Interior Maximum Norm Estimates
for Finite Element Methods

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Abstract. Interior a priori error estimates in the maximum norm are derived from interior Ritz-Galerkin equations which are common to a class of methods used in approximating solutions of second order elliptic boundary value problems. The estimates are valid for a large class of piecewise polynomial subspaces used in practice, which are defined on quasi-uniform meshes.

It is shown that the error in an interior domain Ω_1 can be estimated with the best order of accuracy that is possible locally for the subspaces used plus the error in a weaker norm over a slightly larger domain which measures the effects from outside of the domain Ω_1.

0. Introduction. Let $\mathcal{D}$ be a bounded domain in R^N, $N \geqslant 2$, with boundary $\partial\mathcal{D}$. In order to illustrate the type of results we are seeking, consider a second order elliptic boundary value problem

$$(0.1) \qquad Lu \equiv -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + d(x)u = f \quad \text{in } \mathcal{D},$$

$$(0.2) \qquad\qquad \text{Some boundary conditions on } \partial\mathcal{D}.$$

Assume that this problem has a unique solution u. For $0 < h < 1$, let $S^h(\mathcal{D})$ be a one parameter family of finite element spaces. Many methods have been proposed in the literature for finding an approximation u_h in $S^h(\mathcal{D})$ to the solution of special cases of the problem (0.1), (0.2), and we refer the reader to Bramble [2] for a survey of some of these procedures. Several of these methods differ only in the way they treat the boundary condition (0.2), but have the same interior equations. By this we mean that if we let $\Omega \subset\subset \mathcal{D}$, and let $\overset{\circ}{S}{}^h(\Omega)$ denote the functions in $S^h(\mathcal{D})$ with compact supports in Ω, then

$$(0.3) \qquad A(u - u_h, \chi) = 0 \quad \text{for all } \chi \in \overset{\circ}{S}{}^h(\Omega),$$

where

$$(0.4) \qquad A(v, w) = \int_{\Omega}\left(\sum_{i,j=1}^{N} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} + \sum_{i=1}^{N} b_i\frac{\partial v}{\partial x_i}w + dvw\right)dx.$$

In this paper we shall be concerned with deriving maximum norm estimates for $u - u_h$ on $\Omega_1 \subset\subset \Omega$, where u and u_h are only required to satisfy the interior equation (0.3). This equation disregards what happens outside of Ω. For example,

it disregards the behavior of u outside of Ω, and the particular way a method handles the boundary conditions, in fact, what these boundary conditions are.

We shall now briefly describe some recent work on error estimates for Galerkin methods and then state and discuss our main result. Typically, let $S^h(\Omega) \subseteq W^1_\infty(\Omega)$ consist of the restrictions to Ω of a given class of piecewise polynomials defined on a quasi-uniform partition which covers Ω. Let $r \geqslant 2$ denote the optimal order of h (the "size" of the partition) to which functions in $S^h(\Omega)$ can approximate functions locally in L_q norms. It was shown in Nitsche and Schatz [16] that if (0.3) holds, then for $1 \leqslant q \leqslant \infty$ and $p \geqslant 0$,

$$(0.5) \quad \|u - u_h\|_{L_2(\Omega_1)} \leqslant C(h\|u - \chi\|_{H^1(\Omega)} + \|u - u_h\|_{W_q^{-p}(\Omega)}) \quad \text{for any } \chi \in S^h(\Omega).$$

Here $\| \cdot \|_{W_q^{-p}(\Omega)}$ is the dual norm to that of $\mathring{W}_{q'}^p$, $1/q + 1/q' = 1$.

Interior maximum norm estimates, in cases where the spaces S^h are defined on uniform ("regular", "translation invariant") meshes, were given in Bramble, Nitsche and Schatz [3], Bramble and Schatz [5], Bramble and Thomée [6], and Strang and Fix [20]. Global maximum norm estimates for Dirichlet's problem were discussed in Ciarlet and Raviart [9]. Recently, quasi-optimal global estimates for general quasi-uniform meshes have been obtained. The one-dimensional case of two point boundary value problems was given in Wheeler [21] in the context of continuous, but not continuously differentiable, piecewise polynomial spaces (and then without the assumption of quasi-uniformity), and the general case occurs in Douglas, Dupont and Wahlbin [10]. In Scott [18], the Neumann problem was treated for the equation $-\Delta u + u = f$ and $N = 2$. The Dirichlet problem for $-\Delta u = f$ in the case of $N = 2$ and $r = 2$ was discussed in Natterer [14], and for N arbitrary and $r \geqslant 3$ announced in Nitsche [15]. In [15] and [18] the authors, in their respective situations, obtained the estimate

$$(0.6) \qquad \|u - u_h\|_{L_\infty(\mathcal{D})} \leqslant Ch\left(\ln\frac{1}{h}\right)^{\bar{r}} \|u - \chi\|_{W^1_\infty(\mathcal{D})} \qquad \text{for any } \chi \in S^h(\mathcal{D}),$$

where

$$\bar{r} = \begin{cases} 1 & \text{for } r = 2, \\ 0 & \text{for } r \geqslant 3. \end{cases}$$

Let us return for a moment to estimates in L_2. If $u \in L_2(\Omega)$ and $S^h(\Omega) \subseteq H^2(\Omega)$, then one can show that the estimate (0.5) may be replaced by

$$(0.7) \quad \|u - u_h\|_{L_2(\Omega_1)} \leqslant C(\|u - \chi\|_{L_2(\Omega)} + \|u - u_h\|_{W_q^{-p}(\Omega)}) \quad \text{for any } \chi \in S^h(\Omega).$$

A proof of this will be presented in Appendix 2 for coercive forms A. However, the estimate (0.7) does not hold true if $S^h(\Omega) \not\subseteq H^2(\Omega)$. For example, if the $S^h(\Omega)$ are piecewise linear functions on triangles in the plane, then the equation (0.3) does not in general make sense for $u \in L_2(\Omega)$.

In the present paper we shall show, for S^h defined on quasi-uniform partitions, that (see Theorem 5.1 for the precise assumptions)

$$
(0.8) \quad \|u - u_h\|_{L_\infty(\Omega_1)} \leq C\left(\left(\ln\frac{1}{h}\right)^{\bar{r}} \|u - \chi\|_{L_\infty(\Omega)} + \|u - u_h\|_{W_q^{-p}(\Omega)}\right)
$$

$$
\text{for any } \chi \in S^h(\Omega).
$$

Note that by partial integration over each element, the equation (0.3) makes sense for $S^h(\Omega)$ as above and u continuous on Ω. The estimate (0.8) has been previously proved in Bramble and Schatz [5] in the case that the spaces S^h are smooth splines on a uniform mesh and $r \geq 3$. Their proof relies on Fourier methods, which are not available in the general quasi-uniform case.

Comparing the estimate (0.8) to (0.6), we have reduced the local smoothness requirements on the solution u, and we have also given estimates when $r = 2$ and $N \geq 3$.

The estimate (0.8) essentially says that, except for the factor $\ln(1/h)$ when $r = 2$, the Galerkin method is "locally bounded" in L_∞. This is important since the estimate is then applicable to a large class of problems for which the solutions are not smooth in the interior. For example, if $u \in W_\infty^s(\Omega)$ for $0 < s \leq r$, then the first term on the right of (0.8) can be replaced by $Ch^s(\ln(1/h))^{\bar{r}}$; and the influences of the smoothness of u outside of Ω, and of the treatment of the boundary conditions in the approximate procedure, are contained in the second term. It is often the case that for a particular boundary value problem and approximation method an estimate of the form

$$
\|u - u_h\|_{W_2^{2-r}(D)} \leq Ch^s \|u\|_{W_2^{\max(1,2-r+s)}(D)}
$$

holds. (For a survey of some methods satisfying this inequality, see Bramble and Osborn [4]. In more special cases, cf. (0.7), the norm on the right-hand side can be replaced by the norm in $W_2^{\max(0,2-r+s)}(D)$.) In this case we have that if $u \in W_\infty^s(\Omega) \cap W_2^{\max(1,2-r+s)}(D)$ (or $u \in W_\infty^s(\Omega) \cap W_2^{\max(0,2-r+s)}(D)$), then

$$
\|u - u_h\|_{L_\infty(\Omega_1)} \leq Ch^s \left(\ln\frac{1}{h}\right)^{\bar{r}} \quad \text{for } 0 < s \leq r.
$$

In Section 6 (Theorem 6.1) we consider a model problem, namely the Neumann problem on a smooth domain, and derive pointwise estimates for the Green's function (cf. Bramble and Schatz [5] for a special case). Briefly, if $G^{(y)}(x)$ denotes the Green's function with singularity at y, $G_h^{(y)}(x)$ the usual Galerkin approximation, and if $x \in \Omega_2$, $y \in \Omega_1$ with $\Omega_1 \subset\subset \Omega_2$, then

$$
|G^{(y)}(x) - G_h^{(y)}(x)| \leq C\frac{h^r(\ln(|x-y|/h))^{\bar{r}}}{|x-y|^{N-2+r}}
$$

provided $|x - y| \geq Ch$. Except for the logarithmic factor when $r = 2$, these estimates are "locally optimal". In Theorem 6.1 we also derive an estimate for the error when $|x - y| \leq Ch$. As is well known, cf. [5], estimates of the above type can be used to derive error estimates when the right-hand side f in (0.1) is nonsmooth.

We shall now give a rough outline of the main steps in our analysis leading to the result (0.8). Assume that the form A is coercive, cf. (4.1). (The proof for non-

coercive forms follows from this case via an argument involving a compact perturbation of the elliptic operator, see Appendix 1.) Let $x_0 \in \bar{\Omega}_1$ be such that

$$\sup_{x \in \Omega_1} |(u - u_h)(x)| = |(u - u_h)(x_0)|.$$

Let $D \subset\subset \Omega$ be a sphere with center at x_0, and ω a smooth cut-off function which is 1 in a neighborhood of x_0 and has compact support in D. We set $\tilde{u} = \omega u$ and let $\tilde{u}_h \in S^h(D)$ satisfy

$$A(\tilde{u} - \tilde{u}_h, \chi) = 0 \quad \text{for } \chi \in S^h(D).$$

Here $\tilde{u}_h$ can be thought of as the approximate solution of a Neumann problem with right-hand side equal to $L\tilde{u}$. Taking $w_h = \tilde{u}_h - u_h$, we have

$$(0.9) \qquad (u - u_h)(x_0) = (\tilde{u} - \tilde{u}_h)(x_0) + w_h(x_0),$$

so that the error $u - u_h$ consists of two parts, namely:

(i) The error $\tilde{u} - \tilde{u}_h$ in a projection for a "localized" u. Lemma 5.1 will in particular imply that

$$(0.10) \qquad |(\tilde{u} - \tilde{u}_h)(x_0)| \leqslant C(\ln 1/h)^{\bar{r}} \|\tilde{u} - \chi\|_{L_\infty(D)} \quad \text{for } \chi \in S^h(D).$$

(ii) The function $w_h \in S^h(D)$ which satisfies $A(w_h, \chi) = 0$ for $\chi \in S^h(D)$ having support in a small neighborhood of x_0. In Lemma 5.2 we shall show that

$$(0.11) \qquad |w_h(x_0)| \leqslant C\|w_h\|_{W_q^{-p}(D)}.$$

The desired result now follows from (0.9), (0.10) and (0.11) since

$$\|w_h\|_{W_q^{-p}(D)} \leqslant \|\tilde{u} - \tilde{u}_h\|_{L_\infty(D)} + \|u - u_h\|_{W_q^{-p}(D)}.$$

Here the first term on the right is again estimated via Lemma 5.1.

The proofs of (0.10) and (0.11) rely on the fundamental Lemma 5.3. Briefly, Lemma 5.3 is concerned with error estimates in L_1-based norms for the Galerkin approximation v_h of functions v which satisfy an elliptic differential equation with right-hand side $\varphi \in \overset{\circ}{C}{}^\infty$ whose support is contained in a sphere D_h of radius Ch. For our purposes, any such v may be thought of, after proper normalization, as a smoothed-out Green's function. In Scott [18] maximum norm estimates were obtained by estimating the Green's function in W_1^1; that work motivated some of our considerations. In Lemma 5.3 we shall show that

$$(0.12) \qquad \|v - v_h\|_{W_1^1(D)} \leqslant Ch^{N/2+1}(\ln 1/h)^{\bar{r}} \|\varphi\|_{L_2(D_h)},$$

and for $D_I \subset\subset D$,

$$(0.13) \qquad \|v - v_h\|_{W_1^{2,h}(D_I)} \leqslant Ch^{N/2}(\ln 1/h)^{\bar{r}} \|\varphi\|_{L_2(D_h)},$$

where $W_1^{2,h}(D_I)$ denotes the piecewise norm relative to the element partition.

