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Application of Method of Collocation on Lines for Solving Nonlinear Hyperbolic Problems

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Abstract. A collocation on lines procedure based on piecewise polynomials is applied to initial/boundary value problems for nonlinear hyperbolic partial differential equations. Optimal order a priori estimates are obtained for the error of approximation. The Crank-Nicholson discretization in time is studied and convergence rates of the collocation-Crank-Nicholson procedure are established. Finally, the superconvergence is verified at particular points for linear hyperbolic problems.

Introduction. We consider the nonlinear hyperbolic problem

$$p(x, t, u)D_t^2u - q(x, t, u)D_x^2u = f(x, t, u, D_xu), \qquad (x, t) \in (0, 1) \times (0, T],$$

subject to the initial conditions

$$u(x, 0) = u_0, \qquad D_tu(x, 0) = u_1, \qquad x \in (0, 1),$$

and to Dirichlet boundary conditions for $t > 0$. We examine the convergence of the collocation on lines procedure using piecewise polynomials with continuous first derivatives as the approximating functions.

In Section 4 we obtain optimal-order asymptotic estimates for the error of the approximation in the L_∞-norm. In Section 5, the Crank-Nicholson discretization of the resulting system of ordinary differential equations is studied and convergence rates of the collocation on lines–Crank-Nicholson procedure are established. Finally, in Section 6 the superconvergence phenomenon is established locally for a linear hyperbolic problem.

The method of collocation on lines was proposed first by Kantorovich [7]. The convergence of this method for a problem of mathematical physics was investigated by E. B. Karpilovskaya [8]. Yartsev [11], [10] proved convergence for linear elliptic and biharmonic type problems using trigonometric polynomials as basis functions. Douglas and Dupont [3], have studied the same method using piecewise cubic Hermite polynomials for a nonlinear parabolic problem and in [4] verified the superconvergence locally for the heat equation. Finally, Douglas and Dupont [5] generalized and extended their results in [3], [4]. The results in this paper are from the author's thesis [6].

1. Preliminary Results. Let $\Delta_x = (x_i)_0^N$ be a partition of $[0, 1]$, $I = [0, 1]$, $h_j \equiv |x_{j+1} - x_j|$, $I_j \equiv [x_j, x_{j+1}]$ and $h \equiv \max_j |x_{j+1} - x_j|$. Throughout this paper

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we denote by $\mathbf{P}_r$ the set of polynomials of degree less than r and $\mathbf{P}_{r,\Delta_x}$ the set of functions that are polynomials of degree $r-1$ in each subinterval $[x_i, x_{i+1}]$. We take $-1 < \rho_1 < \rho_2 < \cdots < \rho_k < 1$ and $w_j > 0, j = 1, \ldots, k$, to be Gaussian points and weights, respectively, so that

$$\int_{-1}^{+1} p(x)dx = \sum_{i=1}^{k} p(\rho_i)w_i, \qquad p \in \mathbf{P}_{2k}([-1, 1]).$$

The Gaussian points and weights in the subinterval $[x_j, x_{j+1}]$ are

$$\xi_{kj+i} \equiv (x_j + x_{j+1})/2 + \rho_i h_j/2, \qquad w_i^* = h_j w_i/2, \qquad i = 1, \ldots, k.$$

We introduce two pseudo-inner products corresponding to Gaussian quadrature and composite Gaussian quadrature:

$$(f, g)_{h_j} \equiv \frac{h_j}{2} \sum_{i=1}^{k} w_i f(\xi_{kj+i}) \cdot g(\xi_{kj+i}),$$

and

$$(f, g)_h \equiv \sum_{j=0}^{N-1} (f, g)_{h_j},$$

with

$$|f|_h \equiv \sum_{j=0}^{N-1} (f, f)_{h_j}.$$

For later use, we state without proof the lemmas:

LEMMA 1.1. *The seminorm* $|f|_h$ *is positive definite for all* $f \in \mathbf{P}_{k+2,\Delta_x} \cap C^1[0, 1]$ *with* $f(0) = f(1) = 0$.

LEMMA 1.2. *If* $f, g \in \mathbf{P}_{k+2,\Delta_x} \cap C^1[0, 1]$, *then*

$$-(D_x^2 f, g)_h = (D_x f, D_x g) - D_x f \cdot g|_0^1$$

$$+ \frac{(k+1)k}{(2k)!} \sum_j \frac{D_x^{k+1} f_j}{(k+1)!} \cdot \frac{D_x^{k+1} g_j}{(k+1)!} \int_{x_j}^{x_{j+1}} \prod_{i=1}^{k} (x - \xi_{kj+i})^2 dx. \tag{1.1}$$

LEMMA 1.3. *If* $f \in \{v \in \mathbf{P}_{k+2,\Delta_x} \cap C^1, v(0) = v(1) = 0\}$, *then*

$$(D_x f, D_x f) \leqslant -(D_x^2 f, f)_h \leqslant 2(D_x f, D_x f) \tag{1.2}$$

and

$$|D_x f|_h^2 \leqslant (D_x f, D_x f). \tag{1.3}$$

LEMMA 1.4. *If* $f \in \mathbf{P}_{k+2,\Delta_x} \cap C^1[0, 1]$, *then*

$$|f|_h \leqslant \lambda \|f\|_{L^2(I)}, \tag{1.4}$$

where λ *is the maximum eigenvalue of the matrix* $A_{k+1} \equiv [\Sigma_{l=1}^k w_l L_i(\rho_l) L_j(\rho_l)]$ *and* L_i *denotes the ith degree Legendre polynomials in* $[-1, 1]$.

