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Zeros of Hurwitz Zeta Functions

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Abstract. All complex zeros of each Hurwitz zeta function are shown to lie in a vertical strip. Trivial real zeros analogous to those for the Riemann zeta function are found. Zeros of two particular Hurwitz zeta functions are calculated.

1. Introduction. The Hurwitz zeta function is defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} (a + n)^{-s}, \qquad 0 < a \leqslant 1, (\operatorname{Re} s > 1). \tag{1}$$

Let $s = \sigma + it$. In this paper, zero-free regions are found for $\zeta(s, a)$ and zeros are calculated for $a = 1/3, 2/3, |t| \leqslant 100, \sigma \geqslant -5$.

2. Zero-Free Region on the Right.

THEOREM 1. *If* $\sigma \geqslant 1 + a$, *then* $\zeta(s, a) \neq 0$.

Proof.

$$\begin{aligned} |\zeta(s, a)| &\geqslant a^{-\sigma} - \sum_{n=1}^{\infty} (a + n)^{-\sigma} > a^{-\sigma} - (a + 1)^{-\sigma} - \int_1^{\infty} (a + x)^{-\sigma}\, dx \\ &= a^{-\sigma} - (a + 1)^{-\sigma} - (a + 1)^{1-\sigma}/(\sigma - 1), \end{aligned}$$

which is $\geqslant 0$ provided $(1 + 1/a)^{\sigma} > 1 + (a + 1)/(\sigma - 1)$. Now, for $\sigma > 1$ and $0 < a \leqslant 1$, we have easily $(1 + 1/a)^{\sigma} > 1 + \sigma/a$; so the result holds provided

$$\sigma/a \geqslant (a + 1)/(\sigma - 1) \quad \text{or} \quad \sigma(\sigma - 1) \geqslant a(a + 1).$$

Assuming $\sigma \geqslant 1 + a$, we have $\sigma - 1 \geqslant a$, and multiplying these together, we obtain the last inequality; and Theorem 1 is proved.

3. Zero-Free Region on the Left.

THEOREM 2. *If* $|t| \geqslant 1$ *and* $\sigma \leqslant -1$, *then* $\zeta(s, a) \neq 0$.

Proof. We use the functional equation. We have:

$$\begin{aligned} \zeta(1 - s, a) &= 2\Gamma(s)(2\pi)^{-s} \left\{ \cos\left(\frac{\pi}{2}\, s\right) \sum_{m=1}^{\infty} \frac{\cos 2\pi m a}{m^s} + \sin\left(\frac{\pi}{2}\, s\right) \sum_{m=1}^{\infty} \frac{\sin 2\pi m a}{m^s} \right\} \\ &= 2\Gamma(s)(2\pi)^{-s} \sum_{m=1}^{\infty} \cos\left(\frac{\pi}{2}\, s - 2\pi m a\right) m^{-s} \qquad (2) \\ &= 2\Gamma(s)(2\pi)^{-s} \cos\left(\frac{\pi}{2}\, s - 2\pi a\right) \left\{ 1 + \sum_{m=2}^{\infty} \frac{\cos\left(\frac{\pi}{2}\, s - 2\pi m a\right)}{\cos\left(\frac{\pi}{2}\, s - 2\pi a\right)} m^{-s} \right\}. \end{aligned}$$

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Now, for $|t| \geqslant 1$,

$$\left| \frac{\cos\left(\frac{\pi}{2} s - 2\pi ma\right)}{\cos\left(\frac{\pi}{2} s - 2\pi a\right)} \right| \leqslant \frac{e^{\pi t} + 1}{e^{\pi t} - 1} \leqslant \frac{e^{\pi} + 1}{e^{\pi} - 1} \leqslant 1.09,$$

using

$$|\cos(x + iy)| = (\cos^2 x + \sinh^2 y)^{1/2}.$$

Thus,

$$\left| 1 + \sum_{m=2}^{\infty} \frac{\cos\left(\frac{\pi}{2} s - 2\pi ma\right)}{\cos\left(\frac{\pi}{2} s - 2\pi a\right)} m^{-s} \right| \geqslant 1 - 1.09(\zeta(\sigma) - 1) > 0$$

for $\sigma \geqslant 2$ as $\zeta(2) < 1.645$. Since $\Gamma(s)$, $(2\pi)^{-s}$ and $\cos((\pi/2)s - 2\pi a)$ are all nonzero in $|t| \geqslant 1$, $\sigma \geqslant 2$, Theorem 2 is proved.

4. The Trivial Zeros.

THEOREM 3. *If* $\sigma \leqslant -(4a + 1 + 2[1 - 2a])$ *and* $|t| \leqslant 1$, *then* $\zeta(s, a) \neq 0$ *except for trivial zeros on the negative real axis, one in each interval* $-2n - 4a \pm 1$, $n \geqslant 1 - 2a$.

