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# A Computational Technique for Evaluating $L(1, \chi)$ and the Class Number of a Real Quadratic Field

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**Abstract.** A description is given of a method for estimating $L(1, \chi)$ to sufficient accuracy to determine the class number of a real quadratic field. This algorithm was implemented on an IBM/370-158 computer and the class number, regulator, and value of $L(1, \chi)$ were obtained for each real quadratic field $Q(\sqrt{D})$ ($D = 2, 3, \ldots, 149999$). Several tables, summarizing various results of these computations, are also presented.

**1. Introduction.** Recently Hendy [2] has calculated on a Burroughs B6700 computer the class numbers and number of genera for all the real quadratic fields $Q(\sqrt{D})$, with $10^3 \leqslant D \leqslant 10^5$ and $D$ squarefree. The method he used to do this is a modification of Ince's [3] technique of counting periods. In this paper we describe an entirely different computational procedure for determining the class number of $Q(\sqrt{D})$ via its Dirichlet function $L(1, \chi)$. This algorithm was implemented on an IBM/370-158 computer and used to determine all the class numbers in the range $2 \leqslant D \leqslant 1.5 \times 10^5$.

Our method is based upon the formula

$$h = \sqrt{\Delta}\, L(1, \chi)/2R,$$

where $h$ is the class number, $R$ is the regulator and $\Delta$ is the discriminant of $Q(\sqrt{D})$. The Dirichlet series $L(s, \chi) = \Sigma_{n=1}^{\infty} \chi(n) n^{-s}$ has $\chi(n) = (\Delta | n)$ (the Kronecker Symbol). It should be remarked here that

$$\Delta = \begin{cases} D, & D \equiv 1 \pmod 4, \\ 4D, & D \equiv 2, 3 \pmod 4, \end{cases}$$

and $R = \log \epsilon$, where $\epsilon\ (> 1)$ is the fundamental unit of $Q(\sqrt{D})$.

In the next section we describe a means of evaluating $R$ and in the third section a method of estimating $L(1, \chi)$ to sufficient accuracy to determine the integer $h$. Finally, we give some results of our computations in Section 4.

**2. Evaluation of the Regulator.** The regulator can be evaluated by using the continued fraction algorithm. Put $Q_0 = 1, P_0 = 0, q_0 = [\sqrt{D}]$*, $A_{-1} = B_{-2} = 1$, $A_{-2} = B_{-1} = 0$, and define

$$P_{n+1} = q_n Q_n - P_n, \quad Q_{n+1} = (D - P_{n+1}^2)/Q_n, \quad q_{n+1} = [(P_{n+1} + \sqrt{D})/Q_{n+1}],$$

$$A_{n+1} = q_{n+1} A_n + A_{n-1}, \quad B_{n+1} = q_{n+1} B_n + B_{n-1}.$$

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*We use the usual notation $[\alpha]$ to indicate the greatest integer $\leqslant \alpha$.

Let $r$ be the least nonnegative integer such that $Q_{r+1} = 4$, and let $s$ be the least nonnegative integer such that $Q_{s+1} = 1$ ($s + 1$ is the period length of the continued fraction for $\sqrt{D}$). It is well known that such a value of $s$ always exists; and that if $r$ exists, then $r < s/2$. With these definitions of $r$ and $s$ and $D \neq 5$ we have

$$\epsilon = \begin{cases} (A_r + \sqrt{D}\, B_r)/2 & \text{if } r \text{ exists,} \\ A_s + \sqrt{D}\, B_s & \text{otherwise.} \end{cases}$$

A faster and more convenient way to calculate the regulator, however, can be obtained by first using the following modification of the continued fraction algorithm to evaluate the $P$'s and $Q$'s. We put $Q_{-1} = D, R_0 = 0$ and use the formulas

$$P_{n+1} = [\sqrt{D}\,] - R_n, \quad Q_{n+1} = Q_{n-1} + q_n(P_n - P_{n+1}),$$

$$q_{n+1} = [(P_{n+1} + [\sqrt{D}\,])/Q_{n+1}],$$

$$R_{n+1} = \text{remainder on dividing } P_{n+1} + [\sqrt{D}\,] \text{ by } Q_{n+1}.$$