Let H^k *be the Sobolev space of functions having* L^2*-derivatives of order* k *on* I *and* $H_0^k \equiv \{u \in H^k | u(0) = u(1) = 0\}$.

LEMMA 1.5. *For $f \in H^1$ we have*

$$(D_x f, D_x f) + |f|_h^2 \geqslant \tfrac{1}{4} \|f\|^2_{H^1(I)}. \tag{1.5}$$

The above lemmas are established in [6], proofs also appear in [5]. Lemmas 1.2, 1.3 and 1.5 have been first proved for the case of cubic Hermite polynomials by Douglas and Dupont [3].

2. Approximation Theory. In [6] we show that $R_k(x) \equiv D_x^k(1 - x^2)^{k+2}$, $k = 0, 1, \ldots,$ on $(-1, 1)$ are orthogonal polynomials. By Rodriques' formula we see that $D_x^2 R_k(x) = D_x^{k+2}(1 - x^2)^{k+2}$ is a multiple of the Legendre polynomial on the interval $(-1, 1)$. We now establish some properties of these polynomials.

LEMMA 2.1. *If $k \geqslant 3$,*

$$(D_x^\mu R_{k-2}, x^\nu)_h = 0, \qquad \mu = 0, 1, 2, \nu \leqslant \mu. \tag{2.1}$$

PROOF. Since $D_x^\mu R_{k-2} x^\nu$ is a polynomial of degree $K + 2 - \mu + \nu$, we have for $k \geqslant 3$,

$$(D_x^\mu R_{k-2}, x^\nu)_h = \int_{-1}^1 D_x^\nu R_{k-2} x^\nu dx.$$

Lemma 2.1 now follows by using integration by parts and the fact that $D_x^\mu R_{k-2}$ vanishes at $x = \pm 1$ and $D_x^2 R_{k-2}$ vanishes at the Gaussian points. Note that for $k \geqslant 2$,

$$(D_x R_{k-2}, 1)_h = (D_x^2 R_{k-2}, x^\nu)_h = 0.$$

We define an interpolation operator

$$T_h\colon C^1(I) \longrightarrow \mathbf{P}_{k+2,\Delta_x} \cap C^1(I)$$

such that

$$(T_h v)(x_l) = v(x_l),$$
$$(D_x T_h v)(x_l) = (D_x v)(x_l), \qquad l = 0, 1, \ldots, N,$$
$$(T_h v)(\tau_{i,j}) = v(\tau_{i,j}), \qquad i = 1, \ldots, k, j = 1, \ldots, N,$$

where $\tau_{i,j} \equiv x_j + \sigma_i(x_{j+1} - x_j)$ and the σ_i's are the roots in the interval $(0, 1)$ of the orthogonal polynomials $R_{k-2}(x)$.

LEMMA 2.2. *Assume that $u \in H^{k+4}(I)$ and let $e \equiv u - T_h u$. Then there is a constant K independent of h so that*

$$|D_x^l e|_h \leqslant K h^{k-l+2} \|u\|_{H^{k+2}(I)}, \qquad l = 0, 1,$$

$$|D_x^2 e|_h \leqslant K h^{k-l+1} \|u\|_{H^{k+3}(I)},$$

$$|(D_x e, 1)_h| \leqslant K h^{2k+5/2} \|u\|_{H^{k+3}(I)},$$

$$|(D_x^2 e, 1)_h| \leqslant K h^{2k+5/2} \|u\|_{H^{k+4}(I)}.$$

PROOF. It follows easily from Lemma 2.1 and Peano's Kernel Theorem [9].

3. Collocation on Lines. In this section we consider the problem of approximating the solution of the nonlinear hyperbolic equation

(3.1) $p(x, t, u)D_t^2u - q(x, t, u)D_x^2 u = f(x, t, u, D_x u), \quad (x, t) \in (0, 1) \times (0, T]$,

subject to the initial conditions

(3.2) $$u(x, 0) = \alpha_1(x), \quad D_t(x, 0) = \alpha_2(x), \quad 0 < x < 1,$$

and the boundary conditions

(3.3) $$u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t \leqslant T.$$

Assume that the coefficients satisfy

(3.4) $$0 < c_1 \leqslant p(x, t, u) \leqslant C_1, \quad c_2 \leqslant q(x, t, u) \leqslant C_2,$$

for $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$ and $-\infty < u < +\infty$. Also, we assume that p, q, f are continuously differentiable functions of their arguments and uniformly bounded.

Throughout, we assume that this problem has a solution, u.

Let $S_{\Delta_x} \equiv \mathbf{P}_{k+2,\Delta_x} \cap C^1$ and $S_{\Delta_x} \cap H_0^1$ be spanned by the basis functions $\{B_i\}_1^{kN}$. We seek an approximation $u_h(x, t)$ to u of the form

$$u_h(x, t) = \sum_{i=1}^{kN} \beta_i(t)B_i(x).$$

The coefficients $\{\beta_i(t)\}_{i=1}^{kN}$ as functions of time are the solutions of the nonlinear ordinary differential equations

(3.5) $$\{p(u_h)D_t^2u_h - q(u_h)D_x^2u_h - f(u_h, D_x u_h)\}(\xi_i, t) = 0,$$
$$0 < t \leqslant T,\ i = 1, \ldots, kN,$$

and

(3.6) $$u_h(\xi_i, 0) = \hat{\alpha}_1(\xi_i), \quad D_t u_h(\xi_i, 0) = \hat{\alpha}_2(\xi_i), \quad k = 1, \ldots, kN,$$

where $\hat{\alpha}_1, \hat{\alpha}_2$ are the S_{Δ_x}-interpolants of $\alpha_1(x), \alpha_2(x)$ respectively.