Proof. We will apply Rouché's theorem. Using (2) and

$$u(x) = 2\Gamma(s)(2\pi)^{-s} \cos\left(\frac{\pi}{2} s - 2\pi a\right), \qquad v(x) = u(x) \sum_{m=2}^{\infty} \frac{\cos\left(\frac{\pi}{2} s - 2\pi ma\right)}{\cos\left(\frac{\pi}{2} s - 2\pi a\right)} \frac{1}{m^s},$$

and the rectangle with the four vertices $2n + 1 + 4a \pm 1 \pm i$, and taking $2n + 4a \geqslant 2$, we will have that $\zeta(1 - s, a)$ and $u(x)$ will have the same number of zeros in the rectangle provided $|u(x)| > |v(x)|$ on the boundary. This will be so provided

$$\left| \sum_{m=2}^{\infty} \frac{\cos\left(\frac{\pi}{2} s - 2\pi ma\right)}{\cos\left(\frac{\pi}{2} s - 2\pi a\right)} \frac{1}{m^s} \right| < 1$$

on the boundary. This is certainly the case along the top and bottom, as was shown in Section 3. On the vertical edges we have

$$\left| \frac{\cos\left(\frac{\pi}{2}(2n + 4a + it) - 2\pi ma\right)}{\cos\left(\frac{\pi}{2}(2n + 4a + it) - 2\pi a\right)} \right| = \left\{ \frac{\cos^2\left(\frac{\pi}{2}(2n + 4a) - 2\pi ma\right) + \sinh^2 \frac{\pi}{2} t}{1 + \sinh^2 \frac{\pi}{2} t} \right\}^{1/2} \leqslant 1,$$

so $|v(x)| \leqslant |u(x)|(\zeta(2) - 1) < |u(x)|$ for $\sigma \geqslant 2$. By symmetry, the single zero of $\zeta(s, a)$ in this rectangle must lie on the real axis. This completes the proof of

TABLE I

Zeros of $\zeta(s, 1/3)$

Re	Im	Re	Im
-3.356739	0.0	.382710	59.883303
-1.411510	0.0	.297271	61.558057
.431293	0.0	.087972	63.133469
- .159430	7.184833	.457258	65.195474
.342658	11.431373	.563711	66.783468
.241817	15.189346	.483299	69.515697
- .036837	17.768793	-.067285	70.180061
.591803	20.690440	.401279	72.270589
.193280	23.897873	.328136	74.292420
.127972	25.706324	.520017	75.643415
.334406	28.524914	.330148	77.920206
.429111	30.646264	.495597	79.533738
.462075	33.643477	-.147097	80.830920
- .147835	35.008686	.579068	82.764724
.506383	37.571524	.533603	84.515894
.472657	39.696042	.332095	86.302724
.151364	42.257863	.185455	88.479212
.343298	43.633735	.350208	88.904925
.093732	46.080690	.293211	91.542684
.612256	47.737933	.414854	92.638777
.442315	50.224064	.542324	94.468657
.159285	52.406133	.478417	96.639483
.140729	53.307053	-.016600	97.910995
.580453	56.035147	.316376	99.026723
.322000	57.568636		

Theorem 3. For each a, a rectangle contained in $|t| \leqslant 1, -3 \leqslant \sigma \leqslant -1$ has to be investigated individually.

5. **Zeros of** $\zeta(s, 1/3)$, $\zeta(s, 2/3)$. The zeros were roughly located by tabulating the functions in the strip $-1 \leqslant \sigma \leqslant 2, 0 \leqslant t \leqslant 100$. Then the regula falsi method was applied in the neighborhood of each zero. A final check was made by calculating the change of argument around the boundary of the region. The Euler-Maclaurin formula was used throughout. Table I gives the zeros of $\zeta(s, 1/3)$ and Table II the zeros of $\zeta(s, 2/3)$. There may be an occasional rounding error in the figures given.

Defining $N(a, T)$ to be the number of zeros of $\zeta(s, a)$ in $0 < t \leqslant T$, one should be able to prove, using the method of Berndt [1] that

$$N(a, T) = \frac{T}{2\pi} \log T - \left(\frac{1 + \log(2\pi a)}{2\pi}\right) T + O(\log T),$$

TABLE II

Zeros of $\zeta(s, 2/3)$

Re	Im	Re	Im
-4.582225	0.0	.658788	60.192874
-2.629836	0.0	.136371	62.718934
- .534265	0.0	.694397	65.153529
.166871	10.821929	.145692	66.578130
.570050	16.605888	.510215	69.521528
.002611	20.525222	.799305	71.819557
.850931	24.340409	-.085295	73.824766
- .113795	28.078257	.459084	75.622482
.721490	30.792111	.745029	78.673253
.365790	34.136686	.430076	79.806836
.460172	37.583838	.163703	82.879125
.203952	39.160036	.288050	84.291484
.597874	43.008712	.766662	86.328553
.356658	45.347383	.533808	88.742453
.127852	47.671788	.014239	91.063946
.595766	50.633212	.718618	92.638399
.684428	52.731898	.348818	94.360457
- .235421	55.856118	.359310	97.077760
.775805	57.447893	.742946	98.666525

which for $a = 1$ reduces to the usual formula for the number of zeros of $\zeta(2)$.

According to a theorem of Davenport and Heilbronn [2], $\zeta(s, 1/3)$ and $\zeta(s, 2/3)$ will have zeros with real parts >1, but none appeared in the present calculations. It would be interesting to have plots of the orbits as a varies, as well as plots of Re $\zeta(s, r/3) = 0$ and Im $\zeta(s, r/3) = 0$ for $-1 \leqslant \sigma \leqslant 2, 0 \leqslant t \leqslant 100, r = 1, 2, 3$.

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