We shall now indicate how (0.10) follows from the estimates (0.12) and (0.13).

It suffices to prove it for $\chi = 0$. Instead of representing the error directly in terms of the Green's functions, as was done in Scott [18], we proceed as follows. Taking x_0 to be the center of D_h we have, using inverse properties, that

$$|(\widetilde{u} - \widetilde{u}_h)(x_0)| \leqslant C\{\|\widetilde{u}\|_{L_\infty(D_h)} + h^{-N/2}\|\widetilde{u} - \widetilde{u}_h\|_{L_2(D_h)}\}.$$

We then employ a duality argument in order to estimate the second term on the right-hand side:

$$\|\widetilde{u} - \widetilde{u}_h\|_{L_2(D_h)} = \sup_{\varphi \in \overset{\circ}{C}{}^\infty(D_h)} \frac{(\widetilde{u} - \widetilde{u}_h, \varphi)}{\|\varphi\|_{L_2(D_h)}}$$

$$= \sup_{\varphi \in \overset{\circ}{C}{}^\infty(D_h)} \frac{A(\widetilde{u} - \widetilde{u}_h, v)}{\|\varphi\|_{L_2(D_h)}} = \sup_{\varphi \in \overset{\circ}{C}{}^\infty(D_h)} \frac{A(\widetilde{u}, v - v_h)}{\|\varphi\|_{L_2(D_h)}}.$$

Integrating by parts over each element and taking care of the boundary terms via the trace inequality A.0 of Section 2, one deduces with D_I approximately equal to supp ω,

$$\|\widetilde{u} - \widetilde{u}_h\|_{L_2(D_h)} \leqslant \sup_{\varphi \in \overset{\circ}{C}{}^\infty(D_h)} C \frac{\|\widetilde{u}\|_{L_\infty(D)}}{\|\varphi\|_{L_2(D_h)}} (h^{-1}\|v - v_h\|_{W_1^1(D_I)} + \|v - v_h\|_{W_1^{2,h}(D_I)}).$$

The result (0.10) now follows using (0.12) and (0.13).

.The proof of (0.11) proceeds along somewhat similar lines.

To prove (0.12) (from which (0.13) follows) we start by subdividing D, which for simplicity we assume has radius 1 and center at x_0, into annuli

$$\Omega_j = \{x \mid 2^{-(j-1)} \leqslant |x - x_0| \leqslant 2^{-j}\}, \quad j = 0, \ldots, J,$$

and a sphere Ω_h of radius $C_* h = 2^{-J}$, centered at x_0, so that $D = \Omega_h \cup (\bigcup_{j=0}^J \Omega_j)$. Let $e = v - v_h$. We have

$$\|e\|_{W_1^1(D)} = \sum_{j=0}^J \|e\|_{W_1^1(\Omega_j)} + \|e\|_{W_1^1(\Omega_h)}.$$

We estimate each $\|e\|_{W_1^1(\Omega_j)}$ (and $\|e\|_{W_1^1(\Omega_h)}$) separately. Using the Cauchy-Schwarz inequality

$$\|e\|_{W_1^1(\Omega_j)} \leqslant C d_j^{N/2} \|e\|_{1,\Omega_j},$$

where $d_j = 2^{-j}$. We apply Lemma 3.2, or Lemma 4.4 in the case of $j = 0$, both of which are generalizations of the interior H^1-estimates of Nitsche and Schatz [16], to obtain

$$\|e\|_{W_1^1(\Omega_j)} \leqslant C(d_j^{N/2} h^{r-1} \|v\|_{W_2^r(\Omega_j^1)} + d_j^{-1} \|e\|_{L_1(\Omega_j^1)}),$$

where $\Omega_j^1 = (\Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}) \cap D$. We then sum and estimate the terms on the right-hand side, using in particular a "kickback" argument which involves choosing $C_* = 2^{-J}/h$ sufficiently large. The details are quite technical. We wish to point out again, however, the use in the proof of H^1-estimates which depend on distances, roughly equal to d_j, between domains Ω_j and Ω_j^1, where also d_j is approximately the

distance from $\partial\Omega_j$ to x_0. This idea is related to estimating norms on annuli which are weighted in terms of their distance to x_0. The papers Natterer [14] and Nitsche [15] use weighted norms to prove their results, and similarly Scott [18] employs weighted norms implicitly. The techniques applied in these papers have much in common with the ideas of Nitsche and Schatz [16], as do the techniques of the present paper.

In a forthcoming paper we shall adapt the local techniques developed here to analyze the error in approximating the solution of a Dirichlet problem on a plane polygonal domain. Some of these results were presented at the SYNSPADE III conference at the University of Maryland in May 1975.

An outline of this paper is as follows. In Section 1 we introduce some notation. Assumptions on the finite element spaces are listed in Section 2. In Section 3 we recall the interior H^1-estimates of [16], and extend them somewhat. Section 4 is concerned with some estimates for an auxiliary Neumann problem. Then, in Section 5, we prove the estimate (0.8) in the case of coercive forms A; the proof for noncoercive forms is given in Appendix 1. Lastly, in Section 6, we give the application to pointwise error estimates for the Green's function.

1. **Notation.** Throughout this paper, c and C will denote positive generic constants. Let Ω be a bounded domain in R^N. For $1 \leqslant q \leqslant \infty$ we denote by $L_q(\Omega)$ the usual Banach spaces with norm $\|\cdot\|_{L_q(\Omega)}$. For m a nonnegative integer, $W_q^m(\Omega)$ will be the usual Sobolev spaces with norms

$$\|v\|_{W_q^m(\Omega)} = \begin{cases} \left(\sum_{j=0}^{m} |v|^q_{W_q^j(\Omega)}\right)^{1/q} & \text{for } 1 \leqslant q < \infty, \\[2mm] \max_{j=0,\dots,m} |v|_{W_\infty^j(\Omega)} & \text{for } q = \infty. \end{cases}$$

Here $|\cdot|_{W_q^j(\Omega)}$ denotes the seminorms

$$|v|_{W_q^j(\Omega)} = \begin{cases} \left(\sum_{|\alpha|=j} \|D^\alpha v\|^q_{L_q(\Omega)}\right)^{1/q} & \text{for } 1 \leqslant q < \infty, \\[2mm] \max_{|\alpha|=j} \|D^\alpha v\|_{L_\infty(\Omega)} & \text{for } q = \infty. \end{cases}$$

We define the spaces $\overset{\circ}{W}_q^m(\Omega)$ as the completion of $\overset{\circ}{C}^\infty(\Omega)$ in the norm of $W_q^m(\Omega)$. If $q = 2$, we shall write $W_2^m(\Omega) = H^m(\Omega)$ and use the symbols $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{j,\Omega}$ for the norm and seminorm, respectively.

For m a negative integer we use the notation $\|\cdot\|_{W_q^m(\Omega)}$ for the norm dual to $\overset{\circ}{W}_q^{-m}(\Omega)$, $1/q + 1/q' = 1$, i.e.,

$$\|v\|_{W_q^m(\Omega)} = \sup_{\varphi \in \overset{\circ}{C}^\infty(\Omega)} \frac{(v, \varphi)}{\|\varphi\|_{W_q^{-m}(\Omega)}},$$

where $(v, w) = \int_\Omega vw\, dx$. For $q = 2$ we again write $\|\cdot\|_{m,\Omega}$ for $\|\cdot\|_{W_2^m(\Omega)} =$

$\|\cdot\|_{H^m(\Omega)}$. We note that if $\Omega_1 \subseteq \Omega_2$, then

(1.1)
$$\|v\|_{W_q^m(\Omega_1)} \leqslant \|v\|_{W_q^m(\Omega_2)}.$$

In the proof of Lemma 3.2 we shall also need the following result.

LEMMA 1.1. *Let $1 \leqslant q \leqslant 2$ and $p \geqslant 0$. Furthermore, let Ω_j, $j = 1, \ldots, J$, be disjoint sets and $\Omega = \bigcup_1^J \Omega_j$. Then*

$$\sum_{j=1}^J \|v\|_{W_q^{-p}(\Omega_j)}^2 \leqslant \|v\|_{W_q^{-p}(\Omega)}^2.$$

Proof. Let $\varphi_j \in C_0^\infty(\Omega_j)$, $j = 1, \ldots, J$, be such that $\|\varphi_j\|_{W_{q'}^p(\Omega_j)} = 1$, where $1/q + 1/q' = 1$. Set

$$\theta_j = \|v\|_{W_q^{-p}(\Omega_j)} \quad \text{and} \quad \varphi = \sum_{j=1}^J \theta_j^{q-1} \varphi_j;$$

note that $\varphi \in C_0^\infty(\Omega)$. We have

$$\|v\|_{W_q^{-p}(\Omega)}^q \geqslant \left(\frac{(v, \varphi)}{\|\varphi\|_{W_{q'}^p(\Omega)}} \right)^q = \frac{\left(\sum_{j=1}^J \theta_j^{q-1}(v, \varphi_j) \right)^q}{\left(\sum_{j=1}^J \theta_j^q \right)^{q-1}}.$$

Taking the supremum of the right-hand side over φ_j, we deduce that

$$\|v\|_{W_q^{-p}(\Omega)}^q \geqslant \sum_{j=1}^J \theta_j^q.$$

The lemma now follows since the norm in the sequence space l_q majorizes the norm in l_2 for $1 \leqslant q \leqslant 2$.

We shall also make use of spaces defined relative to partitions of Ω. Let $0 < h < 1$ be a parameter, and for each h let τ_i^h, $0 \leqslant i \leqslant I(h)$ be a finite number of disjoint open sets such that $\Omega \subseteq \bigcup_{i=0}^{I(h)} \bar{\tau}_i^h$. The sets $\tau_i^h \cap \Omega$ induce a partition of Ω, and relative to each such partition, we define $W_q^{m,h}(\Omega)$ $(C^{m,h}(\Omega))$ as the space consisting of those functions which belong to $W_q^m(\tau_i^h \cap \Omega)$ $(C^m(\tau_i^h \cap \Omega))$, $0 \leqslant i \leqslant I(h)$. We introduce the seminorms

$$|v|_{W_q^{m,h}(\Omega)} = \begin{cases} \left(\sum_{i=0}^{I(h)} |v|_{W_q^m(\tau_i^h \cap \Omega)}^q \right)^{1/q} & \text{for } 1 \leqslant q < \infty, \\ \max_{i=0,\ldots,I(h)} |v|_{W_\infty^m(\tau_i^h \cap \Omega)} & \text{for } q = \infty, \end{cases}$$

and the corresponding norms $\|\cdot\|_{W_q^{m,h}(\Omega)}$. Note that if $v \in W_q^m(\Omega)$, then $\|v\|_{W_q^{m,h}(\Omega)} = \|v\|_{W_q^m(\Omega)}$.

2. The Finite Element Spaces.

Let $\Omega \subset\subset R^N$ be fixed throughout this section. We shall describe a class of families of finite-dimensional spaces which have properties that are shared by many quasi-uniform finite element spaces used in practice to approximate solutions of partial differential equations.

For each $0 < h < 1$, $S^h(\Omega)$ will denote a finite-dimensional subspace of $W_\infty^1(\Omega)$ $\cap C^{2,h}(\Omega)$. Our first assumption relates to the geometry of the partitioning sets τ_i^h. We shall assume that a certain trace inequality holds on each of the τ_i^h.

A.0. There exists a constant C such that for $0 < h < 1$, and any $f \in W_1^2(\tau_i^h)$, $i = 0, \ldots, I(h)$,

$$\int_{\partial \tau_i^h} |\nabla f| \, d\sigma \leq C \left\{ \frac{1}{h} |f|_{W_1^1(\tau_i^h)} + |f|_{W_1^2(\tau_i^h)} \right\}.$$

We remark that the assumption A.0 is satisfied for a large class of partitions of Ω. For example, it holds if the τ_i^h are taken to be N-simplices or N-dimensional parallelepipeds of diameter $c_i h$, $c_i \leq C$, provided the ratio of the diameter and the radius of the largest inscribed sphere is uniformly bounded. Briefly, to verify A.0 in these cases one maps each of the τ_i^h onto a standard domain. The inequality can then be proven, with $h = 1$, using integration by parts. The desired inequality is then obtained by mapping back to τ_i^h.