Although these are the equations which one solves in practice, the analysis is more conveniently made if one considers the equivalent problem of finding $u_h \in S_{\Delta_x} \cap H_0^1$ such that

(3.7) $$(p(u_h)D_t^2 u_h - q(u_h)D_x^2u_h - f(u_h, D_x u_h), B_i)_h = 0,$$
$$0 < t \leqslant T,\ i = 1, \ldots, kN,$$

and

(3.8) $$u_h(\xi_i, 0) = \hat{\alpha}_1(\xi_i), \quad D_t u_h(\xi_i, 0) = \hat{\alpha}_2(\xi_i), \quad i = 1, \ldots, kN.$$

LEMMA 3.1. *The collocation method* (3.5), (3.6) *and the discrete Galerkin method* (3.7), (3.8) *each possess a unique solution for* $0 < t \leqslant T$. *Moreover, these solutions are identical if the processes are started from the same initial values.*

PROOF. It follows from Lemma 4.1 in [5].

4. Error Analysis. In this section, we find a priori error bounds for the collocation on lines procedure. We consider the problem of finding $u_h \in S_{\Delta_x} \cap H_0^1$ such that

$$(4.1) \qquad (p(u_h)D_t^2 u_h - D_x^2 u_h - f(u_h, D_x u_h), v)_h = 0, \qquad 0 < t \leqslant T,$$

for all $v \in S_{\Delta_x} \cap H_0^1$.

In order to find estimates for the error $u - u_h$ in the L_∞-norm, we assume that $u(\cdot, t) \in C^1(I)$ and define $w(\cdot, t) \equiv T_h u$ which is in S_{Δ_x}. Then we find a priori bounds for the difference $w - u_h \in S_{\Delta_x}$; and applying known approximation results to the difference $u - w$, we obtain bounds for the error of the collocation on lines procedure.

If X is a normed space and ψ: $[0, T] \longrightarrow X$, define

$$\|\psi\|_{L^2(0,T;X)} = \int_0^T \|\psi(t)\|_X^2\, dt, \qquad \|\psi\|_{L^\infty(0,T;X)} = \sup_{0 \leqslant t \leqslant T} \|\psi(t)\|_X.$$

THEOREM 4.1. *If*

(i) *the coefficients in* (3.1) *have bounded third derivatives and satisfy conditions* (3.4),

(ii) $u \in L^\infty(0, T; H^{k+4})$, $D_t u \in L^2(0, T; H^{k+4})$ *and* $D_t^2 u \in L^2(0, T; H^{k+4})$,

(iii) $u_h(x, 0)$, $D_t u_h(x, 0)$ *are the* S_{Δ_x} *interpolants of* $u(x, 0)$ *and* $D_t u(x, 0)$, *respectively, then for the error of approximation we have*

$$\|u - u_h\|_{L^\infty(0,T;L^\infty)} \leqslant K[\|u\|_{L^\infty(0,T;H^{k+4}(I))} + \|D_t u\|_{L^2(0,T;H^{k+4}(I))} + \|D_t^2 u\|_{L^2(0,T;H^{k+4}(I))}]\, h^{k+2},$$

where K is a constant independent of h and u.

PROOF. Let $\eta \equiv u - w$ and $\zeta \equiv w - u_h$. Then (3.1), (4.1) imply that

$$(4.2) \qquad \begin{aligned} (p(u_h)D_t^2\zeta - D_x^2\zeta, v)_h &= (-p_u^1 \zeta D_t^2 w - p(w)D_t^2\eta - p_u^2 \eta D_x^2 u, v)_h \\ &\quad + (D_x^2\eta, v)_h + ([f(w, D_x u) - f(w, D_x w)], v)_h \\ &\quad + (f_u^1 \eta + f_u^2 \zeta + f_{D_x u}^3 D_x \zeta, v)_h. \end{aligned}$$

In (4.2) we choose $v = D_t\zeta$ and in [6] we show that

$$(4.3) \qquad \begin{aligned} &\tfrac{1}{2}[|\sqrt{p(u_h)}D_t\zeta|_h^2 + |\zeta|_h^2 - (D_x^2\zeta, \zeta)_h] \\ &\quad \leqslant K\int_0^t \{|\zeta|_h^2 + |D_x\zeta|_h^2\} d\tau + \int_0^t \{|\eta|_h^2 + |D_t^2\eta|_h^2\}\, d\tau + K\int_0^t |D_t\zeta|_h^2 \\ &\quad + K\{|\zeta|_h^2(0) - (D_x^2\zeta, \zeta)_h(0) + |\sqrt{p(u_h)}D_t\zeta|_h^2(0)\} \\ &\quad + \int_0^t (D_x^2\eta, D_t\zeta)_h\, d\tau + \int_0^t (f(w, D_x u) - f(w, D_x w), D_t\zeta)_h\, d\tau. \end{aligned}$$

Integration by parts gives

$$\int_0^t (D_x^2\eta, D_t\zeta)_h\, d\tau = (D_x^2\eta, \zeta)_h\big|_0^t - \int_0^t (D_t D_x^2\eta, \zeta)_h\, d\tau,$$

and

$$\int_0^t (f(w, D_x u) - f(w, D_x w), D_t \zeta)_h \, d\tau$$
$$= (f(w, D_x u) - f(w, D_x w), \zeta)_h |_0^t - \int_0^t (D_t\{f(w, D_x u) - f(w, D_x w)\}, \zeta)_h \, d\tau.$$

