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# Growth of Partial Sums of Divergent Series

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**Abstract.** Let $\Sigma f(n)$ be a divergent series of decreasing positive terms, with partial sums $s_n$, where $f$ decreases sufficiently smoothly; let $\varphi(x) = \int_1^x f(t)\,dt$ and let $\psi$ be the inverse of $\varphi$. Let $n_A$ be the smallest integer $n$ such that $s_n \geqslant A$ but $s_{n-1} < A$ ($A = 2, 3, \ldots$); let $\gamma = \lim \{\Sigma_1^n f(k) - \varphi(n)\}$ be the analog of Euler's constant; let $m = [\psi(A - \gamma)]$. Call $\omega$ a Comtet function for $\Sigma f(n)$ if $n_A = m$ when the fractional part of $\psi(A - \gamma)$ is less than $\omega(A)$ and $n_A = m + 1$ when the fractional part of $\psi(A - \gamma)$ is greater than $\omega(A)$. It has been conjectured that $\omega(A) = \frac{1}{2}$ is a Comtet function for $\Sigma 1/n$. It is shown that in general there is a Comtet function of the form

$$\omega(A) = \frac{1}{2} + \frac{1}{24}\{|f'(m)|/f(m)\}(1 + o(1)).$$

For $\Sigma 1/n$ there is a Comtet function of the form $\frac{1}{2} + 1/(24m) - \{1/(48m^2)\}(1 + o(1))$. Some numerical results are presented.

**1. Introduction.** If $\Sigma_{n=1}^\infty f(n)$ is a divergent series of positive terms that approach 0, one can measure how fast it diverges by seeing how fast the partial sums $s_n$ increase. Numerical data for representative series are given in the appendix to [4] (p. 69), but some of them are rather inaccurate. The present note grew out of an attempt to recompute this table. The results are given in the table on p. 259; they correct some of the entries in [4] and give a few more. The entries less than $10^6$ were found by direct machine evaluation of the partial sums; most of these were checked, and the other entries were obtained, by using Theorem 2 below, which is a generalization of known results for the harmonic series [2], [3]. The entries for the harmonic series (no. 4 in the table) were originally calculated by Wrench and published in [2].

A classical theorem of Maclaurin and Cauchy (see [4, p. 45]) states that if $f$ is positive and decreases to 0, then $s_n - \int_1^n f(t)\,dt$ approaches a limit. When $f(n) = 1/n$, this limit is Euler's constant $\gamma$; I use the same notation in the general case. The table includes approximations to $\gamma$ for each series.

*Notation.* $f$ is a positive decreasing function with $f(\infty) = 0$, such that, at least for $n = 1, 2, 3$, $|f^{(n)}(x)|$ decreases for large $x$ and is $O(f(x)x^{-n})$, and with $\Sigma f(n)$ divergent. We define $\varphi(x) = \int_1^x f(t)\,dt$; $\psi(y)$ is the inverse of $y = \varphi(x)$; we assume that $\psi'''$ is eventually monotonic. Let $s_n = \Sigma_{k=1}^n f(k)$ and $\gamma = \lim_{n\to\infty}(s_n - \varphi(n))$. When $A$ is a positive integer, $n_A$ denotes the smallest integer $n$ such that $s_n \geqslant A$ but $s_{n-1} < A$.

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For functions $f$ satisfying these hypotheses, the existence of $\gamma$ suggests that $\psi(A-\gamma)$ ought to be a good estimate of $n_A$.

THEOREM 1. *For sufficiently large $A$, the number $n_A$ is one of the two integers closest to $\psi(A-\gamma)$.*

Theorem 1 (with "sufficiently large" meaning "at least 2") was proved for the harmonic series by Comtet [3]; this seems to have been the first really precise result in this direction.

Because of Theorem 1, $n_A$ is either $[\psi(A-\gamma)]$ or $[\psi(A-\gamma)]+1$. Let us introduce a function $\omega$ such that the first case occurs when the fractional part of $\psi(A-\gamma)$ is less than $\omega(A)$; the second, when the fractional part of $\psi(A-\gamma)$ is greater than $\omega(A)$. Of course, $\omega$ is not uniquely determined. I propose to call such a function a Comtet function for $f$ (or for $\Sigma f(n)$).

It has been conjectured that $\omega(A) = \frac{1}{2}$ is a Comtet function for the harmonic series, and proved [2] that this series has a Comtet function of the form $\omega(A) = \frac{1}{2} + O(e^{-A})$.