For $D \subseteq \Omega$, $S^h(D)$ is defined as the restriction of $S^h(\Omega)$ to D, and

$$\overset{\circ}{S}^h(D) = \{\chi | \chi \in S^h(D), \text{ supp } \chi \subset \subset D\}.$$

Let $r \geq 2$ be a given integer. We shall assume that there exist positive constants C_1, C_2, C_3, C_4, k_0, γ, and $0 < h_0 < 1$ such that the spaces $S^h(\Omega)$ satisfy the following conditions A.1–A.4 for $0 < h \leq h_0$.

A.1. Let $s = 0, 1$, or 2 and let $D_1 \subset \subset D$ with $\text{dist}(D_1, \partial D) \geq k_0 h$ and $\text{dist}(D, \partial\Omega) \geq k_0 h$. Then for each v there exists a $\chi \in S^h(D)$ such that

$$(2.1) \quad \|v - \chi\|_{W_q^{t,h}(D_1)} \leq C_1 h^{l-t} |v|_{W_q^l(D)} \quad \text{for } 0 \leq t \leq s \leq l \leq r, \ 1 \leq q \leq \infty.$$

Furthermore, if $D_2 \subset \subset D_1$ with $\text{dist}(D_2, \partial D_1) \geq k_0 h$ and supp $v \subseteq D_2$, then $\chi \in \overset{\circ}{S}^h(D_1)$.

A.2. *Inverse Properties.* Let $p \geq -1$ be an integer and $D_1 \subset \subset D$ with $\text{dist}(D_1, \partial D) \geq k_0 h$. Then for $\chi \in S^h(D)$,

$$(2.2.a) \qquad \|\chi\|_{1,D_1} \leq C_2 h^{-(p+1)} \|\chi\|_{-p,D},$$

$$(2.2.b) \quad \|\chi\|_{W_q^{s,h}(D_1)} \leq C_2 h^{t-s-N(1/q_1 - 1/q)} \|\chi\|_{W_{q_1}^{t,h}(D)} \quad \text{for } 0 \leq t \leq s \leq 2, \ 1 \leq q_1 \leq q \leq \infty$$

A.3. Let $D_1 \subset \subset D$ with $\text{dist}(D_1, \partial D) \geq k_0 h$, and let $\omega \in \overset{\circ}{C}^\infty(D_1)$. Then for each $\chi \in S^h(D)$ there exists an $\eta \in \overset{\circ}{S}^h(D)$ satisfying

$$(2.3) \qquad \|\omega\chi - \eta\|_{1,D} \leq C_3 h \|\omega\|_{W_\infty^\gamma(D_1)} \|\chi\|_{1,D_1}.$$

Furthermore, let $D_4 \subset \subset D_3 \subset \subset D_2 \subset \subset D_1$ with $\text{dist}(D_4, \partial D_3) \geq k_0 h$ and $\text{dist}(D_3, \partial D_2) \geq k_0 h$. Then if $\omega \equiv 1$ on D_2 we have $\eta \equiv \chi$ on D_3 and

$$(2.4) \qquad \|\omega\chi - \eta\|_{1,D} \leq C_3 h \|\omega\|_{W_\infty^\gamma(D_1)} \|\chi\|_{1,D_1 \backslash D_4}.$$

We shall finally make the assumption that if a sphere or radius d in Ω is transformed by similarity to a sphere of unit size, then the transformed finite element space satisfies A.1, A.2, and A.3 with h replaced by h/d and with the constants occurring the same as before.

A.4. Let $D_d \subset\subset \Omega$ be a sphere of radius $d \geqslant C_4 h$ with center at x_0. The linear transformation $y = (x - x_0)/d$ takes D_d into a sphere $\tilde{D}$ and $S^h(D_d)$ into a new function space $S(\tilde{D})$. Then $S(\tilde{D})$ satisfies A.1, A.2, and A.3 with h replaced by h/d. Furthermore, the constants occurring in A.1, A.2, and A.3 remain unchanged, in particular, independent of d.

Examples of finite element spaces for which the above assumptions obtain are:

(i) The Lagrange and Hermite elements, cf. [8], which include, for example, the restriction to Ω of piecewise linear elements, defined on a sequence of quasi-uniform simplicial partitions.

(ii) The plane triangular elements of Bramble and Zlámal [7].

(iii) The restriction to Ω of tensor products of one-dimensional piecewise polynomials, cf. [3].

In these examples the properties A.1–A.3 are well known, cf. [11] and [19] in the case of A.1, and [3] and [16] for A.3. That A.4 holds with respect to A.1 and A.2 follows via a scaling argument. For A.3, on the transformed spaces $S(\tilde{D})$, one has to consider the original proofs. In the above examples, these proofs give (2.3) and (2.4) with h replaced by h/d.

We shall now state two easy consequences of our assumptions.

PROPOSITION 2.1. *Assume A.1, and let $R_0 > 0$ be fixed. There exists a constant C such that the following holds:*

Let $D \subset\subset \Omega$ be a sphere of radius $R \geqslant R_0$ with $\mathrm{dist}(D, \partial\Omega) \geqslant 2k_0 h$. Let $s = 0, 1$ or 2. Then for each v there exists a $\chi \in S^h(D)$ such that

$$\| v - \chi \|_{W_q^{t,h}(D)} \leqslant Ch^{l-t} \| v \|_{W_q^l(D)} \quad \textit{for } 0 \leqslant t \leqslant s \leqslant l \leqslant r, \ 1 \leqslant q \leqslant \infty.$$

Proof. The function v may be extended continuously in all W_q^l norms to a sphere D_1 of radius $2R$ such that

$$\| v \|_{W_q^l(D_1)} \leqslant C(R_0) \| v \|_{W_q^l(D)}.$$

The result now follows from A.1.

We shall need a way of cutting down functions in the finite element space to have compact support.

PROPOSITION 2.2. *Let A.3 hold and let $D_3 \subset\subset D_2 \subset\subset D_1 \subset\subset D \subset\subset \Omega$. There exists a constant C such that if h is sufficiently small the following holds:*

For each $\chi \in S^h(D)$ there exists an $\eta \in \overset{\circ}{S}{}^h(D_1)$ with $\eta \equiv \chi$ on D_2 and

(2.5) $$\| \chi - \eta \|_{1,D} \leqslant C \| \chi \|_{1, D \backslash D_3}.$$

Proof. Let $\omega \in \overset{\circ}{C}{}^\infty(D_4)$ and $\omega \equiv 1$ on D_5, where $D_2 \subset\subset D_5 \subset\subset D_4 \subset\subset D_1$. By A.3 we can find $\eta \in \overset{\circ}{S}{}^h(D_1)$ with $\eta \equiv \chi$ on D_2 such that

$$\|\omega\chi - \eta\|_{1,D} \leqslant Ch\|\chi\|_{1,D\backslash D_3}.$$

Since $\|(1 - \omega)\chi\|_{1,D} \leqslant C\|\chi\|_{1,D\backslash D_3}$, we obtain the desired result by the triangle inequality.

In what follows we shall set $k = 2k_0$.

3. **Interior Estimates in H^1.** Consider the bilinear form A of (0.4) where, for simplicity, a_{ij}, b_i and d belong to $C^\infty(\bar{\Omega})$ and $a_{ij} = a_{ji}$. Assume that A is uniformly elliptic on $\bar{\Omega}$, i.e., there exists a positive constant C_{ell} such that for all real vectors $\xi = (\xi_1, \ldots \xi_N)$ and $x \in \bar{\Omega}$,

$$(3.1) \qquad \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant C_{\text{ell}} \sum_{i=1}^{N} \xi_i^2.$$

Let $u_h \in S^h(\Omega)$ be such that

$$(3.2) \qquad A(u - u_h, \chi) = 0 \quad \text{for all } \chi \in \overset{\circ}{S}{}^h(\Omega).$$

Interior estimates for $u - u_h$ in H^s norms were given in [16]. Our investigations will rely on the following special case of a result from that paper.

LEMMA 3.1. *Assume that (3.1), A.1 (for $q = 2$), (2.2a) of A.2 and (2.3) of A.3 hold. Let p be a nonnegative integer and $D_1 \subset\subset D \subset\subset \Omega$. There exist constants $h_1 = h_1(p, \text{dist}(D_1, \partial D))$ and C such that the following holds:*

Let u and $u_h \in S^h(\Omega)$ satisfy (3.2). Then for $0 < h \leqslant h_1$, $1 \leqslant q \leqslant \infty$, $s = 0$ or 1, and $1 \leqslant l \leqslant r$,

$$(3.3) \qquad \|u - u_h\|_{s,D_1} \leqslant C\{h^{l-s}|u|_{l,D} + \|u - u_h\|_{W_q^{-p}(D)}\}.$$

For D_1 and D concentric spheres of radii R_1 and R, the constant C depends on p, and monotonically on C_{ell}^{-1}, $(R - R_1)^{-1}$, $\|a_{ij}\|_{W_\infty^\alpha(\Omega)}$, $\|b_i\|_{W_\infty^\beta(\Omega)}$ and $\|d\|_{W_\infty^\gamma(\Omega)}$ for some fixed positive integers α, β, γ.

Lemma 3.1 was proved in [16] for $q = 2$, but follows for all q by use of Sobolev's lemma.

In particular, if u_h satisfies $A(u_h, \chi) = 0$ for all $\chi \in \overset{\circ}{S}{}^h(\Omega)$, then

$$(3.4) \qquad \|u_h\|_{s,D_1} \leqslant C\|u_h\|_{W_q^{-p}(D)}.$$

In Lemma 3.1 the domains D and D_1 were arbitrary but fixed. We shall need the precise behavior of the constants in (3.3) in the case when D and D_1 may vary, and are possibly close, with h. We shall find this behavior by a scaling argument, using A.4, and applying Lemma 3.1 on domains of unit size.

LEMMA 3.2. *Let (3.1), A.1, A.2, A.3 and A.4 hold. Let p be a nonnegative integer. There exist constants C and C_5 such that the following holds:*

Let $D_1 \subset\subset D \subset\subset \Omega$ with $\text{dist}(D, \partial\Omega) \geqslant kh$ and $\text{dist}(D_1, \partial D) = d \geqslant C_5 h$. If u and $u_h \in S^h(\Omega)$ satisfy (3.2), then for $1 \leqslant q \leqslant 2$,

$$(3.5) \quad \|u - u_h\|_{1,D_1} \leqslant C\{h^{r-1}|u|_{r,D} + d^{N(1/2-1/q)-(p+1)}\|u - u_h\|_{W_q^{-p}(D)}\}.$$

Proof. We first note that if d is of unit size, then (3.5) is a special case of (3.3). By (1.1), Lemma 1.1 and a covering argument it suffices to prove (3.5) with D_1 and D taken as spheres of radii $d/2$ and d, respectively, with centers at x_0. Assume $x_0 = 0$ and let x denote the variable on D. Let $y = x/d$ be the new variable on the transformed regions $\tilde{D}_1$ and $\tilde{D}$; note that $\text{dist}(\tilde{D}_1, \partial\tilde{D}) = \frac{1}{2}$. Set

$$e(x) = u(x) - u_h(x)$$

and

$$E(y) = U(y) - U_h(y) \equiv u(yd) - u_h(yd).$$

We have

$$\|e\|_{1,D_1}^2 = d^N(d^{-2}|E|_{1,\tilde{D}_1}^2 + |E|_{0,\tilde{D}_1}^2)$$

so that if $d \leqslant 1$, as may be assumed,

(3.6) $$\|e\|_{1,D_1} \leqslant d^{N/2-1}\|E\|_{1,\tilde{D}_1}.$$

Next note that E satisfies the relation

(3.7) $$A_d(E, \tilde{\chi}) = 0$$

for $\tilde{\chi}$ in the transformed space, where with δ_i denoting differentiation with respect to y_i,

$$A_d(v, w) \equiv \int_{\tilde{D}} \left(\sum_{i,j=1}^{N} (a_{ij}(yd)\delta_i v(y)\delta_j w(y)) \right.$$

$$\left. + \left(d \sum_{i=1}^{N} (b_i(yd)\delta_i v(y)) + d^2 d(yd)v(y) \right) w(y) \right) dy.$$

This form is uniformly elliptic with the same modulus of ellipticity C_{ell} as for the original form A, and the norms of derivatives of the coefficients are decreased compared to those of A. From (3.7), A.4 and Lemma 3.1 we deduce that for $h/d \leqslant h_1 = 1/C_5$,

(3.8) $$\|E\|_{1,\tilde{D}_1} \leqslant C\left(\left(\frac{h}{d}\right)^{r-1} |U|_{r,\tilde{D}} + \|E\|_{W_q^{-p}(\tilde{D})} \right).$$

One easily finds that

(3.9) $$|U|_{j,\tilde{D}} \leqslant d^{-N/2+j}|u|_{j,D};$$

and using the definition of the W_q^{-p} norm, we have for $d \leqslant 1$,

(3.10) $$\|E\|_{W_q^{-p}(\tilde{D})} \leqslant d^{-N/q-p}\|e\|_{W_q^{-p}(D)}.$$

From (3.6), (3.8), (3.9), and (3.10) we obtain the inequality (3.5). This proves the lemma.