Using Poincaré's inequality, the elementary inequality $|cd| \leqslant (\frac{1}{4}p)c^2 + pd^2$ and Lemma 2.2 in [6] we have obtained

$$\left|\int_0^t (D_t D_x^2 \eta, \zeta)_h \, d\tau\right| \leqslant \frac{1}{16}\int_0^t [-(D_x^2 \zeta, \zeta)_h + |\zeta|_h^2] \, d\tau + K \sum_{i=1}^{N} h_j^{2k+4} \int_0^t \|D_t u(\cdot\,, \tau)\|^2_{H^{k+4}(I_j)} d\tau. \tag{4.4}$$

Using Taylor's theorem, we can easily show that

$$(f(w, D_x u) - f(w, D_x w), w - u_h)_h$$
$$= \sum_{j=1}^{N} (f_{D_x u}(w, D_x w)|_{x=\xi_{k(j-1)+1}} D_x \eta + \omega h_j D_x \eta, \zeta)_{h_j},$$

where ω is bounded independent of h_j. It follows from Lemma 2.2

$$|(f_{D_x u}|_{x=\xi_{k(j-1)+1}} D_x \eta, \zeta)_{h_j}|$$
$$\leqslant K h_j^{k+2} [\|u(\cdot\,, t)\|_{H^{k+3}(I_j)} |\zeta|_{h_j} + \|u(\cdot\,, t)\|_{H^{k+3}(I_j)} \|D_x \zeta(\cdot\,, t)\|_{L^2(I_j)}],$$

and

$$|(\omega h_j D_x \eta, \zeta)_{h_j}| \leqslant K h_j^{k+2} \|u(\cdot\,, t)\|_{H^{k+2}(I_j)} |\zeta|_{h_j}.$$

Moreover, we obtain

$$|(f(w, D_x u) - f(w, D_x w), \zeta)_h|$$
$$\leqslant \frac{1}{16} [-(D_x^2 \zeta, \zeta)_h + |\zeta|_h^2] + K \sum_{j=1}^{N} h_j^{2k+4} \|u(\cdot\,, t)\|^2_{H^{k+3}(I_j)}.$$

Following similar arguments as above, we show that

$$\int_0^t (D_t\{f(w, D_x u) - f(w, D_x w)\}, \zeta)_h \, d\tau \leqslant \frac{1}{16}\int_0^t [-(D_x^2 \zeta, \zeta)_h + |\zeta|_h^2] \, d\tau + K \sum_{j=1}^{N} h_j^{2k+4} \int_0^t [\|u(\cdot\,, \tau)\|^2_{H^{k+3}(I_j)} + \|D_t u(\cdot\,, \tau)\|^2_{H^{k+3}(I_j)}] \, d\tau. \tag{4.5}$$

It follows from (4.3)–(4.5), (1.3) and Gronwall's Lemma [6] that

$$\begin{aligned}
&\|w - u_h\|^2_{L^\infty(0,T;L^\infty)} \\
(4.6)\qquad &\leqslant K[\|(w - u_h)(\cdot\,,0)\|^2_{H^1(I)} + \|D_t(w - u_h)(\cdot\,,0)\|^2_{L^2(I)}] \\
&+ \sum_{j=1}^{N} h_j^{2k+4}\{\|u\|^2_{L^\infty(0,T;H^{k+4}(I_j))} \\
&+ \|D_t u\|^2_{L^2(0,T;H^{k+4}(I_j))} + \|D_t^2 u\|^2_{L^2(0,T;H^{k+4}(I_j))}\}.
\end{aligned}$$

It is an elementary consequence of Peano's Kernel Theorem that

$$(4.7)\qquad \|u - w\|^2_{L^\infty(0,T;L^\infty)} \leqslant K \sum_{j=1}^{N} h_j^{2k+4}\|u\|^2_{L^\infty(0,T;H^{k+4}(I_j))}.$$

Finally, from (4.6), (4.7) and assumption (iii) it follows that

$$\begin{aligned}
&\|u - u_h\|_{L^\infty(L^\infty(I))} \\
&\leqslant Kh^{k+2}[\|u\|_{L^\infty(H^{k+4}(I))} + \|D_t u\|_{L^2(H^{k+4}(I))} + \|D_t^2 u\|_{L^2(H^{k+4}(I))}].
\end{aligned}$$

This concludes the proof of Theorem 4.1.

5. Computational Considerations. In this section, we discuss the question of actually solving the system of ordinary differential equations (3.5), (3.6).

Let

$$\begin{aligned}
&u_h^j \equiv u_h^j(x) = u_h^j(x, t^j), \quad t^j \equiv j\Delta t, \quad \Delta t = T/N, \\
(5.1)\qquad &v^{j+1/2} \equiv (v^{j+1} + v^j)/2, \quad v^{j,1/4} \equiv \frac{1}{4}v^{j+1} + \frac{1}{2}v^j + \frac{1}{4}v^{j-1}, \\
&\partial_t v^{j+1/2} \equiv (v^{j+1} - v^j)/\Delta t, \quad \partial_t^2 v^j \equiv (v^{j+1} - 2v^j + v^{j-1})/(\Delta t)^2.
\end{aligned}$$