THEOREM 2. *Every series of the form $\Sigma f(n)$ (with the hypotheses stated above) has a Comtet function of the form*

$$\omega(A) = \frac{1}{2} + \frac{1}{24}\,(|f'(m)|/f(m))(1+o(1)),$$

*where $m = [\psi(A-\gamma)]$.*

For any specific $f$ we can improve Theorem 2 by more detailed calculation. We shall do this for the harmonic series.

THEOREM 3. *For $\Sigma 1/n$ there is a Comtet function of the form $\frac{1}{2} + 1/(24m) - (1/(48m^2))(1+o(1))$. For $A \geqslant 2$ there is a Comtet function between $\frac{1}{2} + 1/(24m) - 1/(49m^2)$ and $\frac{1}{2} + 1/(24m) - 1/(47m^2)$.*

For larger values of $A$ the coefficients of $m^{-2}$ can be taken much closer together.

Theorem 3 does not disprove the conjecture that $\omega(A) = \frac{1}{2}$ is a Comtet function for the harmonic series, but it does seem to make it less plausible. It is conceivable that the fractional part of $e^{A-\gamma}$ never falls between $\frac{1}{2}$ and $\frac{1}{2} + 1/(24m) - 1/(48m^2)$. A machine computation for $A = 20(1)200$ found no exceptions; in fact, the cruder Comtet function found in [2] was more than adequate to determine $n_A$ for $A \leqslant 200$. The values of $n_A$ for $A = 1(1)20$ are given in [2] and reproduced in [9], sequence 1385; $n_{21}$ and $n_{22}$, calculated by H. P. Robinson, are given in a supplement to [9]. After the present paper had been submitted for publication, Robert Spira communicated to me the results of his computations in which he obtained $n_A$ for $A = 100(100)1000$, and also showed that there are no exceptions to the conjecture for $A \leqslant 1000$. Since $1/(24m)$ is about $2 \times 10^{-436}$ at this point, any exception to the conjecture will have the fractional part of $e^{A-\gamma}$ closer to $\frac{1}{2}$ than this, so that it seems unlikely that the conjecture will be disproved by computation.

For the series $\Sigma n^{-1/2}$, the corresponding conjecture is that $n_A$ is the closest integer to $(A-\gamma+2)^2/4$, where now $\gamma = 0.53964\ 54911\ 9 = 2 + \zeta(\frac{1}{2})$ (as pointed out to me by John W. Wrench, Jr., who also provided me with the decimal approximation). I found no exceptions for $A = 2(1)1000$.

(1) $\sum_1^\infty \frac{1}{\log\log(n+2)}$ (2) $\sum_1^\infty \frac{1}{\log(n+1)}$ (3) $\sum_1^\infty \frac{1}{n^{1/2}}$

(4) $\sum_1^\infty \frac{1}{n}$ (5) $\sum_1^\infty \frac{1}{(n+1)\log(n+1)}$ (6) $\sum_1^\infty \frac{1}{(n+2)\log(n+2)\log\log(n+2)}$

| | | Number of terms to make the sum greater than | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Series | $\gamma$ | 3 | 4 | 5 | 6 | 7 | 10 | 20 | 100 | 1000 | 1000000 |
| 1 | 7.21848 | 1 | 1 | 1 | 1 | 1 | 1 | 6 | 112 | 1812 | $2.62 \times 10^6$ (a) |
| 2(b) | 0.80193 | 3 | 5 | 7 | 9 | 12 | 20 | 56 | 489 | 7764 | $1.55 \times 10^7$ |
| 3 | 0.53964549 | 5 | 7 | 10 | 14 | 18 | 33 | 115 | 2574 | 250731 | $2.50 \times 10^{11}$ (c) |
| 4 | 0.57721566 | 11 | 31 | 83 | 227 | 616 | 12367 | $2.7 \times 10^8$ | $1.5 \times 10^{43}$ | $1.1 \times 10^{434}$ | $T(4.3 \times 10^5)$ |
| 5 | 0.42816572 | 8717 | $5.1 \times 10^{10}$ | $1.3 \times 10^{29}$ | $1.4 \times 10^{79}$ | $1.4 \times 10^{215}$ | $1.6 \times 10^{4321}$ | $T_2(8)$ | $T(5 \times 10^{42})$ | $T(4 \times 10^{433})$ | $T_2(4.3 \times 10^5)$ |
| 6 | 2.29992697 | 1 | 3 | 56 | $3.1 \times 10^{19}$ | $T(1.3 \times 10^4)$ | $T(7 \times 10^{89})$ | $T_2(2 \times 10^6)$ | $T_2(1.1 \times 10^{41})$ | $T_2(8 \times 10^{431})$ | $T_3(4.3 \times 10^5)$ |

*Notes*: To simplify the typography, I write $T(x) = T_1(x) = 10^x$, $T_n(x) = T(T_{n-1}(x))$.