We can then avoid calculating the $A$'s and $B$'s by using a multiplicative formulation (see Smith [5]) wherein

$$(-1)^{n+1}(A_n - \sqrt{D}\, B_n) = \left(\prod_{i=1}^{n+1} (P_i + \sqrt{D})/Q_i\right)^{-1}.$$

Since $A_r^2 - DB_r^2 = 4(-1)^{r+1}$, we get

$$(A_r + \sqrt{D}\, B_r)/2 = 2 \prod_{i=1}^{r+1} (P_i + \sqrt{D})/Q_i.$$

Now during the development of the $P$'s and $Q$'s there must be an integer $k$ such that $Q_k = Q_{k+1}$ or an integer $j$ such that $P_j = P_{j+1}$. If $k$ exists, then $s + 1 = 2k + 1$; if $j$ exists, $s + 1 = 2j$; also, $r + 1 \leqslant k, j$. Using these facts to determine $k$ or $j$, we can cut the calculation of $A_s + \sqrt{D}\, B_s$ in half by using the symmetry properties of the $P$'s and $Q$'s and the fact that $A_s^2 - DB_s^2 = (-1)^{s+1}$. We have

$$A_s + \sqrt{D}\, B_s = (P_{k+1} + \sqrt{D}) \left[\prod_{i=1}^{k} (P_i + \sqrt{D})/Q_i\right]^2 \quad \text{when } s + 1 = 2k + 1$$

or

$$A_s + \sqrt{D}\, B_s = Q_j \left[\prod_{i=1}^{j} (P_j + \sqrt{D})/Q_i\right]^2 \quad \text{when } s + 1 = 2j;$$

hence,

$$R = \log \epsilon = \begin{cases} \log 2 + \displaystyle\sum_{i=1}^{r+1} \log(P_i + \sqrt{D})/Q_i & \text{when } r \text{ exists,} \\ \log(P_{k+1} + \sqrt{D}) + 2 \displaystyle\sum_{i=1}^{k} \log(P_i + \sqrt{D})/Q_i & \text{when } r \text{ does not exist and } s + 1 = 2k + 1, \\ \log Q_j + 2 \displaystyle\sum_{i=1}^{j} \log(P_i + \sqrt{D})/Q_i & \text{when } r \text{ does not exist and } s + 1 = 2j. \end{cases}$$

**3. Estimation of $L(1, \chi)$.** In order to estimate $L(1, \chi)$ we make use of a device used by Barrucand [1]. We first note that $L(s, \chi)$ satisfies the functional equation

$$L(1-s, \chi) = A^{-s+1/2} \frac{\Gamma(s/2)}{\Gamma((1-s)/2)} L(s, \chi),$$

where $A = \pi/\Delta$. Putting $A = \alpha\beta$, we get

$$\alpha^{-s+1/2}\Gamma(s/2)L(s, \chi) = \beta^{s-1/2}\Gamma((1-s)/2)L(1-s, \chi).$$

If we let

$$B(x) = \sum_{n=1}^{\infty} \chi(n)e^{-n^2x}$$

and use the method of [1], we obtain the functional equation

$$\sqrt{\alpha}\, B(\alpha^2 x) = \sqrt{\beta}\, x^{-1/2} B(\beta^2/x).$$

Hence,

$$\alpha^{-s}L(s, \chi)\Gamma(s/2) = \int_0^{\infty} B(\alpha^2 x)x^{s/2-1}\, dx$$

$$= \int_1^{\infty} B(\alpha^2 x)x^{s/2-1}\, dx + \sqrt{\beta/\alpha} \int_0^1 B(\beta^2/x)x^{(s-3)/2}\, dx$$

$$= \sum_{n=1}^{\infty} \chi(n) \int_1^{\infty} e^{-\alpha^2 n^2 x} x^{s/2-1}\, dx + \sqrt{\beta/\alpha} \sum_{n=1}^{\infty} \chi(n) \int_0^1 x^{(s-3)/2} e^{-\beta^2 n^2/x}\, dx.$$

