

Uniform Convergence of Galerkin's Method for Splines on Highly Nonuniform Meshes

By Frank Natterer

Abstract. Different sets of conditions for an estimate of the form

$$\|y - y^\pi\|_{L_\infty(a,b)} \leq C \max_i h_i^{r+1} \|y^{(r+1)}\|_{L_\infty(I_i)}$$

to hold are given. Here, y^π is the Galerkin approximation to the solution y of a boundary value problem for an ordinary differential equation, the trial functions being polynomials of degree $\leq r$ on the subintervals $I_i = [x_i, x_{i+1}]$ of length h_i .

The sequence of subdivisions $\pi: x_0 < x_1 < \dots < x_n$ need not be quasi-uniform.

1. Introduction. This note is concerned with the numerical solution of the boundary value problem

$$(1.1) \quad \begin{aligned} Ly &= y^{(2m)} + \sum_{\nu=0}^{2m-1} a_\nu y^{(\nu)} = f \quad \text{in } (a, b), \\ y^{(\nu)}(a) &= y^{(\nu)}(b) = 0, \quad \nu = 0, \dots, m-1, \\ a_\nu &\in C^\nu(a, b), \quad \nu = 0, \dots, 2m-1, \end{aligned}$$

by projection methods such as Galerkin's method or collocation using splines as trial functions. By splines we mean the elements of

$$S(r, k, \pi) = \{v \in C^k(a, b) : v \in P_r \text{ in each subinterval of } \pi\}.$$

Here, $r > k \geq 0$ are integers, P_r denotes the set of polynomials of degree $\leq r$, and $\pi: a = x_0 < x_1 < \dots < x_n = b$ is a subdivision of $[a, b]$. With each π we associate the quantities

$$h_i = x_{i+1} - x_i, \quad I_i = [x_i, x_{i+1}], \quad |\pi| = \max_i h_i.$$

By Π we denote a set of subdivisions, and we put $\Pi_h = \{\pi \in \Pi : |\pi| < h\}$.

If r, k, Π are suitably chosen, then a typical error estimate for an approximate solution y^π calculated by some projection method reads as follows: There are $h > 0$, $C < \infty$ such that y^π is well defined for all $y \in C^{r+1}(a, b)$ and all $\pi \in \Pi_h$, and

$$(1.2) \quad \|y - y^\pi\|_{L_\infty(a,b)} \leq C |\pi|^{r+1} \|y^{(r+1)}\|_{L_\infty(a,b)}.$$

Such results have been obtained by de Boor-Swartz [3] for collocation methods and by Wheeler [8], Douglas-Dupont-Wahlbin [4] for Galerkin methods. The corresponding multidimensional results are due to Scott [7] and Nitsche [6].

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The exponent $r + 1$ of $|\pi|$ in (1.2) is best possible. This follows from the fact that the estimate

$$\inf_{v \in S(r, k, \pi)} \|y - v\|_{L_\infty(a, b)} \leq C |\pi|^{r+1} \|y^{(r+1)}\|_{L_\infty(a, b)}$$

with C independent of π , y is optimal as far as the exponent of $|\pi|$ is concerned. However, it has been shown by de Boor [1] that the sharper estimate

$$(1.3) \quad \inf_{v \in S(r, k, \pi)} \|y - v\|_{L_\infty(a, b)} \leq C \max_i h_i^{r+1} \|y^{(r+1)}\|_{L_\infty(I_i)}$$

holds with C independent of π , y , if either $r \geq 2k + 1$ or the mesh ratio h_i/h_j , $|i - j| \leq 1$ remains bounded. This estimate makes sure that local refinement of π at points where $y^{(r+1)}$ is large reduces the overall error. Thus it would be highly desirable to sharpen the estimate (1.2) in a corresponding fashion, i.e. to prove that we can replace (1.2) by

$$(1.4) \quad \|y - y^\pi\|_{L_\infty(a, b)} \leq C \max_i h_i^{r+1} \|y^{(r+1)}\|_{L_\infty(I_i)}.$$

For the Galerkin method, (1.4) follows easily from the work of Wheeler [8] in the case $m = 1$, $k = 0$. We will obtain (1.4) for arbitrary m , $r \geq 2m - 1$, $k = m - 1$. For $r \geq 2m - 1$, $k \geq m - 1$; we will prove (1.4) under some mild assumptions on Π which are satisfied e.g. for the highly nonuniform family of subdivisions of $[0, 1]$ given by $x_i = (i/n)^\alpha$, $i = 0, \dots, n$ with $\alpha \geq 1$ arbitrarily large. This proof will be based on estimates in weighted Sobolev spaces as used in Natterer [5] and Nitsche [6] in connection with L_∞ -estimates for the finite element method.

The estimate (1.4) follows immediately from (1.3) if the projection method is quasi-optimal in $L_\infty(a, b)$, i.e. if there is a constant C independent of π , y such that

$$\|y - y^\pi\|_{L_\infty(a, b)} \leq C \inf_{v \in S(r, k, \pi)} \|y - v\|_{L_\infty(a, b)}.$$

Unfortunately, there are many quite reasonable projection methods which are not quasi-optimal in $L_\infty(a, b)$. We therefore introduce in Section 2 a weaker condition, called local optimality, which still implies (1.4). In Sections 3 and 4 we give different sets of conditions for the local optimality of the Galerkin method.

We will show by an example (see Section 2) that, in general, collocation fails to be locally optimal. This matches with the fact that the best estimate for collocation methods obtained so far is

$$\|y - y^\pi\|_{L_\infty(a, b)} \leq C \max_i h_i^{r+1} \|y^{(r+1)}\|_{L_\infty(I_i)} + O(|\pi|^{r+2})$$

(see de Boor [2]). We therefore feel that for highly nonuniform meshes (as needed e.g. in an adaptive code), Galerkin's method may be superior to collocation. However, the numerical experience available so far does not allow any definite conclusion.