Then the Crank-Nicholson-Collocation approximation $\{u_h^j\}_0^N$ is defined such that

$$\begin{aligned}
&\text{(i)}\ \ \{p(t^j, u_h^{j,1/4})\partial_t^2 u_h^j - q(t^j, u_h^{j,1/4})D_x^2 u_h^{j,1/4} - f(t^j, u_h^{j,1/4}, D_x u_h^{j,1/4})\}(\xi_i) = 0, \\
(5.2)\qquad &\qquad\qquad\qquad i = 1, \ldots, kN,\ j = 0, \ldots, N-1, \\
&\text{(ii)}\ \ u_h^j(0) = u_h^j(1) = 0, \quad j = 0, \ldots, N.
\end{aligned}$$

At the end of this section we discuss the choice of u_h^0, u_h^1. In order to analyze the convergence of the solution of (5.2) we consider the equivalent to (5.2) normalized problem

$$\begin{aligned}
&(p(t^j, u_h^{j,1/4})\partial_t^2 u_h^j, v)_h - (D_x^2 u_h^{j,1/4}, v)_h = (f(t^j, u_h^{j,1/4}, D_x u_h^{j,1/4}), v)_h, \\
(5.3)\qquad &\qquad\qquad\qquad v \in S_{\Delta_x} \cap H_0^1,\ 0 \leqslant j < N.
\end{aligned}$$

Also, we introduce the notation

$$\|u\|^2_{L^2_{\Delta t}(0,T;X)} \equiv \sum_{0 \leqslant t^j \leqslant T} \|u^j\|_X^2\,\Delta t,$$

$$\|u\|^2_{L^\infty_{\Delta t}(0,T;X)} \equiv \max_{0 \leqslant t^j \leqslant T} \|u^j\|_X^2,$$

$$\|u\|^2_{\tilde{L}^2_{\Delta t}(0,T;X)} \equiv \sum_{0<t^j<T} \|u^j\|^2_X \Delta t.$$

THEOREM 5.1. *Assume the hypotheses* (i), (ii) *of Theorem* 4.1 *hold. Further, assume* $D_t^3 u$, D_t^4, u *are in* $L^\infty(0, T; L^2(I))$ *and*

$$\|(u_h - w)^{1/2}\|_{H^1(I)} + \|\partial_t(u_h - w)^{1/2}\|_{L^2(I)} = O(h^{k+2}).$$

For Δt *sufficiently small there exists a unique solution of the Crank-Nicholson-Collocation equations* (5.2) *and for the error of approximation we have*

$$\|u - u_h\|_{L^\infty_{\Delta t}(0,T;L^\infty)} \leqslant C(h^{k+2} + (\Delta t)^2),$$

where C depends on u and is independent of h, Δt.

PROOF. It is easily seen that a unique solution of (5.2) exists under assumption (i) and (3.3) for Δt sufficiently small. Throughout this proof we use the notation $w \equiv T_h u$, $\eta \equiv u - w$ and $\zeta \equiv u_h - w$. First, we observe that u satisfies

$$(5.4)\qquad (p(u^{j,1/4})\partial_t^2 u^j)_h - (D_x^2 u^{j,1/4}, v)_h = (f(u^{j,1/4}, D_x u^{j,1/4}), v)_h + (e^j, v)_h$$

for $v \in S_{\Delta_x} \cap H_0^1$, where $\|e^j\|_{L^2(I)} = O(\Delta t^2)\|D_t^4 u\|_{L^2(I)}$.

After straightforward calculations and the application of the Mean Value Theorem, we obtain

$$(5.5)\qquad \begin{aligned} &(p(u_h^{j,1/4})\partial_t^2 \zeta^j, v)_h - (D_x^2 \zeta^j, v) \\ &\quad = (p^* \zeta^{j,1/4} \partial_t^2 w^j, v)_h + (p(w^{j,1/4})\partial_t^2 \eta^j, v)_h \\ &\qquad + (p^{**}\eta^{j,1/4}\partial_t^2 u^j, v)_h + (e^j, v)_h - (D_x^2 \eta^{j,1/4}, v)_h \\ &\qquad + (f_u^* \eta^{j,1/4} + f_u^{**}\zeta^{j,1/4} + f^*_{D_x u} D_x \zeta^{j,1/4}, v)_h \\ &\qquad + (f(w^{j,1/4}, D_x w^{j,1/4}) - f(w^{j,1/4}, D_x u^{j,1/4}), v)_h. \end{aligned}$$

In (5.5), we choose as test function $v = (\zeta^{j+1} - \zeta^{j-1})/2t$ and then we obtain

$$(5.6)\qquad \begin{aligned} &\frac{1}{2\Delta t}\{[|\sqrt{p(u_h^{j,1/4})}\partial_t \zeta^{j+1/2}|_h^2 + |\zeta^{j+1/2}|_h^2 - (D_x^2 \zeta^{j+1/2}, \zeta^{j+1/2})_h] \\ &\qquad - [|\sqrt{p(u^{j,1/4})}\partial_t \zeta^{j-1/2}|_h^2 + |\zeta^{j-1/2}|_h^2 - (D_x^2 \zeta^{j-1/2}, \zeta^{j-1/2})_h]\} \\ &\leqslant C[|\zeta^{j+1/2}|_h^2 + |\zeta^{j-1/2}|_h^2 + |\partial_t^2 \eta^j|_h^2 + |\eta^{j,1/4}|_h^2 + |\partial_t \zeta^{j+1/2}|_h^2 \\ &\qquad + |\partial_t \zeta^{j-1/2}|_h^2 + |D_x \zeta^{j,1/4}|_h^2 + |e^j|_h^2] \\ &\quad + \left|\left(D_x^2 \eta^{j,1/4}, \frac{\zeta^{j+1} - \zeta^{j-1}}{2\Delta t}\right)_h\right| \\ &\quad + \left|\left(f(w^{j,1/4}, D_x w^{j,1/4}) - f(w^{j,1/4}, D_x u^{j,1/4}), \frac{\zeta^{j+1} - \zeta^{j-1}}{2\Delta t}\right)_h\right|, \end{aligned}$$

where C is a generic constant.