If we put $\alpha = \beta = \sqrt{A}$, $s = 1$,

$$E(x) = \int_x^{\infty} e^{-t}/t\, dt, \qquad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt,$$

we get

$$L(1, \chi) = \frac{1}{\sqrt{\Delta}} \sum_{n=1}^{\infty} \chi(n)E(An^2) + \sum_{n=1}^{\infty} (\chi(n)/n) \operatorname{erfc}(n\sqrt{A}).$$

Let

$$C(m) = \frac{1}{\sqrt{\Delta}} \sum_{n=1}^{m} \chi(n)E(An^2) + \sum_{n=1}^{m} (\chi(n)/n) \operatorname{erfc}(n\sqrt{A}),$$

$$T(m) = \frac{1}{\sqrt{\Delta}} \sum_{n=m+1}^{\infty} \chi(n)E(An^2) + \sum_{n=m+1}^{\infty} (\chi(n)/n) \operatorname{erfc}(n\sqrt{A}).$$

Since we wish to approximate $L(1, \chi)$ by the partial sum $C(m)$, we want to be able to determine $m$ such that the remainder $T(m)$ is small enough that the integer $h$ can be determined unequivocally. Now if $x > 0$,

$$0 < \operatorname{erfc}(x) < \frac{1}{x\sqrt{\pi}} e^{-x^2} \quad \text{and} \quad 0 < E(x) < e^{-x}/x;$$

thus, since $|\chi(n)| \leqslant 1$, we get

$$|T(m)| < \frac{2\sqrt{\Delta}}{\pi} \sum_{n=m+1}^{\infty} e^{-An^2}/n^2 < \frac{2\sqrt{\Delta}}{\pi} \int_m^{\infty} e^{-At^2}/t^2\, dt < \frac{\Delta^{3/2}}{\pi^2} e^{-Am^2}/m^3.$$

If we let $h^* = \sqrt{\Delta}\, C(m)/2R$, then

$$|h - h^*| < \frac{\sqrt{\Delta}}{2R}\,|T(m)| < \frac{A^2}{2Rm^3}\,e^{-Am^2}.$$

Since $h$ is an integer and $2^{t-\lambda}|h$, where $t$ is the number of distinct prime factors of $\Delta$ and

$$\lambda = \begin{cases} 1 & \text{if all prime divisors of } D \text{ are congruent to } 1, 2 \pmod 4, \\ 2 & \text{otherwise,} \end{cases}$$

we must find $m$ such that $A^2e^{-Am^2}/m^3 < 2^{t-\lambda}R$. When this is done, $h$ is the unique integer, which is divisible by $2^{t-\lambda}$, in the interval $[h^* - 2^{t-\lambda-1}, h^* + 2^{t-\lambda-1}]$.

If we put $c = \sqrt{A}\,m$ and $l = \log(\sqrt{A}/2^{t-\lambda}R)$, we must find $c$ such that $c^2 + 3\log c > l$. If $l > 1$ and $\Delta < 10^7$, then $1 < c < 2$; thus, $c = 1 + \epsilon$, $\epsilon < 1$ and $\log c > \epsilon - \epsilon^2/2$. If $l < 1$, then $\frac{1}{2} < c < 1$ and $c = 1 - \epsilon > 1 - 2\epsilon^2 > e^{-2\epsilon}$; hence, $\log c > 2(c - 1)$. If we use these results, we see that we may put $m = [c\sqrt{\Delta}/\sqrt{\pi}] + 1$, where

$$c = \begin{cases} 6 - \sqrt{27 - 2l}, & l > 1, \\ \sqrt{15 + l} - 3, & l < 1. \end{cases}$$

For $\Delta < 6 \times 10^5$ it is rarely necessary to go beyond 500 terms in the series $C(m)$ in order to evaluate $h$.