(a) The function $\varphi$ for series 1 has apparently not been tabulated before; I tabulated it in order to get $\gamma$ and $\psi(A - \gamma)$. The value $2.6 \times 10^6$ given in [4] corresponding to $A = 10^6$ was probably arrived at by arguing that $\varphi(x)$ is nearly $x/\log\log x$, so $\psi(x)$ is nearly $x \log\log x$.

(b) Here $\varphi(x)$ was sufficiently well tabulated [5], [7], [8].

(c) It is easy to find this entry exactly.

I am indebted to Dr. Wrench for the 150D value of $e^{-\gamma}$ which made the computations for the harmonic series possible. I am also indebted to Lester M. Carlyle, Jr., for communicating the results of his calculations which suggested the possibility of a result like Theorem 3.

I take this opportunity to note the following errata to [2]: In Theorem 1, last line, read $m$ for $n$ (twice). On p. 866, in the line before formula (1), read $-\frac{1}{8}n^{-2}$. On p. 868, lines 9 and 10 (statements (ii) and (iii)) read $m$ for $n$. On p. 865, first line, read "for $A = 5, 10, 100$ his values are somewhat inaccurate."

2. **Proof of Theorems 1 and 2.** By the Euler-Maclaurin formula we can write

$$s_n = \gamma + \varphi(n) + \frac{1}{2}f(n) + \frac{1}{12}f'(n) + R_n, \tag{2.1}$$

where

$$R_n = -\int_n^\infty f'''(t)P_3(t)\,dt,$$

and $P_3$ is the function of period 1 that is equal on (0, 1) to the Bernoulli polynomial $B_3(x)/6$. (Notation for the $B$'s as in [6] or [1].) We can estimate $R_n$ as in [6, pp. 538–539]; it turns out that

$$0 < R_n < \frac{1}{720}|f'''(n)| = O(f(n)/n^3). \tag{2.2}$$

Suppose now that $n$ is any integer such that $s_n \geqslant A$. Put $\delta_n = \frac{1}{2}f(n) + f'(n)/12 + R_n$; then from (2.1) we have $\varphi(n) + \delta_n > A - \gamma$, whence

$$\psi\{\varphi(n) + \delta_n\} > \psi(A - \gamma). \tag{2.3}$$

We have $\varphi(n) \longrightarrow \infty$ and $\delta_n \longrightarrow 0$, so that it is reasonable to expand the left-hand side of (2.3) in a Taylor series with remainder of order 3,

$$\psi\{\varphi(n) + \delta_n\} = \psi(\varphi(n)) + \delta_n\psi'(\varphi(n)) + \tfrac{1}{2}\delta_n^2\psi''(\varphi(n)) + E_n, \tag{2.4}$$

where we may assume that

$$|E_n| \leqslant \frac{1}{6}\delta_n^3 \max\{|\psi'''(\varphi(n))|, |\psi'''(\varphi(n+1))|\}, \tag{2.5}$$

when $n$ is large enough (since we assumed that $|\psi'''|$ is monotonic). But $\psi(\varphi(n)) = n$, $\psi'(\varphi(n)) = 1/f(n)$, $\psi''(\varphi(n)) = -f'(n)/f(n)^3 = O(n^{-1}f(n)^{-2})$, and

$$\psi'''(\varphi(n)) = \{3f'(n)^2 - f(n)f''(n)\}/f(n)^5 = O(n^{-2}f(n)^{-3})$$

(and similarly for $\psi'''(\varphi(n+1))$). Hence, (2.4) becomes

$$\psi\{\varphi(n) + \delta_n\} = n + \delta_n/f(n) - \tfrac{1}{2}\delta_n^2 f'(n)/f(n)^3 + E_n, \tag{2.6}$$

where $E_n = O(n^{-2})$.

Now write $\delta_n = \frac{1}{2}f(n) + f'(n)/12 + R_n$ and multiply out $\delta_n^2$ in (2.6). We get

$$\psi\{\varphi(n) + \delta_n\} = n + \frac{1}{2} - \frac{1}{24}f'(n)/f(n) + O(n^{-2}), \tag{2.7}$$

where the $O$-term can be calculated more precisely in any particular case. Thus, if $n$ is large enough, we can combine (2.3) and (2.7) to get

$$n + \frac{1}{2} - \frac{1}{24} f'(n)/f(n) + O(n^{-2}) > \psi(A - \gamma), \qquad n > \psi(A - \gamma) - \frac{1}{2} + O(n^{-1}).$$

Consequently, if $m = [\psi(A - \gamma)]$ and $A$ is large enough, we have $n > m - ½$. Since $n$ is an integer, this means that $n \geqslant m$. Now it was assumed that $s_n \geqslant A$; in particular, $n$ can be $n_A$, the smallest such index, and we conclude that $n_A \geqslant m$.