4. An Auxiliary Neumann Problem. In order to prove our interior estimates we shall employ an auxiliary Neumann problem. In this section we state this problem and collect some facts relating to it.

Throughout this section we shall assume that if $D \subseteq \Omega$, then the form A is coercive over $H^1(D)$, i.e., there exists a positive constant c independent of D such that

(4.1)
$$c\|v\|_{1,D}^2 \leqslant A(v, v) \quad \text{for all } v \in H^1(D).$$

Here the region of integration for A is D. In the sequel the appropriate region of integration will be clear from the context.

Let $D \subseteq \Omega$ be a sphere. Then for each $\varphi \in L_2(D)$ there exists a unique function v in $H^1(D)$ such that

(4.2)
$$A(v, w) = (\varphi, w) \quad \text{for all } w \in H^1(D).$$

This may be thought of as the variational formulation of a Neumann problem for a second order operator with the natural boundary conditions being the vanishing of the conormal derivative. The following a priori estimates are well known, cf. [17].

LEMMA 4.1. *Assume that (3.1), (4.1), and (4.2) hold, let $1 < q < \infty$, and let m be a nonnegative integer. Then there exists a constant C such that*

$$\|v\|_{W_q^{m+2}(D)} \leqslant C\|\varphi\|_{W_q^m(D)}.$$

We shall be particularly concerned with the problem (4.2) when φ has small support. Let D have radius 1 and center at the origin, and let

$$D_{[a,b]} = \{x: a \leqslant |x| \leqslant b\}.$$

Write B_ϵ for $D_{[0,\epsilon]}$. We have the following estimates.

LEMMA 4.2. *There exists a constant C such that if v and φ satisfy (4.2), then the following hold:*

(i) *If* supp $\varphi \subseteq B_\epsilon$, *then*

(4.3)
$$\|v\|_{r,D_{[a,2a]}} \leqslant Ca^{-N/2+2-r}\epsilon^{N/2}\|\varphi\|_{0,B_\epsilon} \quad \text{for } \tfrac{1}{4} \geqslant a \geqslant 2\epsilon, \, r \geqslant 1,$$

(4.4)
$$\|v\|_{W_1^r(D\backslash B_{2\epsilon})} \leqslant C\epsilon^{2-r}(\ln 1/\epsilon)^{\bar{r}}\|\varphi\|_{L_1(B_\epsilon)}, \quad r \geqslant 2,$$

(4.5)
$$\|v\|_{W_\infty^1(D\backslash B_{2\epsilon})} \leqslant C\epsilon^{1-N}\|\varphi\|_{L_1(B_\epsilon)},$$

(4.6)
$$\|v\|_{W_1^1(B_{l\epsilon})} \leqslant C(l)\epsilon\|\varphi\|_{L_1(B_\epsilon)}.$$

(ii) *If* supp $\varphi \subseteq D_{[a,2a]}$, *then*

(4.7)
$$\|v\|_{W_\infty^2(D\backslash D_{[a/2,4a]})} \leqslant C\|\varphi\|_{L_\infty(D_{[a,2a]})}.$$

(iii) *For $\varphi \in L_1(D)$,*

(4.8)
$$\|v\|_{W_1^1(D)} \leqslant C\|\varphi\|_{L_1(D)}.$$

Proof. We have

$$v(x) = \int_D G^{(x)}(y)\varphi(y)\,dy,$$

where G is the Green's function for the problem (4.2). By [12] and [13],

(4.9) $\quad |D_x^\alpha G^{(x)}(y)| \leqslant \begin{cases} C(1 + |\ln|x - y||) & \text{for } |\alpha| = 0, N = 2, \\ C|x - y|^{2-N-|\alpha|} & \text{for } N \geqslant 3 \text{ or } N = 2, |\alpha| > 0. \end{cases}$

Let us show how this implies (4.3); the other estimates are proved in a similar fashion. For $x \in D_{[a,2a]}$ and $1 \leqslant |\alpha| \leqslant r$, we have

$$|D^\alpha v(x)| \leqslant \int_{B_\epsilon} |D_x^\alpha G^{(x)}(y)||\varphi(y)|\,dy$$

$$\leqslant \left(\int_{B_\epsilon} |x - y|^{-2(N-2+|\alpha|)}dy \right)^{1/2} \|\varphi\|_{0,B_\epsilon} \leqslant C\epsilon^{N/2}a^{-N+2-r}\|\varphi\|_{0,B_\epsilon}.$$

The same estimate is obviously true for $|\alpha| = 0$. Squaring and integrating with respect to x we obtain the estimate (4.3).

The classical Ritz-Galerkin approximation $v_h \in S^h(D)$ to v is defined by

(4.10) $A(v - v_h, \chi) = 0 \quad \text{for all } \chi \in S^h(D).$

We recall the following error estimates, cf. [1] and Proposition 2.1.

LEMMA 4.3.   *Assume* (3.1), (4.1) *and* A.1 *(for $q = 2$). There exists a constant C such that the following holds:*

Let v and $v_h \in S^h(D)$ satisfy (4.10). *Then for $2 - r \leqslant s \leqslant 1, 1 \leqslant l \leqslant r$,*

$$\|v - v_h\|_{s,D} \leqslant Ch^{l-s}\|v\|_{l,D}.$$

We shall also need a local estimate near the boundary, corresponding to Lemma 3.1, for the projection with respect to the form A. The proof will follow along the lines of Lemma 5.1 in [16] with some simplifications due to the fact that A is coercive. Let $D_0 \subset\subset D$, $D_0' = D\backslash D_0$, and define

$$\overset{\text{C}}{S}{}^h(D_0') = \{\chi \in S^h(D), \chi \equiv 0 \text{ in a neighborhood of } D_0\}.$$

LEMMA 4.4.   *Assume that* (3.1), (4.1), A.1 *(for $q = 2$),* (2.2.a) *of* A.2 *and* A.3 *hold. Let $D_0 \subset\subset D_1 \subset\subset D \subset\subset \Omega$ be concentric spheres, and $D_i' = D\backslash D_i$, $i = 0, 1$. There exists a constant C such that for h sufficiently small the following holds:*

Let $v \in H^r(D_0')$ and $v_h \in S^h(D)$ satisfy $A(v - v_h, \chi) = 0$ for all $\chi \in \overset{\text{C}}{S}{}^h(D_0')$. Then

(4.11) $\|v - v_h\|_{1,D_1'} \leqslant C(h^{r-1}\|v\|_{r,D_0'} + \|v - v_h\|_{L_1(D_1\backslash D_0)}).$

Here C in general depends on the radii of D_0, D_1, D, the quantities C_{ell} and the constant in (4.1), *and $\|a_{ij}\|_{W_\infty^\alpha(D)}$, $\|b_i\|_{W_\infty^\beta(D)}$, $\|d\|_{W_\infty^\gamma(D)}$ for some fixed α, β, γ.*

Proof. We shall first prove (4.11) in the case that $v \equiv 0$ on D_0', i.e. that if v_h satisfies

(4.12) $\qquad A(v_h, \chi) = 0 \quad \text{for all } \chi \in \overset{C}{S}{}^h(D_0'),$

then

(4.13) $\qquad \|v_h\|_{1,D_1'} \leqslant C\|v_h\|_{L_1(D_1 \backslash D_0)}.$

In order to show this let $D_0 \subset\subset D_2 \subset\subset D_3 \subset\subset D_4 \subset\subset D_5 \subset\subset D_1 \subset\subset D$ be concentric spheres and set $D_j' = D\backslash D_j$. Let $\omega \in C^\infty(D)$ with $\omega \equiv 1$ on D_4' and $\omega \equiv 0$ on D_3. Then using (4.1), we have

(4.14) $\qquad \|v_h\|_{1,D_1'}^2 \leqslant \|\omega v_h\|_{1,D}^2 \leqslant CA(\omega v_h, \omega v_h).$

A straightforward computation yields that

(4.15) $\qquad A(\omega v_h, \omega v_h) = A(v_h, \omega^2 v_h) + I,$

where

$$I = \int_D v_h^2 \sum_{i,j=1}^N a_{ij} D_i \omega D_j \omega \, dx + \int_D \omega v_h^2 \sum_{i=1}^N b_i D_i \omega \, dx.$$

Since each $D_i\omega$ vanishes outside of $D_4\backslash D_3$, we have that

(4.16) $\qquad |I| \leqslant C\|v_h\|_{0,D_4\backslash D_3}^2.$

Now from (4.12), we have for any $\chi \in \overset{C}{S}{}^h(D_0')$,

(4.17) $\qquad A(v_h, \omega^2 v_h) = A(v_h, \omega^2 v_h - \chi).$

By A.3 we can find an $\eta \in \overset{\partial h}{S}{}^h(D_5)$ with $\eta \equiv v_h$ on D_2 such that

(4.18) $\qquad \|(1 - \omega^2)v_h - \eta\|_{1,D} \leqslant Ch\|v_h\|_{1,D_5\backslash D_2}.$

Choosing χ in (4.17) as $\chi = v_h - \eta$, we have that $\omega^2 v_h - \chi$ vanishes outside of $D_5\backslash D_2$ and

$$\|\omega^2 v_h - \chi\|_{1,D_5\backslash D_2} \leqslant \|(1 - \omega^2)v_h - \eta\|_{1,D}.$$

Hence from (4.17) and (4.18),

(4.19) $\qquad A(v_h, \omega^2 v_h) \leqslant Ch\|v_h\|_{1,D_5\backslash D_2}^2.$

From (4.14), (4.15), (4.16) and (4.19) we obtain

(4.20) $\qquad \|v_h\|_{1,D_1'} \leqslant C\|v_h\|_{1,D_5\backslash D_2},$

and the desired result (4.13) follows on applying (3.4) of Lemma 3.1 to the right-hand side of (4.20).

We are now in position to prove (4.11). With ω as before, let $(\omega v)_h \in S^h(D)$ satisfy $A(\omega v - (\omega v)_h, \chi) = 0$ for all $\chi \in S^h(D)$. Then

(4.21) $\qquad \|v - v_h\|_{1,D_1'} \leqslant \|\omega v - (\omega v)_h\|_{1,D_1'} + \|(\omega v)_h - v_h\|_{1,D_1'}.$

From Lemma 4.3 we obtain

(4.22) $\|\omega v - (\omega v)_h\|_{1,D} \leqslant Ch^{r-1}\|\omega v\|_{r,D} \leqslant Ch^{r-1}\|v\|_{r,D_0'}.$

Since $A((\omega v)_h - v_h, \chi) = 0$ for all $\chi \in \overset{\circ}{S}^h(D_5')$, it follows from (4.13) (with D_0 replaced by D_5), the triangle inequality and (4.22) that

$$\|(\omega v)_h - v_h\|_{1,D_1'} \leqslant C\|(\omega v)_h - v_h\|_{L_1(D_1 \backslash D_5)}$$

(4.23)
$$\leqslant C(\|\omega v - (\omega v)_h\|_{L_1(D_1 \backslash D_5)} + \|v - v_h\|_{L_1(D_1 \backslash D_5)})$$

$$\leqslant C(h^{r-1}\|v\|_{r,D_0'} + \|v - v_h\|_{L_1(D_1 \backslash D_5)}).$$

The inequality (4.11) now follows from (4.21), (4.22) and (4.23). This completes the proof of the lemma.

5. Interior Maximum Norm Estimates. In this section we shall state the main result of this paper. We shall prove it here for coercive forms. The case of noncoercive forms will be treated in Appendix 1.