**4. Results of the Computations.** The program which evaluated $C(m)$ and then $h$, was written, using double precision, in FORTRAN. A special subroutine to evaluate $E(x)$ in double precision was written in assembler language, and the FORTRAN function DERFC was used to evaluate $\operatorname{erfc}(x)$. In about seven hours of CPU time the computer calculated the class numbers of $Q(\sqrt{D})$ for all squarefree $D$ such that $1 < D \leqslant 1.5 \times 10^5$. These class numbers for $10^3 \leqslant D \leqslant 10^5$ probably agree entirely with those in Hendy's table since the number of $D$, between these limits, having a given $h$ agree in the two tables. Once $h$ had been calculated for $Q(\sqrt{D})$, the value of $L(1, \chi)$ was calculated more precisely by using $L(1, \chi) = 2Rh/\sqrt{\Delta}$.

A large table, listing for each of the 91189 $Q(\sqrt{D})$, the regulator, the class number, and the value of $L(1, \chi)$ has been deposited in the UMT file. In this section we present some excerpts from that table. In Table 1 we give each value of $h$ which occurs in the large table, the frequency $f(h)$ with which this $h$ occurs, and the least value of $D$ such that $h$ is the class number for $Q(\sqrt{D})$.

Denote by $R(d)$ the regulator of $Q(\sqrt{d})$ and by $L(1, \chi_\delta)$ the value of $L(1, \chi)$ when $\chi(n) = (\delta|n)$. In Table 2 we give those values of $D$ and $R(D)$ where $R(D)$ attains a new maximum:

$$R(D) > R(d) \qquad \text{for all } 2 \leqslant d < D.$$

In his examination of Littlewood's bounds on $L(1, \chi)$, Shanks [4] considered the function

$$L_{-\delta}(1) = \sum_{m=1}^{\infty} (4\delta|m)m^{-1}.$$

TABLE 1

| h | f(h) | least D | h | f(h) | least D |
|---|---|---|---|---|---|
| 1 | 20574 | 2 | 2 | 26427 | 10 |
| 3 | 2677 | 79 | 4 | 18573 | 82 |
| 5 | 943 | 401 | 6 | 3453 | 235 |
| 7 | 462 | 577 | 8 | 6898 | 226 |
| 9 | 311 | 1129 | 10 | 1237 | 1111 |
| 11 | 176 | 1297 | 12 | 2434 | 730 |
| 13 | 124 | 4759 | 14 | 563 | 1594 |
| 15 | 115 | 9871 | 16 | 1970 | 2305 |
| 17 | 62 | 7054 | 18 | 385 | 4954 |
| 19 | 48 | 15409 | 20 | 788 | 3601 |
| 21 | 43 | 7057 | 22 | 163 | 4762 |
| 23 | 20 | 23593 | 24 | 838 | 9634 |
| 25 | 30 | 24859 | 26 | 110 | 13321 |
| 27 | 20 | 8761 | 28 | 324 | 5626 |
| 29 | 16 | 49281 | 30 | 113 | 11665 |
| 31 | 4 | 97753 | 32 | 397 | 15130 |
| 33 | 11 | 55339 | 34 | 47 | 19882 |
| 35 | 8 | 25601 | 36 | 165 | 18226 |
| 37 | 7 | 24337 | 38 | 33 | 19834 |
| 39 | 6 | 41614 | 40 | 179 | 16899 |
| 41 | 1 | 55966 | 42 | 30 | 47959 |
| 43 | 3 | 14401 | 44 | 82 | 11026 |
| 45 | 7 | 32401 | 46 | 14 | 49321 |
| 47 | 1 | 78401 | 48 | 92 | 21610 |
| 49 | 1 | 70969 | 50 | 8 | 54769 |
| 51 | 1 | 69697 | 52 | 28 | 23410 |
| 53 | 1 | 69694 | 54 | 8 | 49834 |
| 55 | 1 | 106537 | 56 | 38 | 39909 |
| 57 | 2 | 41617 | 58 | 7 | 27226 |
| 60 | 18 | 78745 | 61 | 1 | 126499 |
| 62 | 3 | 68179 | 63 | 1 | 57601 |
| 64 | 23 | 71290 | 66 | 3 | 87271 |
| 68 | 12 | 53362 | 70 | 5 | 56011 |
| 72 | 11 | 45511 | 74 | 1 | 38026 |
| 76 | 7 | 93619 | 78 | 1 | 136159 |
| 80 | 3 | 94546 | 84 | 3 | 77779 |
| 86 | 2 | 110926 | 87 | 2 | 90001 |
| 88 | 3 | 56170 | 94 | 2 | 99226 |
| 96 | 4 | 50626 | 100 | 2 | 131770 |
| 108 | 1 | 140626 | 110 | 1 | 125434 |
| 116 | 1 | 116554 | | | |