Following the same arguments as in Section 4 and using Lemma 2.2, we get

$$
\begin{aligned}
&\Delta t \sum_{j=1}^{n-1} \left(D_x^2 \eta^{j,1/4}, \frac{\zeta^{j+1} - \zeta^{j-1}}{2\Delta t} \right)_h \\
&\leqslant \frac{1}{\epsilon} \{ - (D_x^2 \zeta^{n-1/2}, \zeta^{n-1/2})_h + |\zeta^{n-1/2}|_h^2 - (D_x^2 \zeta^{1/2}, \zeta^{1/2})_h + |\zeta^{1/2}|_h^2 \} \\
&\quad + K \max_{0 \leqslant t^j \leqslant T} \sum_{i=1}^{N-1} h_i^{2k+4} \| u^j \|_{H^{k+4}(I_i)} + \max_{0 \leqslant t^j \leqslant T} \| e_1^j \|^2_{L^2(I)} \\
&\quad + \frac{1}{2\epsilon} \Delta t \sum_{j=1}^{n-1} \{ - (D_x^2 \zeta^{j+1/2}, \zeta^{j+1/2})_h - (D_x^2 \zeta^{j-1/2}, \zeta^{j-1/2})_h \\
&\qquad\qquad + |\zeta^{j+1/2}|_h^2 + |\zeta^{j-1/2}|_h^2 \} \\
&\quad + K \Delta t \sum_{j=1}^{n-1} \sum_{i=0}^{N-1} h_i^{2k+4} \| D_t^2 u^j \|^2_{H^{k+4}(I_i)} \\
&\quad + \Delta t \sum_{j=1}^{N-1} \| e_2^j \|^2_{L^2(I)},
\end{aligned} \tag{5.7}
$$

where $\|e_s^j\|_{L^2(I)} = O(\Delta t^2)$ for $s = 1, 2, K$ is a generic constant and ϵ a constant that can be small enough.

Finally, by arguments similar to those of Section 4 we can show that

$$
\begin{aligned}
&\Delta t \sum_{j=1}^{n-1} \left(f(w^{j,1/4}, D_x u^{j,1/4}) - f(w^{j,1/4}, D_x w^{j,1/4}), \frac{\zeta^{j+1/2} - \zeta^{j-1/2}}{\Delta t} \right)_h \\
&\leqslant \frac{1}{\epsilon} [- (D_x^2 \zeta^{n-1/2}, \zeta^{n-1/2})_h + |\zeta^{n-1/2}|_h^2 - (D_x^2 \zeta^{1/2}, \zeta^{1/2})_h + |\zeta^{1/2}|_h] \\
&\quad + K \max_{0 \leqslant t^j \leqslant T} \sum_{j=0}^{N-1} h_i^{2k+4} \| u^j \|_{H^{k+4}(I_i)} + \max_{0 \leqslant t^j \leqslant T} \| e_1^j \|^2_{L^2(I)} \\
&\quad + \frac{1}{\epsilon} \Delta t \sum_{j=1}^{n-1} [- (D_x^2 \zeta^{j+1/2}, \zeta^{j+1/2})_h + |\zeta^{j+1/2}|_h^2 \\
&\qquad\qquad - (D_x^2 \zeta^{j-1/2}, \zeta^{j-1/2})_h + |\zeta^{j-1/2}|_h^2] \\
&\quad + \Delta t \sum_{j=1}^{n-1} \| e_3^j \|^2_{H^{k+3}(I)} \\
&\quad + K \Delta t \sum_{j=1}^{n-1} \sum_{i=0}^{N-1} h_i^{2k+4} [\| u^j \|^2_{H^{k+3}(I_i)} + \| D_t^2 u^j \|^2_{H^{k+3}(I_i)}],
\end{aligned} \tag{5.8}
$$

where $e_3^j = O(\Delta t^2)$. From (5.6)–(5.8) and the discrete form of the Gronwall Lemma we derive in [6] the relation

$$
\begin{aligned}
&|\partial_t \zeta^{n-1/2}|_h^2 + |\zeta^{n-1/2}|_h^2 - (D_x^2 \zeta^{n-1/2}, \zeta^{n-1/2})_h \\
&\leq C\{-(D_x^2 \zeta^{1/2}, \zeta^{1/2})_h + |\zeta^{1/2}|_h^2 |\partial_t \zeta^{1/2}|_h^2\} \\
&+ K\Delta t \sum_{j=1}^{n-1} \sum_{i=0}^{N-1} h_i^{2k+4} [\|u^j\|_{H^{k+3}(I_i)}^2 + \|D_t^2 u^j\|_{H^{k+4}(I_i)}^2] \\
&+ C\Delta t \sum_{j=1}^{n-1} \{|\partial_t^2 \eta^j|_h^2 + |\eta^{j-1/2}|_h^2\} + \Delta t \sum_{j=1}^{n-1} [\|e_2^j\|_{L^2(I)}^2 + \|e_3^j\|_{L^2(I)}^2] \\
&+ K\left[\max_{0 \leq t^j \leq T} \sum_{i=0}^{N-1} h_i^{2k+4} \|u^j\|_{H^{k+4}(I_i)}^2 + \max_{0 \leq t^j \leq T} \|e_1^j\|_{L^2(I)}^2\right].
\end{aligned} \tag{5.9}
$$