THEOREM 5.1.   *Assume that* A.0, A.1, A.2, A.3, A.4 *and* (3.1) *hold. Let* $\Omega_1 \subset\subset \Omega$, *let* p *be a nonnegative integer, and* $1 \leqslant q \leqslant \infty$. *There exists a constant* C *and an* $0 < h_1 < 1$ *such that the following holds:*

Let u *be continuous on* Ω *and* $u_h \in S^h(\Omega)$ *satisfy*

$$A(u - u_h, \chi) = 0 \quad \text{for all } \chi \in \overset{\circ}{S}^h(\Omega).$$

Then for $0 < h \leqslant h_1$ *and any* $\chi \in S^h(\Omega)$,

(5.1) $\|u - u_h\|_{L_\infty(\Omega_1)} \leqslant C\{(\ln 1/h)^{\bar{r}}\|u - \chi\|_{L_\infty(\Omega)} + \|u - u_h\|_{W_q^{-p}(\Omega)}\},$

where

$$\bar{r} = \begin{cases} 1 & \text{if } r = 2, \\ 0 & \text{if } r \geqslant 3. \end{cases}$$

Proof of Theorem 5.1 for Coercive Forms A. We shall make the additional hypothesis that the form A is coercive, i.e., satisfies (4.1). The case of noncoercive forms will be given in Appendix 1. In what follows, $D \subset\subset \Omega$ will be a sphere of radius $R > 0$ with center at x_0, where $x_0 \in \bar{\Omega}_1$ is such that $\|u - u_h\|_{L_\infty(\Omega_1)} = |(u - u_h)(x_0)|.$ We shall need the following lemmas.

LEMMA 5.1.   *Under the above hypotheses, there exists a constant* C *such that the following holds:*

Let $\widetilde{u}$ *have compact support in the sphere* $\frac{1}{2}D$. *If* $\widetilde{u}_h \in S^h(D)$ *satisfies*

(5.2) $A(\widetilde{u} - \widetilde{u}_h, \chi) = 0 \quad \text{for all } \chi \in S^h(D),$

then for $0 < h \leqslant h_1$,

(5.3) $\|\widetilde{u} - \widetilde{u}_h\|_{L_\infty(\frac{1}{4}D)} \leqslant C(\ln 1/h)^{\bar{r}}\|\widetilde{u}\|_{L_\infty(D)}.$

LEMMA 5.2.   *Under the above hypotheses, there exists a constant* C *such that if* $w_h \in S^h(D)$ *satisfies*

(5.4) $$A(w_h, \chi) = 0 \quad \text{for all } \chi \in \overset{\circ}{S}{}^h(D),$$

then

(5.5) $$|w_h(x_0)| \leqslant C\|w_h\|_{W_q^{-p}(D)}.$$

Before proving Lemmas 5.1 and 5.2, let us show how Theorem 5.1 follows in the coercive case. Let $\omega \in \overset{\circ}{C}{}^\infty(\tfrac{1}{2}D)$ with $\omega \equiv 1$ on $\tfrac{1}{4}D$ and set $\widetilde{u} = \omega u$. Then taking $\widetilde{u}_h \in S^h(D)$ satisfying (5.2), we have from (5.3) that

(5.6) $$|u(x_0) - \widetilde{u}_h(x_0)| \leqslant C(\ln 1/h)^{\overline{r}}\|\widetilde{u}\|_{L_\infty(D)} \leqslant C(\ln 1/h)^{\overline{r}}\|u\|_{L_\infty(D)}.$$

For $\eta \in \overset{\circ}{S}{}^h(\tfrac{1}{4}D)$ we obtain

$$A(\widetilde{u}_h - u_h, \eta) = A(u - u_h, \eta) = 0.$$

Hence taking $w_h = \widetilde{u}_h - u_h$, it follows from (5.5) (with D replaced by $\tfrac{1}{4}D$) and (5.3) that

$$|(\widetilde{u}_h - u_h)(x_0)| \leqslant C\|\widetilde{u}_h - u_h\|_{W_q^{-p}(\tfrac{1}{4}D)}$$

$$\leqslant C\{\|\widetilde{u} - \widetilde{u}_h\|_{L_\infty(\tfrac{1}{4}D)} + \|u - u_h\|_{W_q^{-p}(\tfrac{1}{4}D)}\}$$

$$\leqslant C\{(\ln 1/h)^{\overline{r}}\|u\|_{L_\infty(D)} + \|u - u_h\|_{W_q^{-p}(D)}\}.$$

Using this and (5.6), we obtain via the triangle inequality

$$\|u - u_h\|_{L_\infty(\Omega_1)} \leqslant C\{(\ln 1/h)^{\overline{r}}\|u\|_{L_\infty(\Omega)} + \|u - u_h\|_{W_q^{-p}(\Omega)}\}$$

and the theorem follows upon writing $u - u_h = (u - \chi) - (u_h - \chi)$ for $\chi \in S^h(\Omega)$.

The proofs of Lemmas 5.1 and 5.2 will depend on the following technical lemma.

LEMMA 5.3. *Under the hypotheses of Theorem 5.1, let $D_h \subset\subset \tfrac{1}{4}D$ be a sphere of radius $C'h$. There exists a constant C such that for $0 < h \leqslant h_1$ the following holds.*

Let $\varphi \in \overset{\circ}{C}{}^\infty(D_h)$, and let v and $v_h \in S^h(D)$ satisfy

(5.7) $$A(\psi, v) = (\psi, \varphi) \quad \text{for all } \psi \in H^1(D),$$

(5.8) $$A(\chi, v - v_h) = 0 \quad \text{for all } \chi \in S^h(D).$$

Then

(5.9) $$\|v - v_h\|_{W_1^1(D)} \leqslant Ch^{N/2+1}(\ln 1/h)^{\overline{r}}\|\varphi\|_{0,D_h}$$

and for $D_1 \subset\subset D$,

(5.10) $$\|v - v_h\|_{W_1^{2,h}(D_1)} \leqslant Ch^{N/2}(\ln 1/h)^{\overline{r}}\|\varphi\|_{0,D_h}.$$

Assuming Lemma 5.3 for the moment, we shall prove Lemmas 5.1 and 5.2.

Proof of Lemma 5.1. Let $\|\widetilde{u} - \widetilde{u}_h\|_{L_\infty(\tfrac{1}{4}D)} = |(\widetilde{u} - \widetilde{u}_h)(x_1)|$. For simplicity in notation we shall assume that x_1 is the center of D. Let D_h be a sphere of radius $C'h$, $C' > k$, with center at x_1. Then

$$\begin{aligned}
|(\tilde{u} - \tilde{u}_h)(x_1)| &\leq |\tilde{u}(x_1)| + |\tilde{u}_h(x_1)| \\
&\leq |\tilde{u}(x_1)| + Ch^{-N/2}\|\tilde{u}_h\|_{L_2(D_h)} \\
&\leq |\tilde{u}(x_1)| + Ch^{-N/2}\|\tilde{u}_h - \tilde{u}\|_{L_2(D_h)} + Ch^{-N/2}\|\tilde{u}\|_{L_2(D_h)} \\
&\leq C\{\|\tilde{u}\|_{L_\infty(D_h)} + h^{-N/2}\|\tilde{u}_h - \tilde{u}\|_{L_2(D_h)}\}.
\end{aligned}$$

Here we used (2.2.b) and the triangle inequality.

Now (5.3) would follow once we had shown that

$$(5.11) \qquad h^{-N/2}\|\tilde{u} - \tilde{u}_h\|_{0,D_h} \leq C(\ln 1/h)^{\bar{r}}\|\tilde{u}\|_{L_\infty(D)}.$$

We have

$$(5.12) \qquad \|\tilde{u} - \tilde{u}_h\|_{0,D_h} = \sup_{\varphi \in \overset{\circ}{C}{}^\infty(D_h)} \frac{(\tilde{u} - \tilde{u}_h, \varphi)}{\|\varphi\|_{0,D_h}}.$$

For each such φ, let v and v_h satisfy (5.7) and (5.8). Then

$$(5.13) \qquad (\tilde{u} - \tilde{u}_h, \varphi) = A(\tilde{u} - \tilde{u}_h, v) = A(\tilde{u}, v - v_h).$$

Let L denote the second order operator corresponding to A (given by (0.1)), L^* its formal adjoint, and $\partial/\partial n$ the conormal derivative. Integrating by parts, we have

$$A(\tilde{u}, v - v_h) = \sum_{i \in I} \int_{\tau_i^h} \tilde{u} L^*(v - v_h)\,dx + \sum_{i \in I} \int_{\partial \tau_i^h} \tilde{u}\,\frac{\partial}{\partial n}(v - v_h)\,d\sigma,$$

where, since $\tilde{u}$ vanishes outside $\frac{1}{2}D$, we may take $I = \{i \mid \tau_i^h \cap \frac{1}{2}D \neq \varnothing\}$. Let $D_I = \bigcup_{i \in I} \bar{\tau}_i^h$. Then

$$\left| \sum_{i \in I} \int_{\tau_i^h} \tilde{u} L^*(v - v_h)\,dx \right| \leq C\|\tilde{u}\|_{L_\infty(D)}\|v - v_h\|_{W_1^{2,h}(D_I)}.$$

In view of our assumption A.0,

$$\left| \sum_{i \in I} \int_{\partial \tau_i^h} \tilde{u}\,\frac{\partial(v - v_h)}{\partial n}\,d\sigma \right| \leq C\|\tilde{u}\|_{L_\infty(D)}\left(h^{-1}\|v - v_h\|_{W_1^1(D_I)} + \|v - v_h\|_{W_1^{2,h}(D_I)}\right),$$

and therefore,

$$|A(\tilde{u}, v - v_h)| \leq C\|\tilde{u}\|_{L_\infty(D)}\left(h^{-1}\|v - v_h\|_{W_1^1(D_I)} + \|v - v_h\|_{W_1^{2,h}(D_I)}\right).$$

Now using Lemma 5.3 we obtain

$$|A(\tilde{u}, v - v_h)| \leq C\|\tilde{u}\|_{L_\infty(D)}h^{N/2}(\ln 1/h)^{\bar{r}}\|\varphi\|_{0,D_h},$$

and (5.11) follows from this, (5.13) and (5.12). This proves Lemma 5.1.

Proof of Lemma 5.2. Using Proposition 2.2, there exists a function $\eta_h \in \overset{\circ}{S}{}^h(\frac{3}{4}D)$ such that $\eta_h \equiv w_h$ on $\frac{1}{2}D$ and $\|\eta_h\|_{1,\frac{3}{4}D} \leq C\|w_h\|_{1,D}$. Then

$$|w_h(x_0)| = |\eta_h(x_0)| \leq Ch^{-N/2}\|\eta_h\|_{0,D_h} = Ch^{-N/2} \sup_{\varphi \in \overset{\circ}{C}{}^\infty(D_h)} \frac{(\eta_h, \varphi)}{\|\varphi\|_{0,D_h}},$$

where diam $D_h \leq Ch$. Let $\varphi \in \overset{\circ}{C}{}^\infty(D_h)$ be fixed and let v and v_h satisfy (5.7) and (5.8) so that

$$(\eta_h, \varphi) = A(\eta_h, v) = A(\eta_h, v_h).$$

Let $\chi_h \in \overset{\circ}{S}{}^h(\tfrac{1}{2}D)$ with $\chi_h \equiv v_h$ on $\tfrac{1}{4}D$ be as η in Proposition 2.2. Since $A(\eta_h, \chi_h) = A(w_h, \chi_h) = 0$, and using (2.5), we have

(5.15)
$$(\eta_h, \varphi) = A(\eta_h, v_h - \chi_h) \leqslant C \|\eta_h\|_{1,3D/4} \|v_h\|_{1,3D/4 \setminus D/8}$$
$$\leqslant C \|w_h\|_{1,D} \|v_h\|_{1,3D/4 \setminus D/8}.$$

Since v_h satisfies $A(\chi, v_h) = 0$ for $\chi \in \overset{\circ}{S}{}^h(D \setminus D_h)$, (3.4) yields

(5.16)
$$\|v_h\|_{1,3D/4 \setminus D/8} \leqslant C \|v_h\|_{W_1^1(D)}.$$

Using Lemma 5.3 and (4.8),

$$\|v_h\|_{W_1^1(D)} \leqslant \|v - v_h\|_{W_1^1(D)} + \|v\|_{W_1^1(D)} \leqslant C h^{N/2} \|\varphi\|_{0,D_h}.$$

Thus, from (5.16), (5.15) and (5.14) we obtain

$$|w_h(x_0)| \leqslant C \|w_h\|_{1,D}.$$

Replacing D by $\tfrac{1}{2}D$, the desired result (5.5) follows from (3.4). This proves Lemma 5.2.