Here we have

$$L_{-\delta}(1) = \begin{cases} \tfrac{1}{2}L(1, \chi_\delta), & \delta \equiv 1 \pmod 8, \\ 3L(1, \chi_\delta)/2, & \delta \equiv 5 \pmod 8, \\ L(1, \chi_\delta), & \text{otherwise.} \end{cases}$$

In Table 3 we give the values of $D$, $L_{-D}(1)$ (also the "Upper Littlewood Index" ULI [4]) such that

$$L_{-D}(1) > L_{-\delta}(1) \quad \text{for all } 2 \leqslant \delta < D.$$

This gives us an extension of Shanks' table of Hichamps [4, Table 6].

TABLE 2

| D | R(D) | D | R(D) |
|---|---|---|---|
| 2 | 0.88137359 | 9619 | 239.95274415 |
| 3 | 1.31695790 | 10399 | 255.84851576 |
| 6 | 2.29243167 | 10651 | 270.87206891 |
| 7 | 2.76865038 | 12919 | 283.24482085 |
| 11 | 2.99322285 | 13126 | 298.64260332 |
| 14 | 3.40008441 | 15031 | 303.73613093 |
| 19 | 5.82893697 | 16699 | 306.31406366 |
| 22 | 5.97634447 | 16879 | 318.45171155 |
| 31 | 8.01961269 | 17494 | 335.65693960 |
| 43 | 8.84850928 | 17614 | 336.73823980 |
| 46 | 10.79281810 | 18379 | 367.19773204 |
| 67 | 11.48949306 | 21319 | 392.01026227 |
| 94 | 15.27100210 | 23566 | 397.85610155 |
| 109 | 16.69360526 | 23599 | 400.38847076 |
| 139 | 18.85975147 | 25939 | 415.06367196 |
| 151 | 21.96346336 | 26959 | 423.40328648 |
| 199 | 24.20550214 | 27934 | 433.05457646 |
| 211 | 27.04530804 | 28414 | 447.08037149 |
| 214 | 27.96084155 | 31606 | 456.89547593 |
| 331 | 36.25638320 | 32839 | 458.83193172 |
| 379 | 37.79233938 | 32971 | 502.24984001 |
| 526 | 46.57116319 | 34654 | 508.58627196 |
| 571 | 47.33886269 | 38119 | 525.24115870 |
| 631 | 52.93846995 | 42046 | 532.11987985 |
| 739 | 53.63256141 | 42571 | 538.69114448 |
| 751 | 57.94214806 | 43726 | 550.86782494 |
| 886 | 58.00204637 | 46006 | 585.68371690 |
| 919 | 64.36292549 | 48799 | 619.57038850 |
| 991 | 68.80184250 | 53299 | 645.02269054 |
| 1291 | 69.42731847 | 55819 | 646.58791328 |
| 1366 | 77.50745983 | 56611 | 647.63115251 |
| 1699 | 77.68763324 | 58774 | 649.84304001 |
| 1726 | 91.48344937 | 60811 | 653.01948610 |
| 1999 | 91.88716165 | 61051 | 700.82741506 |
| 2011 | 100.53300453 | 67846 | 725.32530214 |
| 2311 | 110.30316856 | 72934 | 737.97224426 |
| 2326 | 111.22886783 | 76651 | 754.29325713 |
| 2566 | 114.05602902 | 78094 | 795.25078126 |
| 2671 | 119.59493590 | 78439 | 813.56346791 |
| 3019 | 127.49681351 | 82471 | 817.86184239 |
| 3259 | 132.12648968 | 84991 | 822.16136682 |
| 3691 | 137.39824137 | 85999 | 826.11841497 |
| 3931 | 147.25726673 | 87151 | 841.01248095 |
| 4174 | 153.01734303 | 90931 | 867.19521570 |
| 4846 | 162.46487523 | 98011 | 867.76246000 |
| 4951 | 166.65898164 | 100291 | 879.44151133 |
| 5119 | 172.50838882 | 102859 | 894.92275682 |
| 6211 | 174.49073086 | 104311 | 907.83373877 |
| 6379 | 175.31521179 | 106591 | 922.69477242 |
| 6406 | 188.37917309 | 111094 | 971.04549162 |
| 6451 | 196.13099779 | 122719 | 982.46753897 |
| 7606 | 215.68131176 | 132694 | 986.17025244 |
| 8254 | 221.19253648 | 133519 | 1029.90983807 |
| 8779 | 231.75791826 | 139591 | 1063.76482684 |