Similarly, if $n = n_A - 1$, we have $s_n < A$, and so

$$n_A - 1 < \psi(A - \gamma) - \frac{1}{2} + O(n^{-1}), \qquad n_A < m + \frac{3}{2} + O(n^{-1}),$$

whence $n_A \leqslant m + 1$.

Consequently, we have shown that $m = [\psi(A - \gamma)] \leqslant n_A \leqslant m + 1$ for large $A$, and this is the conclusion of Theorem 1.

To go further, suppose that

$$\psi(A - \gamma) > m + \frac{1}{2} + \left(\frac{1}{24} + \epsilon\right) |f'(m)|/f(m), \qquad \epsilon > 0. \tag{2.8}$$

By definition, $s_n \geqslant A$ for $n = n_A$ and hence by (2.7), (2.3) and (2.8)

$$n_A + \frac{1}{2} + \frac{1}{24} |f'(n_A)|/f(n_A) + O(n_A^{-2}) > m + \frac{1}{2} + \left(\frac{1}{24} + \epsilon\right)|f'(m)|/f(m).$$

Thus,

$$n_A > m + \frac{1}{24}\left\{\frac{|f'(m)|}{f(m)} - \frac{|f'(n_A)|}{f(n_A)}\right\} + \epsilon \frac{|f'(m)|}{f(m)} + O(n_A^{-2}). \tag{2.9}$$

We know that $m + 1 \geqslant n_A \geqslant m$; since $|f'(x)|/f(x)$ decreases, the expression in braces is nonnegative and so $n_A > m$ if $A$ is large enough, and (2.8) holds.

Similarly, if $s_n < A$ (as it is when $n = n_A - 1$), we have

$$n + \frac{1}{2} + \frac{1}{24} \frac{|f'(n)|}{f(n)} + O(n^{-2}) < \psi(A - \gamma).$$

Supposing that

$$\psi(A - \gamma) < m + \frac{1}{2} + \left(\frac{1}{24} - \epsilon\right)|f'(m)|/f(m), \qquad \epsilon > 0, \tag{2.10}$$

we get

$$n < m + \frac{1}{24}\left\{\frac{|f'(m)|}{f(m)} - \frac{|f'(n)|}{f(n)}\right\} - \epsilon \frac{|f'(m)|}{f(m)} + O(m^{-2}).$$

Here $n = n_A - 1 < m$, so the expression in braces is not positive and consequently $n < m$, i.e., $n_A < m + 1$. Therefore, $n_A = m$ under (2.10) if $A$ is large enough.

3. **Proof of Theorem 3.** We have $\varphi(x) = \log x$, $\psi(x) = e^x$, $\gamma = 0.57721\ 56649 \ldots$ . Then (2.1) becomes

$$s_n = \gamma + \log n + \frac{1}{2n} - \frac{1}{12n^2} + R_n,$$

where

$$R_n = 6\int_n^\infty t^{-4}P_3(t)\,dt,$$

and by (2.2)

$$0 < R_n < \frac{1}{120}n^{-4}. \tag{3.1}$$

We now proceed as in Theorem 1 but take one more term in the Taylor series for $\psi(x) = e^x$. Here $\delta_n = (2n)^{-1} - (12n^2)^{-2} + R_n$ and

$$\psi\{\varphi(n) + \delta_n\} = ne^{\delta_n} = n\left(1 + \delta_n + \frac{1}{2}\delta_n^2 + \frac{1}{6}\delta_n^3 + \epsilon_n n^{-4}\right),$$

where

$$0 < \epsilon_n \leqslant \frac{1}{24}e^{\delta_n}\delta_n^4 < \frac{1}{384}e^{1/(2n)} < 0.0034 \tag{3.2}$$