Proof of Lemma 5.3. We shall first prove (5.9). For simplicity in notation, let D_h and D be concentric spheres with centers at the origin and with radii h and 1, respectively. Let Ω_j denote the annuli

$$\Omega_j = \{x \,|\, 2^{-j-1} \leqslant |x| \leqslant 2^{-j}\},$$

and let J be the largest integer such that $2^{-J} \geqslant C_* h$, where C_* is to be chosen later (sufficiently large). Set $d_j = 2^{-j}$, $\Omega_h = D \setminus \bigcup_{j=0}^J \Omega_j$, and let

$$\Omega_j^l = (\Omega_{j-l} \cup \overline{\Omega}_{j-l+1} \cup \cdots \cup \Omega_{j+l}) \cap D, \qquad l = 1, 2, \ldots .$$

Furthermore, set $e = v - v_h$. We have

(5.17)
$$\|e\|_{W_1^1(D)} = \sum_{j=0}^J \|e\|_{W_1^1(\Omega_j)} + \|e\|_{W_1^1(\Omega_h)}.$$

Using Lemmas 4.3 and 4.1, we obtain

(5.18)
$$\|e\|_{W_1^1(\Omega_h)} \leqslant C C_*^{N/2} h^{N/2} \|e\|_{1,\Omega_h} \leqslant C C_*^{N/2} h^{N/2+1} \|\varphi\|_{0,D_h}.$$

By Lemma 3.2, or Lemma 4.4 in the case of $j = 0$, it follows that

$$\|e\|_{W_1^1(\Omega_j)} \leqslant C d_j^{N/2} \|e\|_{1,\Omega_j} \leqslant C d_j^{N/2} \{ h^{r-1} \|v\|_{r,\Omega_j^1} + d_j^{-N/2-1} \|e\|_{L_1(\Omega_j^1)} \}.$$

By (4.3) we have

$$\|v\|_{r,\Omega_j^1} \leqslant C d_j^{-N/2+2-r} h^{N/2} \|\varphi\|_{0,D_h},$$

and thus

$$\sum_{j=0}^{J} \|e\|_{W_1^1(\Omega_j)} \leqslant C\left(\sum_{j=0}^{J} d_j^{2-r} h^{N/2+r-1}\right) \|\varphi\|_{0,D_h} + C\sum_{j=0}^{J} d_j^{-1} \|e\|_{L_1(\Omega_j^1)}$$

(5.19)

$$\leqslant Ch^{N/2+1}(\ln 1/h)^{\bar r} \|\varphi\|_{0,D_h} + C \cdot I,$$

where

$$I = \sum_{j=0}^{J} d_j^{-1} \|e\|_{L_1(\Omega_j^1)}.$$

We shall now estimate I. Let

$$\tilde I = \frac{1}{C_* h} \|e\|_{L_1(\Omega_h)} + \sum_{j=0}^{J} d_j^{-1} \|e\|_{L_1(\Omega_j)}$$

and note that $I \leqslant 8\tilde I$. Now,

$$\frac{1}{C_* h} \|e\|_{L_1(\Omega_h)} \leqslant C(C_*)^{N/2-1} h^{N/2-1} \|e\|_{L_2(\Omega_h)}$$

and by Lemma 4.3 and 4.1,

(5.20) $$\frac{1}{C_* h} \|e\|_{L_1(\Omega_h)} \leqslant C(C_*)^{N/2-1} h^{N/2+1} \|\varphi\|_{0,D_h}.$$

Next write

$$\|e\|_{L_1(\Omega_j)} = \sup_{\eta \in \overset{\circ}{C}^\infty(\Omega_j)} \frac{(e,\eta)}{\|\eta\|_{L_\infty(\Omega_j)}}.$$

Letting $A(w,\psi) = (w,\eta)$ for $w \in H^1(D)$ and letting $\chi \in S^h(D)$,

(5.21) $$\|e\|_{L_1(\Omega_j)} = \sup_{\eta \in \overset{\circ}{C}^\infty(\Omega_j)} \frac{A(e,\psi-\chi)}{\|\eta\|_{L_\infty(\Omega_j)}}.$$

We have

(5.22) $$A(e,\psi-\chi) \leqslant \|e\|_{W_1^1(D\backslash\Omega_j^2)} \|\psi-\chi\|_{W_\infty^1(D\backslash\Omega_j^2)} + \|e\|_{1,\Omega_j^2} \|\psi-\chi\|_{1,\Omega_j^2}.$$

By A.1 (cf. Proposition 2.2 for $j = 0$) and by (4.7) we obtain for a suitable χ,

(5.23) $$\|\psi-\chi\|_{W_\infty^1(D\backslash\Omega_j^2)} \leqslant Ch\|\psi\|_{W_\infty^2(D\backslash\Omega_j^1)} \leqslant Ch\|\eta\|_{L_\infty(\Omega_j)}.$$

Furthermore, by Lemma 3.2 and (4.3),

$$\|e\|_{1,\Omega_j^2} \leqslant Ch\|v\|_{2,\Omega_j^3} + Cd_j^{-N/2-1} \|e\|_{L_1(\Omega_j^3)}$$

$$\leqslant Chd_j^{-N/2} h^{N/2} \|\varphi\|_{0,D_h} + Cd_j^{-N/2-1} \|e\|_{L_1(\Omega_j^3)};$$

and we can take χ such that

$$\|\psi-\chi\|_{1,\Omega_j^2} \leqslant Ch\|\psi\|_{2,\Omega} \leqslant Ch\|\eta\|_{0,\Omega_j} \leqslant Chd_j^{N/2} \|\eta\|_{L_\infty(\Omega_j)}.$$

Thus,

$$\|e\|_{1,\Omega_j^2}\|\psi - \chi\|_{1,\Omega_j^2} \leqslant C\{h^{N/2+2}\|\varphi\|_{0,D_h} + hd_j^{-1}\|e\|_{L_1(\Omega_j^3)}\}\|\eta\|_{L_\infty(\Omega_j)},$$

and from this and (5.21), (5.22) and (5.23),

$$\|e\|_{L_1(\Omega_j)} \leqslant C\{h\|e\|_{W_1^1(D\setminus\Omega_j^2)} + hd_j^{-1}\|e\|_{L_1(\Omega_j^3)} + h^{N/2+2}\|\varphi\|_{0,D_h}\}.$$

Now using (5.20), we have

$$\widetilde{I} \leqslant C\left\{C_*^{N/2-1}h^{N/2+1} + h^{N/2+2}\sum_{j=0}^J d_j^{-1}\right\}\|\varphi\|_{0,D_h}$$

$$+ Ch\sum_{j=0}^J (d_j^{-1}\|e\|_{W_1^1(D\setminus\Omega_j^2)}) + C\sum_{j=0}^J \left(\frac{h}{d_j}\right)d_j^{-1}\|e\|_{L_1(\Omega_j^3)}$$

$$\leqslant CC_*^{N/2-1}h^{N/2+1}\|\varphi\|_{0,D_h} + \frac{C}{C_*}\|e\|_{W_1^1(D)} + \frac{C}{C_*}\widetilde{I}.$$

Thus, choosing C_* large enough we obtain

$$\widetilde{I} \leqslant CC_*^{N/2-1}h^{N/2+1}\|\varphi\|_{0,D_h} + \frac{C}{C_*}\|e\|_{W_1^1(D)},$$

and from this, (5.17), (5.18) and (5.19),

$$\|e\|_{W_1^1(D)} \leqslant CC_*^{N/2}h^{N/2+1}(\ln 1/h)^{\bar{r}}\|\varphi\|_{0,D_h} + \frac{C}{C_*}\|e\|_{W_1^1(D)}.$$

Hence,

$$\|e\|_{W_1^1(D)} \leqslant Ch^{N/2+1}(\ln 1/h)^{\bar{r}}\|\varphi\|_{0,D_h},$$

which completes the proof of (5.9).

We shall now prove (5.10). Let $D_h \subset\subset D_{2h} \subset\subset D_{3h} \subset\subset D_{4h} \subset\subset D_{5h} \subset\subset D_{6h}$ be concentric spheres with $\text{dist}(D_{jh}, \partial D_{(j+1)h}) = kh, j = 1, \ldots, 5$. We have

$$(5.24) \quad \|v - v_h\|_{W_1^{2,h}(D_1)} \leqslant \|v - v_h\|_{W_1^{2,h}(D_1\setminus D_{4h})} + \|v - v_h\|_{W_1^{2,h}(D_{4h})}.$$

For $\chi \in S^h(D)$ it follows from (2.2.b) that

$$(5.25) \quad \|v - v_h\|_{W_1^{2,h}(D_{4h})} \leqslant \|v - \chi\|_{W_1^{2,h}(D_{4h})} + Ch^{-1}\|\chi - v_h\|_{W_1^1(D_{5h})}$$

$$\leqslant \|v - \chi\|_{W_1^{2,h}(D_{4h})} + Ch^{-1}\|v - \chi\|_{W_1^1(D_{5h})} + Ch^{-1}\|v - v_h\|_{W_1^1(D)}$$

Choosing χ to satisfy (2.1), we have

$$\|v - \chi\|_{W_1^{2,h}(D_{4h})} + Ch^{-1}\|v - \chi\|_{W_1^1(D_{5h})}$$

$$\leqslant C\|v\|_{W_1^2(D_{6h})} \leqslant Ch^{N/2}\|v\|_{2,D_{6h}} \leqslant Ch^{N/2}\|\varphi\|_{0,D_h}.$$

From this, (5.25) and (5.9) we obtain

(5.26) $$\|v - v_h\|_{W_1^{2,h}(D_{4h})} \leqslant Ch^{N/2}(\ln 1/h)^{\bar{r}}\|\varphi\|_{0,D_h}.$$

As in (5.25), we have that for any $\eta \in S^h(D)$

(5.27) $$\|v - v_h\|_{W_1^{2,h}(D \setminus D_{4h})}$$

$$\leqslant \|v - \eta\|_{W_1^{2,h}(D \setminus D_{4h})} + Ch^{-1}\|v - \eta\|_{W_1^1(D \setminus D_{3h})} + Ch^{-1}\|v - v_h\|_{W_1^1(D)}.$$

Choosing η to satisfy (2.1) and using (4.4), we arrive at

$$\|v - \eta\|_{W_1^{2,h}(D \setminus D_{4h})} + Ch^{-1}\|v - \eta\|_{W_1^1(D \setminus D_{3h})}$$

$$\leqslant Ch^{r-2}\|v\|_{W_1^r(D \setminus D_{2h})} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar{r}}\|\varphi\|_{L_1(D_h)} \leqslant Ch^{N/2}\left(\ln \frac{1}{h}\right)^{\bar{r}}\|\varphi\|_{0,D_h}.$$

This together with (5.9), (5.24), (5.26) and (5.27) proves (5.10).

This completes the proof of Lemma 5.3.

The proof of Theorem 5.1 in the case that the form A is coercive is now accomplished. As mentioned previously, the case of noncoercive A will be given in Appendix 1.

Theorem 5.1 was proved under the assumption that the domains Ω_1 and Ω were fixed. Analogously to Lemma 3.2 we have the following result when the domains may vary and be close with h.

COROLLARY 5.1. *Assume that the conditions of Theorem 5.1 hold and let* $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$. *There exist constants* $C_6 > 0$ *and* C *such that if* $C_6 h \leqslant d$, $\text{dist}(\Omega_1, \partial\Omega_2) \geqslant d$, *and* $\text{dist}(\Omega_2, \partial\Omega) \geqslant d$, *then for* $0 \leqslant l \leqslant r$,

$$\|u - u_h\|_{L_\infty(\Omega_1)} \leqslant C\left\{h^l\left(\ln \frac{d}{h}\right)^{\bar{r}}|u|_{W_\infty^l(\Omega_2)} + d^{-N/q-p}\|u - u_h\|_{W_q^{-p}(\Omega_2)}\right\},$$

where C *is independent of* h, u, u_h, Ω_1, Ω_2 *and in general depends on* p, C_{ell}, $\|a_{ij}\|_{W_\infty^\alpha(\Omega)}$, $\|b_i\|_{W_\infty^\beta(\Omega)}$, $\|d\|_{W_\infty^\gamma(\Omega)}$ *for some* α, β, γ.

