1} \qquad (= \epsilon_1).$$

Units like ϵ_1 have been considered by Yamamoto [10], who showed that for special infinitudes of cases, ϵ_1 is fundamental. Such units satisfy $\log \epsilon \approx (\log d)^2$, so these are not "small" units. It is unfortunately *not* true that for the cases of Table I,

ϵ_1 is fundamental (say) for the ring-discriminant d. For example, $\epsilon_1 = \epsilon^2$ for the case [2; 3, 1] right in Table II, as well as other cases where $r = 3$. Actually, the unit ϵ_1 is a special case of the more general type

$$\left[\frac{A + (A^2 \pm 8)^{1/2}}{2}\right]^{u+1} \bigg/ \left[\frac{A + 2t + (-1)^t(A^2 \pm 8)^{1/2}}{2}\right], \tag{6.3}$$

where $A > 0$ is odd, $(A^2 \pm 8 \neq 9)$, and $|(A + 2t)^2 - A^2 \mp 8| = 2^{u+1}$.

Several other units were used in numerical experiments. In a few cases $|f(\pm 3)|$ (one sign) turned out to be the exact power of $2^{T \pm 3}$. This yields a "bonus" unit

$$\Omega_{\pm 3} = (\xi - 1)^{T \pm 3}/(\xi \mp 3). \tag{6.4}$$

The highest d for which this occurred was 41 (i.e., [2; 1, 0], the example (1.1)). Only in this case do both signs of $|f(\pm 3)|$ give powers of two. Another type of unit which occurred naturally was

$$\Omega_{22} = (\xi + 2)/(\xi - 2) \tag{6.5}$$

for $e = -4$, (where $f(2) = f(-2)$ by (3.4)). Here, however, there is seen to be a further requirement that $(\xi - 2)\,|\,4$, which excludes the case [−4; E] only. Numerical calculations involving the dependence of such further units are excluded for brevity.

The more challenging problem is to find *fundamental* units, (i.e., r generators of the unit group, ignoring sign). This shall be left to later experimentation. We might, nevertheless, remark on the relative scarcity (see [3], [11]) of nonabelian fields for which independent units are readily available in parametric form.

City College of New York
138 Street and Convent Avenue
New York, New York 10031

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