if $n \geqslant 2$. Expanding the powers of $\delta_n$, we get

$$\begin{aligned}\psi\{\varphi(n) + \delta_n\} = n\Bigg\{1 &+ \frac{1}{2}n^{-1} - \frac{1}{12}n^{-2} + R_n \\ &+ \frac{1}{2}\left(\frac{1}{4}n^{-2} + \frac{1}{144}n^{-4} + R_n^2 - \frac{1}{12}n^{-3} + n^{-1}R_n - \frac{1}{6}n^{-2}R_n\right) \\ &+ \frac{1}{6}\Bigg[\frac{1}{8}n^{-3} + \frac{3}{4}n^{-2}\left(R_n - \frac{1}{12}n^{-2}\right) + \frac{3}{2}n^{-1}\left(R_n - \frac{1}{12}n^{-2}\right)^2 \\ &\qquad + \left(R_n - \frac{1}{12}n^{-2}\right)^3 + \epsilon_n n^{-4}\Bigg]\Bigg\} \\ = n &+ \frac{1}{2} + \frac{1}{24}n^{-1} - \frac{1}{48}n^{-2} + E_n,\end{aligned}$$

where

$$\begin{aligned}n^3E_n = {}&\frac{1}{6}\epsilon_n - \frac{1}{144} + \frac{1}{576}n^{-1} - \frac{1}{10368}n^{-2} \\ &+ R_n\left(n + \frac{1}{2} + \frac{1}{24}n^{-1} - \frac{1}{24}n^{-2} + \frac{1}{288}n^{-3}\right) \\ &+ R_n^2\left(\frac{1}{2}n + \frac{1}{4} - \frac{1}{24}n^{-1} + \frac{1}{6}nR_n\right).\end{aligned}$$

Since each of the expressions in parentheses is positive for $n \geqslant 2$, we get an upper bound for $n^3E_n$ by replacing $\epsilon_n$ and $R_n$ by their upper bounds from (3.1) and (3.2). The result is a decreasing function of $n$, so it is largest at $n = 2$ and we get, after some calculation, $n^3E_n < 0.005$. To get a lower bound for $n^3E_n$ we have only to replace $R_n$ and $\epsilon_n$ by 0, and then we get

$$n^3E_n > -\frac{1}{144} - \frac{1}{41472} > -0.007.$$

Using the upper bound, we obtain, for $n = n_A$,

$$n + \frac{1}{2} + \frac{1}{24}n^{-1} - \frac{1}{48}n^{-2} + 0.005n^{-3} > e^{A-\gamma}.$$

Consequently, with $m = [e^{A-\gamma}]$, if

(3.3) $$e^{A-\gamma} > m + \frac{1}{2} + \frac{1}{24} m^{-1} - \frac{1}{49} m^{-2}$$

we have

(3.4) $$n > m + \frac{1}{24}(m^{-1} - n^{-1}) + \frac{1}{48}(n^{-2} - m^{-2}) + \left(\frac{1}{48} - \frac{1}{49}\right) m^{-2} - 0.005 n^{-3}.$$

But we know that $n \geqslant m$; if we had $n = m$, (3.4) would yield

$$0 > \left(\frac{1}{48} - \frac{1}{49}\right) m^{-2} - 0.005 m^{-3}.$$

Now suppose that $A \geqslant 4$; then $m = [e^{A-\gamma}] \geqslant 30$, and so we would have

$$0 > \left(\frac{1}{48} - \frac{1}{49}\right) - (0.005)/30 > 0.000425 - 0.00016.$$

This contradiction shows that $n > m$, so that $n = n_A = m + 1$ under (3.3).

On the other hand, with $n = n_A - 1$ we have

$$n + \frac{1}{2} + \frac{1}{24} n^{-1} - \frac{1}{48} n^{-2} - 0.007 n^{-3} < e^{A-\gamma}.$$

If $n < m$, we have $n_A < m + 1$ and so $n_A = m$, so we have only to exclude the possibility that $n = m$. If we suppose that $n = m$ and

(3.5) $$e^{A-\gamma} < m + \frac{1}{2} + \frac{1}{24} m^{-1} - \frac{1}{47} m^{-2},$$

we then have

$$m + \frac{1}{2} + \frac{1}{24} m^{-1} - \frac{1}{48} m^{-2} - 0.007 m^{-3} < m + \frac{1}{2} + \frac{1}{24} m^{-1} - \frac{1}{47} m^{-2},$$

that is, $1/47 - 1/48 < 0.007 m^{-1}$. If $A \geqslant 4$, we again have $m \geqslant 30$, and the last inequality says that $0.00043 < 0.00024$. Thus, the assumption that $n_A = m + 1$ leads to a contradiction if (3.5) holds.

This establishes the second part of the theorem for $A \geqslant 4$; but it also holds, by direct computation, for $A = 2, 3$.

If we replace $1/47$ and $1/49$ by $1/48 \pm \epsilon$, we can take $\epsilon$ as small as we like if we take $A \geqslant A_0$, sufficiently large, and the first part of Theorem 3 follows.

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