Proof. We shall mimic the proof of Lemma 3.2. Let $x_0 \in \bar{\Omega}_1$ be such that $|(u - u_h)(x_0)| = \|u - u_h\|_{L_\infty(\Omega_1)}$. Without loss of generality we assume that Ω_1 and Ω_2 are concentric spheres with center at x_0 and diam $\Omega_1 = 2d$. Let $y = (x - x_0)/d$ and transform the problem to the new variables y on $\tilde{\Omega}_2$. The new sets $\tilde{\tau}_i^h$ satisfy A.0 with h replaced by h/d. From A.4 it follows that A.1, A.2 and A.3 are satisfied on the new domain $\tilde{\Omega}_2$ with h replaced by h/d. To verify that A.4 is satisfied on $\tilde{\Omega}_2$, let $y_0 \in \tilde{\Omega}_2$ and transform a sphere of radius $\tilde{d} \geqslant C_4 h/d$ in $\tilde{\Omega}_2$ via the transformation $z = (y - y_0)/\tilde{d}$. It is seen that this is equivalent to transforming a sphere of radius $d\tilde{d}$ in Ω_2 via the transformation $z = (x - x_1)/d\tilde{d}$ for some point $x_1 \in \Omega_2$. Since $d\tilde{d} \geqslant C_4 h$, A.4 yields that A.1, A.2, and A.3 also hold with h/d replaced by $h/d\tilde{d}$. Hence Corollary 5.1 is valid on the domains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, and the desired result follows by transforming back to the original domains.

6. Pointwise Error Estimates for the Green's Function Near the Singularity. In this section we shall apply the results of Section 5 to derive estimates for the error in

the Green's function. We shall restrict our attention to the Neumann problem on a smooth domain, but the techniques given below are suitable for investigating other boundary value problems and methods for solving them.

Let $\mathcal{D}$ be a bounded domain in R^N with a smooth boundary $\partial \mathcal{D}$, and let A be a bilinear form of the type (0.4), where A is coercive over H^1, i.e., there exists a constant $c > 0$ such that

$$(6.1) \qquad c\|v\|_{1,\mathcal{D}}^2 \leqslant A(v, v) \quad \text{for all } v \in H^1(\mathcal{D}).$$

Let $S^h(\mathcal{D})$ be a one parameter family of subspaces of $W_\infty^1(\mathcal{D})$ having the global property that

$$(6.2) \qquad \inf_{\chi \in S^h(\mathcal{D})} \|v - \chi\|_{1,\mathcal{D}} \leqslant Ch^{l-1} \|v\|_{l,\mathcal{D}} \quad \text{for } 1 \leqslant l \leqslant r.$$

In addition, the spaces will be required to satisfy the conditions of Theorem 5.1 on interior subdomains.

Let $y \in \mathcal{D}$ and $G^{(y)}(x)$ and $G_h^{(y)}(x) \in S^h(\mathcal{D})$, respectively, be the Green's function and approximate Green's function defined by

$$A(G^{(y)}, v) = v(y) \quad \text{for all } v \in W_\infty^1(\mathcal{D}),$$

and

$$A(G_h^{(y)}, \chi) = \chi(y) \quad \text{for all } \chi \in S^h(\mathcal{D}).$$

We wish to estimate $|G^{(y)}(x) - G_h^{(y)}(x)|$ where y is in the interior of $\mathcal{D}$ and x may be close to y. More precisely we have:

THEOREM 6.1. *Assume* (6.1) *and* (6.2). *Let* $\Omega_1 \subset\subset \Omega_2 \subset\subset \mathcal{D}$ *and suppose that the conditions of Corollary 5.1 are satisfied. There exist constants C and C_7 such that if h is sufficiently small, then for* $y \in \Omega_1$, $x \in \Omega_2$.

(i) *If* $|x - y| \geqslant C_7 h$,

$$(6.3) \qquad |G^{(y)}(x) - G_h^{(y)}(x)| \leqslant Ch^r \left(\ln\left(\frac{|x-y|}{h} \right) \right)^{\overline{r}} \Big/ |x-y|^{N-2+r}.$$

(ii) *If* $0 < |x - y| \leqslant C_7 h$,

$$(6.4) \qquad |G^{(y)}(x) - G_h^{(y)}(x)| \leqslant C \begin{cases} \ln \dfrac{1}{|x-y|} + 1 & \text{for } N = 2, \\[2ex] \dfrac{1}{|x-y|^{N-2}} & \text{for } N \geqslant 3. \end{cases}$$

This theorem can be applied, e.g. in the following situation, cf. [5]. Let f have compact support in $\mathcal{D}$. Then

$$(u - u_h)(y) = \int_{\text{supp } f} (G^{(y)}(x) - G_h^{(y)}(x)) f(x) \, dx,$$

and using the results (6.3) and (6.4), the error can be estimated in various domains under weak regularity assumptions on f.

Proof. Let $d = |x - y|$ and denote by $B_r(x)$ the sphere of radius r around x. Furthermore, set $e^{(y)} = G^{(y)} - G_h^{(y)}$. Note that there exist a fixed domain Ω_3

independent of d, $\Omega_2 \subset\subset \Omega_3 \subset\subset \mathcal{D}$, and a constant c_1, $0 < c_1 \leqslant \frac{1}{2}$ such that for any $x \in \Omega_2$,

$$D \equiv B_{c_1 d}(x) \subset\subset \Omega_3.$$

Let $D_1 = B_{\frac{1}{2} c_1 d}(x) \subset\subset D$. Applying Corollary 5.1 to $D_1 \subset\subset D$, we obtain

$$(6.5) \quad |e^{(y)}(x)| \leqslant C\left\{ h^r \left(\ln \frac{d}{h} \right)^{\overline{r}} |G^{(y)}|_{W_\infty^r(D)} + d^{-N-r+2} \|e^{(y)}\|_{W_1^{2-r}(D)} \right\}.$$

Since $\operatorname{dist}(D, y) \geqslant \frac{1}{2} d$, it follows from (4.9) that

$$(6.6) \quad |G^{(y)}|_{W_\infty^r(D)} \leqslant C d^{-N-r+2}.$$

We shall next estimate the second term on the right in (6.5). We have

$$(6.7) \quad \|e^{(y)}\|_{W_1^{2-r}(D)} = \sup_{\varphi \in \overset{\circ}{C}^\infty(D)} \frac{(e^{(y)}, \varphi)}{\|\varphi\|_{W_\infty^{r-2}(D)}}.$$

Let $v \in W_\infty^1(\mathcal{D})$ satisfy $A(w, v) = (w, \varphi)$ for all $w \in W_1^1(\mathcal{D})$. Then

$$(e^{(y)}, \varphi) = A(e^{(y)}, v) = v(y) - v_h(y),$$

where $v_h \in S^h(\mathcal{D})$ satisfies $A(\chi, v - v_h) = 0$ for all $\chi \in S^h(\mathcal{D})$. Since $y \in \Omega_1$, we may apply Corollary 5.1 to $(v - v_h)(y)$ on the spheres $D_1' \subset\subset D'$ of diameter $d/8$ and $d/4$, respectively, with center at y. Then

$$|(v - v_h)(y)| \leqslant C\left\{ h^r \left(\ln \left(\frac{d}{h} \right) \right)^{\overline{r}} |v|_{W_\infty^r(D')} + d^{-N/2} \|v - v_h\|_{0,\mathcal{D}} \right\}.$$

Note that $\operatorname{dist}(D', D) \geqslant d/4$. Using the fact that $\varphi \in \overset{\circ}{C}^\infty(D)$, we have, cf. the proof of Lemma 4.2, that

$$\|v\|_{W_\infty^r(D')} \leqslant C\|\varphi\|_{W_\infty^{r-2}(D)}.$$

Furthermore, from (6.2), cf. Lemma 4.3,

$$\|v - v_h\|_{0,\mathcal{D}} \leqslant C h^r \|\varphi\|_{r-2,\mathcal{D}} \leqslant C h^r d^{N/2} \|\varphi\|_{W_\infty^{r-2}(D)}.$$

Hence

$$|(e^{(y)}, \varphi)| \leqslant C h^r \left(\ln \left(\frac{d}{h} \right) \right)^{\overline{r}} \|\varphi\|_{W_\infty^{r-2}(D)}.$$

This together with (6.7), (6.6) and (6.5) proves (6.3).

We shall next show (6.4). Let $d = |x - y| \leqslant C_7 h$. In view of (4.9) it suffices to show that

$$(6.8) \quad |G_h^{(y)}(x)| \leqslant \begin{cases} C \ln \dfrac{1}{h} & \text{for } N = 2, \\ C h^{2-N} & \text{for } N \geqslant 3. \end{cases}$$

We define x_0 by

(6.9)
$$\sup_{x \in \Omega_2} |G_h^{(y)}(x)| = |G_h^{(y)}(x_0)|.$$

We may assume that $B_{kh}(x_0) \subseteq \Omega_2$; otherwise $|x_0 - y| \geqslant \frac{1}{2} \operatorname{dist}(\Omega_1, \Omega_2)$ and (6.8) follows from (6.3) and (4.9).

Let A_D denote the form A with the integration extended over the domain D. We have with ϵ to be chosen later,

(6.10)
$$G_h^{(y)}(x_0) = A(G_h^{(x_0)}, G_h^{(y)}) = A(G^{(x_0)}, G_h^{(y)})$$

$$= A_{B_{\epsilon h}(x_0)}(G^{(x_0)}, G_h^{(y)}) + A_{D \setminus B_{\epsilon h}(x_0)}(G^{(x_0)}, G_h^{(y)}) = I_1 + I_2.$$

Using the inverse property (2.2.b) and (4.9), we have

(6.11)
$$I_1 \leqslant C \|G_h^{(y)}\|_{W_\infty^1(B_{\epsilon h}(x_0))} \|G^{(x_0)}\|_{W_1^1(B_{\epsilon h}(x_0))}$$

$$\leqslant \frac{C}{h} |G_h^{(y)}(x_0)| \cdot \epsilon h = C\epsilon |G_h^{(y)}(x_0)|.$$

For I_2, the Cauchy-Schwarz inequality and (4.9) give

$$I_2 \leqslant C \|G^{(x_0)}\|_{1, D \setminus B_{\epsilon h}(x_0)} \|G_h^{(y)}\|_{1, D}$$

$$\leqslant \begin{cases} C(\epsilon)(\ln 1/h)^{1/2} \|G_h^{(y)}\|_{1, D} & \text{for } N = 2, \\ C(\epsilon) h^{-N/2 + 1} \|G_h^{(y)}\|_{1, D} & \text{for } N \geqslant 3. \end{cases}$$

Note that by (6.1) and (6.9),

$$\|G_h^{(y)}\|_{1, D}^2 \leqslant CA(G_h^{(y)}, G_h^{(y)}) = CG_h^{(y)}(y) \leqslant C|G_h^{(y)}(x_0)|;$$

thus

(6.12)
$$I_2 \leqslant \begin{cases} \widetilde{C}(\epsilon)\ln \frac{1}{h} + \epsilon |G_h^{(y)}(x_0)| & \text{for } N = 2, \\ \widetilde{C}(\epsilon) h^{-N+2} + \epsilon |G_h^{(y)}(x_0)| & \text{for } N \geqslant 3. \end{cases}$$

Choosing ϵ small enough we obtain the desired inequality (6.8) from (6.9)–(6.12). This completes the proof of the theorem.

Appendix 1. Proof of Theorem 5.1 for Noncoercive Forms A. Assume that we have proven the following weaker version of Theorem 5.1.