TABLE 3

| D | $L_{-D}(1)$ | ULI | D | $L_{-D}(1)$ | ULI |
|---|---|---|---|---|---|
| 2 | 0.62322524 | 0.4780 | 2146 | 2.31103546 | 0.5888 |
| 3 | 0.76034600 | 0.4690 | 2479 | 2.31375725 | 0.5853 |
| 6 | 0.93588131 | 0.4544 | 2599 | 2.34972306 | 0.5931 |
| 7 | 1.04645488 | 0.4881 | 3826 | 2.45090149 | 0.6074 |
| 10 | 1.15008652 | 0.4947 | 5014 | 2.45196600 | 0.6003 |
| 19 | 1.33724985 | 0.5122 | 5251 | 2.47339296 | 0.6044 |
| 31 | 1.44036496 | 0.5142 | 7459 | 2.53759015 | 0.6108 |
| 34 | 1.45715183 | 0.5140 | 8551 | 2.54931771 | 0.6102 |
| 46 | 1.59131421 | 0.5410 | 9454 | 2.57982512 | 0.6150 |
| 79 | 1.71299181 | 0.5495 | 10651 | 2.62463570 | 0.6227 |
| 106 | 1.74607012 | 0.5446 | 13666 | 2.67064271 | 0.6275 |
| 151 | 1.78736130 | 0.5404 | 18379 | 2.70856368 | 0.6293 |
| 211 | 1.86187579 | 0.5479 | 22234 | 2.76477214 | 0.6380 |
| 214 | 1.91136378 | 0.5619 | 32971 | 2.76601001 | 0.6295 |
| 274 | 1.91926263 | 0.5538 | 39274 | 2.77587028 | 0.6279 |
| 331 | 1.99283105 | 0.5673 | 45046 | 2.79115252 | 0.6285 |
| 394 | 2.06430094 | 0.5806 | 48799 | 2.80469210 | 0.6299 |
| 631 | 2.10744721 | 0.5748 | 61051 | 2.83638181 | 0.6324 |
| 751 | 2.11433901 | 0.5706 | 62386 | 2.84175734 | 0.6332 |
| 919 | 2.12313701 | 0.5662 | 74299 | 2.85281091 | 0.6321 |
| 991 | 2.18556256 | 0.5803 | 78439 | 2.90486139 | 0.6425 |
| 1054 | 2.24501069 | 0.5940 | 84319 | 2.91010275 | 0.6423 |
| 1486 | 2.26360155 | 0.5878 | 111094 | 2.91336081 | 0.6376 |
| 1654 | 2.27963311 | 0.5886 | 127906 | 2.93902278 | 0.6406 |

No further information was found to add to Shanks' table of Lochamps [4, Table 4]. There is no value of $D$ in the interval $2 \times 10^3 < D < 1.5 \times 10^5$ for which

$$L_{-D}(1) < L_{-398}(1) = 0.33494376.$$

That is, $D = 398$ is such a strong Lochamp that it cannot be beaten for $D < 1.5 \times 10^5$. In that respect, it is similar to the imaginary quadratic field with the notorious $D = -163$; see [4, Table 3].

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