Finally from Lemma 1.4, 2.2 and inequality (5.9), we conclude that

$$
\begin{aligned}
\|\zeta\|_{L^\infty_{\Delta t}(0,T;L^\infty)} &\leq C[\|\zeta^{1/2}\|_{H^1(I)} + \|\partial_t \zeta^{1/2}\|_{L^2(I)}] \\
&+ Kh^{k+2}[\|u\|_{\tilde{L}^2_{\Delta t}(0,T;H^{k+3}(I))} + \|D_t^2 u\|_{\tilde{L}^2(0,T;H^{k+4}(I))} \\
&+ \|u\|_{L^\infty_{\Delta t}(0,T;H^{k+4}(I))}] \\
&+ c(u)\Delta t^2,
\end{aligned} \tag{5.10}
$$

where C and K are generic constants independent of u, h, Δt and $c(u)$ independent of h, Δt. From the results of Section 2 we easily see that

$$
\|\eta\|_{L^\infty_{\Delta t}(0,T;L^\infty)} \leq Ch^{k+2}\|u\|_{L^\infty_{\Delta t}(0,T;H^{k+2})}. \tag{5.11}
$$

Therefore, the inequalities (5.10) and (5.11) imply

$$
\|u - u_h\|_{L^\infty_{\Delta t}(0,T;L^\infty)} \leq c(u)(h^{k+2} + (\Delta t)^2),
$$

provided

$$
\|\zeta^{1/2}\|_{H^1(I)} + \|\partial_t \zeta^{1/2}\|_{L^2(I)} \leq ch^{k+2},
$$

where $c(u)$ is independent of h and Δt. This concludes the proof of Theorem 5.1.

It remains to discuss the choice of u_h^0 and u_h^1. We choose $u_h^0 \equiv T_h u(x, 0)$ and $u_h^1 \equiv T_h \tilde{u}$ where

$$
\tilde{u} \equiv u(x, 0) + \Delta t D_t u(x, 0) + \frac{(\Delta t)^2}{2} D_t^2 u(x, 0) + \frac{(\Delta t)^3}{6} D_t^3 u(x, 0);
$$

the derivatives $D_t^2 u$ and $D_t^3 u$ are evaluated using the differential equation.

6. The Superconvergence Phenomenon. Consider the linear hyperbolic problem

$$
p(x, t)D_t^2 u - D_x^2 u = f(x, t), \quad (x, t) \in (0, 1) \times (0, T), \tag{6.1}
$$

subject to initial conditions

(6.2) $$u(x,0)=\varphi_1(x),\quad D_t u(x,0)=\varphi_2(x),\qquad 0\leqslant x\leqslant 1,$$

and boundary conditions

(6.3) $$u(0,t)=0,\quad u(1,t)=0,\qquad 0<t\leqslant T.$$

Also, we assume for all $(x,t)\in[0,1]\times[0,T]$,

(6.4) $$0<m<p(x,t)\leqslant M,\quad 0<m\leqslant q(x,t)\leqslant M.$$

Let u_h denote the collocation on lines approximation defined from (3.5) and (3.6) where p, q and f are independent of u. Throughout we denote by $L\equiv pD_t^2-D_x^2$, $\|u\|_{j,i}\equiv \sup\{D_x^\alpha D_t^\beta u(x,t)|x\in I, \alpha\leqslant j, \beta\leqslant i\}$ and $x_{i-1/2}\equiv(i-½)h_j$. By Peano's Kernel Theorem [9] we obtain

$$L(u-T_hu)(\zeta_{kj+i},t)$$
$$=\sum_{l=1}^{s-2}\{D_x^{k+l+1}D_t^2u(x_{j-½})\psi_l(\rho_i)-D_x^{k+l+3}u(x_{j-½})\psi''_{l+2}(\rho_i)\}h_j^{k+l+1}$$
$$-D_x^{k+3}u(x_{j-½})\psi''_2(\rho_i)h_j^{k+1}+O(h_j^{k+s}[\|u\|_{k+s+2,0}+\|u\|_{k+s+2,2}]),$$

where

$$\psi_i(x)=\frac{1}{(k+i+1)!}A_i(x)R_{k-2}(x)$$

with A_i a polynomial of degree $i-1$. In order to cancel the term of h_j^{k+1} accuracy we make a correction to T_hu defined locally by the following relations, $\delta_0(\cdot,t)\in \mathbf{P}_{k+2,\Delta_x}\cap C^1$ with

$$h^{s-1}D_x^2\delta_0(\xi_{kj+i},t)=D_x^{k+3}u(x_{j-½})\psi''_2(\rho_i),\qquad i=1,\ldots,k,\ j=0,\ldots,N-1,$$
$$\delta_0(x_j,t)=D_x\delta(x_j,t)=0,\qquad j=0,1,\ldots,N.$$

Now, in order to cancel the h_j^{k+l+1} order terms we define a new correction in the following way: first we introduce the function

$$v(y)=\begin{cases}0, & y\leqslant 0,\\ 3y^2-2y^3, & 0\leqslant y\leqslant 1,\\ 1, & 1\leqslant y,\end{cases}$$

which obviously belongs to C^1 and define for $x\in I_j$

$$E_j(x,t)\equiv\lambda_{1,l}D_x^{k+l+1}D_t^2u(x_{j-½})v\left(\frac{x-x_j}{h_j}\right)-\lambda_{2,l}D_x^{k+l+3}u(x_{j-½})v\left(\frac{x-x_j}{h_j}\right),$$

where $\lambda_{1,l}\equiv-\psi_l(\rho_i)/v''(\rho_i)$, $\lambda_{2,l}\equiv-\psi''_{l+2}(\rho_i)/v''(\rho_i)$. Also, we define

$$\delta_l(x, t) \equiv \sum_{j=0}^{N-1} h_j^{l+3-s}\{E_j(x, t) - xE_j(1, t)\}$$

$$= \sum_{j=0}^{N-1} \{\lambda_{1,l} D_x^{k+l+1} D_t^2 u(x_{j-1/2}) - \lambda_{2,l} D_x^{k+l+3} u(x_{j-1/2})\}\left(v\left(\frac{x - x_j}{h_j}\right) - x\right).$$