LEMMA A.1. *Under the assumptions of Theorem 5.1, if $u \in W_\infty^1(D)$ and*

$$A(u - u_h, \chi) = 0 \quad \text{for } \chi \in \mathring{S}^h(D),$$

then

(A.1)
$$\|u - u_h\|_{L_\infty(\frac{1}{4}D)} \leqslant C(\|u\|_{W_\infty^1(D)} + \|u - u_h\|_{W_q^{-p}(D)}).$$

We shall first show how Theorem 5.1 follows from this. Let K be a positive number such that the form A_K, $A_K(v, w) \equiv A(v, w) + K(v, w)$ is coercive, i.e., satisfies (4.1). Let $\omega \in \mathring{C}^\infty(\frac{1}{2}D)$, $\omega \equiv 1$ on $\frac{1}{4}D$, let $\tilde{u} = \omega u$ and let $\tilde{u}_h \in S^h(D)$ be

given by

$$A_K(\widetilde{u} - \widetilde{u}_h, \chi) = 0 \quad \text{for } \chi \in S^h(D).$$

By Lemma 5.1,

(A.2) $$\|\widetilde{u} - \widetilde{u}_h\|_{L_\infty(\frac{1}{4}D)} \leqslant C\left(\ln \frac{1}{h}\right)^{\overline{r}} \|u\|_{L_\infty(D)}.$$

Next note that

$$A(\widetilde{u}_h - u_h, \chi) = -K(\widetilde{u}_h - \widetilde{u}, \chi) \quad \text{for } \chi \in \overset{\circ}{S}{}^h(\frac{1}{4}D).$$

For $D' = cD$ with sufficiently small c, the form A is coercive over $\overset{\circ}{H}{}^1(D')$. Let then $\psi \in \overset{\circ}{H}{}^1(D')$ be such that

$$A(\psi, v) = K(\widetilde{u}_h - \widetilde{u}, v) \quad \text{for } v \in \overset{\circ}{H}{}^1(D').$$

Thus,

$$A(\psi - (u_h - \widetilde{u}_h), \chi) = 0 \quad \text{for all } \chi \in \overset{\circ}{S}{}^h(D');$$

and hence using Lemma A.1 and (A.2),

$$\|\psi - (u_h - \widetilde{u}_h)\|_{L_\infty(\frac{1}{4}D')}$$
$$\leqslant C\|\psi\|_{W_\infty^1(D')} + C\|\psi - (u_h - \widetilde{u}_h)\|_{W_q^{-p}(D')}$$
$$\leqslant C\|\psi\|_{W_\infty^1(D')} + C\|u_h - u\|_{W_q^{-p}(\frac{1}{4}D)} + C(\ln 1/h)^{\overline{r}}\|u\|_{L_\infty(D)}.$$

Thus, by (A.2) and the triangle inequality,

$$\|u - u_h\|_{L_\infty(\frac{1}{4}D')} \leqslant C\|\psi\|_{W_\infty^1(D')} + C\|u_h - u\|_{W_q^{-p}(\frac{1}{4}D)} + C\left(\ln \frac{1}{h}\right)^{\overline{r}}\|u\|_{L_\infty(D)}.$$

By elliptic regularity, cf. Lemma 4.1, and (A.2),

$$\|\psi\|_{W_\infty^1(D')} \leqslant C\|\widetilde{u}_h - \widetilde{u}\|_{L_\infty(D')} \leqslant C\left(\ln \frac{1}{h}\right)^{\overline{r}}\|u\|_{L_\infty(D)},$$

so that finally

$$\|u - u_h\|_{L_\infty(\frac{1}{4}D')} \leqslant C\left(\ln \frac{1}{h}\right)^{\overline{r}}\|u\|_{L_\infty(D)} + C\|u - u_h\|_{W_q^{-p}(D)}.$$

Theorem 5.1 follows from this.

It remains to prove Lemma 4.1. We shall, under the general assumptions of Theorem 5.1, prove the following:

LEMMA A.2. *Let $D' \subset\subset D$ be concentric spheres. If $2 \leqslant q < p \leqslant \infty$ with $1/q - 1/p < 1/N$, then*

$$\|u - u_h\|_{L_p(D')} \leqslant C(\|u\|_{W_\infty^1(D)} + \|u - u_h\|_{L_q(D)}).$$

Lemma A.1 follows from Lemma A.2, for, by iteration of Lemma A.2 one has

$$\|u - u_h\|_{L_\infty(\frac{1}{4}D)} \leqslant C\|u\|_{W_\infty^1(\frac{1}{2}D)} + C\|u - u_h\|_{L_2(\frac{1}{2}D)}.$$

By Lemma 3.1,

$$\|u - u_h\|_{L_2(\frac{1}{2}D)} \leqslant C\|u\|_{W^1_\infty(D)} + \|u - u_h\|_{W^{-q}_p(D)},$$

and thus, Lemma A.1 obtains.

Proof of Lemma A.2. Let K be such that A_K is coercive. Now,

$$A_K(u - u_h, \chi) = K(u - u_h, \chi) \quad \text{for } \chi \in \mathring{S}^h(D).$$

Let ψ be such that

$$A_K(\psi, v) = K(u - u_h, v) \quad \text{all } v \in H^1(D),$$

and $\psi_h \in S^h(D)$ such that $A_K(\psi - \psi_h, \chi) = 0$ for all $\chi \in S^h(D)$. Note that

$$A_K(u - u_h - \psi_h, \chi) = 0 \quad \text{for } \chi \in \mathring{S}^h(D),$$

and we may apply Theorem 5.1 in the coercive case to deduce

$$\|u - u_h - \psi_h\|_{L_\infty(D')} \leqslant C\|u\|_{W^1_\infty(D)} + C\|u - u_h\|_{0,D} + C\|\psi_h\|_{0,D}.$$

Since

$$\|\psi_h\|_{0,D} \leqslant C\|\psi_h\|_{1,D} \leqslant C\|u - u_h\|_{0,D}$$

and

$$\|u - u_h\|_{L_p(D')} \leqslant \|u - u_h - \psi_h\|_{L_p(D')} + \|\psi_h\|_{L_p(D')},$$

the desired result will follow if we can show that

(A.3) $\|\psi_h\|_{L_p(D')} \leqslant C\|u - u_h\|_{L_q(D)}$ for $2 \leqslant p \leqslant \infty$, $0 < 1/q - 1/p < 1/N$.

Since

$$\|\psi - \psi_h\|_{0,D} \leqslant \|\psi - \psi_h\|_{1,D} \leqslant C\|\psi\|_{1,D} \leqslant C\|\psi\|_{W^1_\infty(D)},$$

we obtain using Sobolev's lemma and elliptic regularity,

(A.4)
$$\|\psi_h\|_{L_\infty(D')} \leqslant \|\psi - \psi_h\|_{L_\infty(D')} + \|\psi\|_{L_\infty(D')}$$
$$\leqslant Ch\left(\ln \frac{1}{h}\right)^{\bar{r}} \|\psi\|_{W^1_\infty(D)} + C\|\psi - \psi_h\|_{0,D} + C\|\psi\|_{L_\infty(D')}$$
$$\leqslant C\|\psi\|_{W^1_\infty(D)} \leqslant C\|\psi\|_{W^2_s(D)} \leqslant C\|u - u_h\|_{L_s(D)} \quad \text{for } 1/s < 1/N.$$

Similarly,

(A.5)
$$\|\psi_h\|_{0,D'} \leqslant \|\psi\|_{1,D} \leqslant C\|\psi\|_{W^2_s(D)}$$
$$\leqslant C\|u - u_h\|_{L_s(D)} \quad \text{for } 1/s - 1/2 < 1/N.$$

By interpolation between (A.4) and (A.5) we obtain (A.3). As noted, this proves Lemma A.2.

This completes the proof of Theorem 5.1 in the case of noncoercive forms.

Appendix 2. Proof of (0.7). As mentioned in the Introduction, we shall take the form A to be coercive, i.e., satisfy (4.1). We assume that A.1, A.2 and A.3 hold (see Section 2), and furthermore that $S^h(\Omega) \subseteq H^2(\Omega)$. We shall then prove that if $A(u - u_h, \chi) = 0$ for all $\chi \in \overset{\circ}{S}{}^h(\Omega)$, then for $\Omega_1 \subset\subset \Omega$, $p \geqslant 0$, $1 \leqslant q \leqslant \infty$,

$$(0.7) \quad \|u - u_h\|_{L_2(\Omega_1)} \leqslant C(\|u - \chi\|_{L_2(\Omega)} + \|u - u_h\|_{W_q^{-p}(\Omega)}) \quad \text{for any } \chi \in S^h(\Omega).$$

Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega_4 \subset\subset \Omega$ where we may assume that $\partial\Omega_4$ is smooth. Let $\omega \in \overset{\circ}{C}{}^\infty(\Omega_3)$ with $\omega \equiv 1$ on Ω_2, and put $\tilde{u} = \omega u$. Let $\tilde{u}_h \in S^h(\Omega_4)$ satisfy $A(\tilde{u} - \tilde{u}_h, \chi) = 0$ for all $\chi \in S^h(\Omega_4)$, where the form A is now taken over Ω_4. Then

$$(A.6) \quad \|u - u_h\|_{0,\Omega_1} \leqslant \|\tilde{u} - \tilde{u}_h\|_{0,\Omega_1} + \|\tilde{u}_h - u_h\|_{0,\Omega_1}.$$

We shall estimate the two terms on the right-hand side. We have

$$(A.7) \quad \|\tilde{u} - \tilde{u}_h\|_{0,\Omega_4} = \sup_{\varphi \in \overset{\circ}{C}{}^\infty(\Omega_4)} \frac{|(\tilde{u} - \tilde{u}_h, \varphi)|}{\|\varphi\|_{0,\Omega_4}}.$$

For each such φ let $v \in H^2(\Omega_4)$ satisfy $A(\psi, v) = (\psi, \varphi)$ for all $\psi \in H^1(\Omega_4)$. Hence if $v_h \in S^h(\Omega_4)$ satisfies $A(\chi, v - v_h) = 0$ for all $\chi \in S^h(\Omega_4)$, we have

$$(\tilde{u} - \tilde{u}_h, \varphi) = A(\tilde{u} - \tilde{u}_h, v) = A(\tilde{u}, v - v_h).$$

Integrating by parts and using Schwarz' inequality, we obtain

$$(A.8) \quad |(\tilde{u} - \tilde{u}_h, \varphi)| = |(\tilde{u}, L^*(v - v_h))| \leqslant C\|\tilde{u}\|_{0,\Omega_3}\|v - v_h\|_{2,\Omega_3}.$$

For a suitable $\chi \in S^h(\Omega_4)$, given by Proposition 2.1, we have on using the inverse property A.2 and the fact that $\|v - v_h\|_{1,\Omega_4} \leqslant C \inf_{\chi \in S^h(\Omega_4)} \|v - \chi\|_{1,\Omega_4}$,

$$\|v - v_h\|_{2,\Omega_3} \leqslant \|v - \chi\|_{2,\Omega_3} + \|\chi - v_h\|_{2,\Omega_3}$$

$$\leqslant \|v - \chi\|_{2,\Omega_3} + Ch^{-1}\|\chi - v_h\|_{1,\Omega_4}$$

$$(A.9) \qquad \leqslant \|v - \chi\|_{2,\Omega_3} + Ch^{-1}\|v - \chi\|_{1,\Omega_4} + Ch^{-1}\|v - v_h\|_{1,\Omega_4}$$

$$\leqslant \|v - \chi\|_{2,\Omega_3} + Ch^{-1}\|v - \chi\|_{1,\Omega_4}$$

$$\leqslant C\|v\|_{2,\Omega_4} \leqslant C\|\varphi\|_{0,\Omega_4}.$$

In the last step we used elliptic regularity, cf. Lemma 4.1. It follows from (A.7), (A.8) and (A.9) that

$$(A.10) \quad \|\tilde{u} - \tilde{u}_h\|_{0,\Omega_4} \leqslant C\|u\|_{0,\Omega_4} \leqslant C\|u\|_{0,\Omega}.$$

For the second term on the right of (A.6), we note that $A(u_h - \tilde{u}_h, \chi) = 0$ for all $\chi \in \overset{\circ}{S}{}^h(\Omega_2)$. By (3.4) and (A.10) (and assuming without loss of generality that $q \leqslant 2$ below), we have

$$\|\widetilde{u}_h - u_h\|_{0,\Omega_1} \leqslant C\|\widetilde{u}_h - u_h\|_{W_q^{-p}(\Omega_2)}$$

(A.11) $$\leqslant C(\|\widetilde{u} - \widetilde{u}_h\|_{0,\Omega_2} + \|u - u_h\|_{W_q^{-p}(\Omega)})$$

$$\leqslant C(\|u\|_{0,\Omega} + \|u - u_h\|_{W_q^{-p}(\Omega)}).$$

Inserting (A.10) and (A.11) into (A.6) we obtain the result (0.7) with $\chi = 0$. The general case follows by writing $u - u_h = (u - \chi) - (u_h - \chi)$. This completes the proof of (0.7).

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