In [6] we show that the $\lambda_{\alpha,l}$ for $\alpha = 1, 2$ are well defined and easily obtain

$$L(u - \bar{u})\,(\xi_{kj+i}, t) = O(h_j^{k+s}[\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}]),$$

where

$$\bar{u} = T_h u + h_j^{k+s} \sum_{l=0}^{s-2} \delta_l.$$

THEOREM 6.1. *Let u denote the solution of the problem* (6.1) *to* (6.4) *such that $u \in L^\infty(0, T; H^{k+s+4})$, $s \leqslant k$ and u_h is the collocation on lines approximation of u defined by* (3.5), (3.6). *Then the error of approximation at the nodes satisfies*

$$\max_j \|(u - u_h)\,(x_j, \cdot)\|_{L^\infty(0,T)} \leqslant Ch^{k+s}[\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}]$$
$$+ C[\|D_t(u_h - \bar{u})\|_{L^2(I)}(0) + \|u_h - \bar{u}\|_{H^1(I)}(0)],$$

where C is a constant independent of u and h and $s \leqslant k$.

PROOF. We define

$$\rho(\xi_{kj+i}, t) \equiv L(u_h - \bar{u})\,(\xi_{kj+i}, t),$$

where

$$|\rho(\xi_{kj+i}, t)| \leqslant Ch_j^{k+s}[\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}];$$

and we form the relation

$$(\rho, D_t(u_h - \bar{u}))_h = (D_t^2(u_h - \bar{u}), D_t(u_h - \bar{u}))_h - (D_x^2(u_h - \bar{u}), D_t(u_h - \bar{u}))_h.$$

We apply the elementary inequality $\tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 \geqslant ab$ to obtain

$$|\rho|_h^2 + |D_t(u_h - \bar{u})|_h^2 \geqslant D_t|D_t(u_h - \bar{u})|_h^2 - D_t(D_x^2(u_h - \bar{u}), u_h - \bar{u})_h.$$

In the above inequality we add the inequality

$$\frac{1}{2}D_t|u_h - \bar{u}|_h^2 \leqslant \frac{1}{2}|D_t(u_h - \bar{u})|_h^2 + \frac{1}{2}|u_h - \bar{u}|_h^2$$

to obtain

$$|\rho|_h^2 + |u_h - \bar{u}|_h^2 + 2|D_t(u_h - \bar{u})|_h^2$$
$$\geqslant D_t\{|u_h - \bar{u}|_h^2 + |D_t(u_h - \bar{u})|_h^2\} - D_t(D_x^2(u_h - \bar{u}), u_h - \bar{u})_h.$$

We integrate from 0 to t and apply Gronwall's Lemma to get

$$C\int_0^T |\rho|_h^2(\tau)d\tau + |u_h - \bar{u}|_h^2(0) + |D_t(u_h - \bar{u})|_h^2(0) - (D_x^2(u_h - \bar{u}), u_h - \bar{u})_h(0)$$

$$\geqslant |u_h - \bar{u}|_h^2 + |D_t(u_h - \bar{u})|_h^2 - (D_x^2(u_h - \bar{u}), u_h - \bar{u})_h.$$

It follows from Lemmas 1.3, 1.4, 1.5 that

$$C\left\{\max_t |\rho|_h + \|D_t(u_h - \bar{u})\|_{L^2(I)}(0) + \|u_h - \bar{u}\|_{H^1(I)}(0)\right\}$$

$$\geqslant \|u_h - \bar{u}\|_{H^1(I)} + |D_t(u_h - \bar{u})|_h^2.$$

In particular, we have

$$C\left\{\max_t |\rho|_h + \|D_t(u_h - \bar{u})\|_{L^2(I)}(0) + \|u_h - \bar{u}\|_{H^1(I)}(0)\right\} \geqslant \|u_h - \bar{u}\|_{L^\infty(I)}.$$

It is easy to see that

$$|(u - \bar{u})(x_i, t)| \leqslant Ch^{k+s}[\|u\|_{k+s-1,2} + \|u\|_{k+s+1,0}],$$

where $h = \max_j h_j$. Consequently, we have

$$\max_t \max_{0\leqslant j\leqslant N} |(u - u_h)(x_j, t)| \leqslant Ch^{k+s}[\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}]$$

$$+ C\{\|D_t(u_h - \bar{u})\|_{L^2}(0) + \|u_h - \bar{u}\|_{H^1(I)}(0)\}.$$

This completes the proof of Theorem 6.1.

Now, we consider the problem of choosing initial values, in order to obtain maximum accuracy. It is clear that the following

$$u_h(x, 0) = T_h\varphi_1 + h^{k+s}\delta_0(x, 0), \quad D_tu_h(x, 0) = T_h\varphi_2 + h^{k+s}\delta_0(x, 0)$$

yields $(u_h - \bar{u})(0) = O(h^{k+s})$ in H^1 norm, and $D_t(u_h - \bar{u})(0) = O(h^{k+s})$ in L^2 norm.

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