2. Locally Optimal Projection Methods. We begin with some notation:

E, F are normed linear spaces. We always assume that $E \subseteq C(a, b)$ is defined by smoothness and boundary conditions only. E is normed by

$$\|y\| = \|y\|_{L_\infty(a,b)} = \max_{a \leq x \leq b} |y(x)|.$$

For each subdivision π of $[a, b]$ we define the spaces

$$C_\pi^k = \{v \in C(a, b): v \in C^k(I_i), i = 0, \dots, n - 1\}$$

and the seminorms, respectively, norms

$$|y|_{l,\pi} = \max_i h_i^l \|y^{(l)}\|_{L_\infty(I_i)}, \quad \|y\|_{l,\pi} = \sum_{\nu=0}^l |y|_{\nu,\pi}.$$

A projection method for the approximate solution of $Ly = f$, $L: E \rightarrow F$ being a linear map, is defined by a set of subdivisions Π , integers $r > k \geq 0$, and a family of linear maps $(\psi^\pi)_{\pi \in \Pi}$, $\psi^\pi: F \rightarrow R^{d(\pi)}$, $d(\pi) = \dim(S^\pi)$, where $S^\pi = S(r, k, \pi) \cap E$. If for arbitrary $f \in F$ there is a unique $y^\pi \in S^\pi$ such that $\psi^\pi Ly^\pi = \psi^\pi f$, then we take y^π as an approximation to y . In that case we put $P^\pi y = y^\pi$. It is obvious that $P^\pi: E \rightarrow S^\pi$ is a projection.

Definition 2.1. A projection method is said to be locally optimal of order l if

- (i) $\forall \pi \in \Pi \ E \subseteq C_\pi^l$,
- (ii) there are constants $C < \infty, h > 0$ such that

$$\forall \pi \in \Pi_h \ \forall y \in E \quad \|P^\pi y - y\| \leq C \inf_{z \in S^\pi} \|z - y\|_{l,\pi}.$$

Remark. A projection method is locally optimal of order 0 if and only if it is quasi-optimal in L_∞ .

Definition 2.2. A set Π of subdivisions is called locally quasi-uniform if there is a constant C such that

$$\forall \pi \in \Pi \quad h_i/h_j \leq C \quad \text{if } |i - j| \leq 1.$$

The proofs of the following lemmas are based on Hermite interpolation in the case $r \geq 2k + 1$ and on the use of a local basis for $S(r, k, \pi)$ in the general case.

LEMMA 2.1. Assume that Π is locally quasi-uniform or $r \geq 2k + 1$. Then the norms $\|\cdot\|, \|\cdot\|_{l,\pi}$ are equivalent on $S(r, k, \pi)$ uniformly for $\pi \in \Pi$, i.e. there is a constant C such that

$$\forall \pi \in \Pi \quad \forall y \in S(r, k, \pi) \quad \|y\|_{l,\pi} \leq C \|y\|.$$

LEMMA 2.2. If Π is locally quasi-uniform or $r \geq 2k + 1$ and if a projection method for $Ly = f$ is locally optimal of order $l \leq r + 1$, then there are constants $C < \infty, h > 0$ independent of y such that

$$\forall \pi \in \Pi_h \ \forall y \in C^{r+1}(a, b) \quad \|y - y^\pi\| \leq C \max_i h_i^{r+1} \|y^{(r+1)}\|_{L_\infty(I_i)}.$$

Remark. The proof follows from Lemma 2.4 if $y \in C^{r+1}(a, b)$. A more careful analysis along the lines of the proof of Lemma 4.3 shows that $y \in C^{m-1}(a, b) \cap C_\pi^{r+1}$ is sufficient.

LEMMA 2.3. Assume that $E \subseteq C_\pi^l$ for each $\pi \in \Pi$. Then a projection method is locally optimal of order l if and only if there are constants $C < \infty, h > 0$ such that

$$\forall \pi \in \Pi_h \quad \|P^\pi y\| \leq C \|y\|_{I, \pi}.$$

As an easy consequence of Theorem 4.1 of [1] we obtain

LEMMA 2.4. *Let Π be locally quasi-uniform and $0 \leq \mu \leq \nu \leq r$. Then there are constants t, C such that for all $g \in C^\nu(a, b)$, $\pi \in \Pi$ there is $z \in S(r, k, \pi)$ satisfying*

$$\forall i \quad |(g - z)^{(\mu)}|_{L_\infty(I_i)} \leq C h_i^{\nu - \mu} \omega(g^{(\nu)}, h_i, I_i),$$

where I'_i is the union of I_i and at most t adjacent intervals, and

$$\omega(f, h, I) = \sup \{ |f(x) - f(x')| : |x - x'| \leq h, x, x' \in I \}.$$

The idea for the proof of the following lemma is well known and can be found e.g. in de Boor-Swartz [3].

LEMMA 2.5. *Assume that Π is locally quasi-uniform or $r \geq 2k + 1$ and $E \subseteq C^l_\pi$ for each $\pi \in \Pi$. Let $L = L_0 + L_1$ and assume that*

- (i) L_0^{-1}, L^{-1} are defined on F ,
- (ii) There is a constant C such that with $K = L_0^{-1}L_1$

$$\forall \pi \in \Pi \quad \forall y \in E$$

$$\|(Ky)^{(\nu)}\|_{L_\infty(I_i)} \leq C \left\{ \sum_{\mu=0}^{\nu-1} \|y^{(\mu)}\|_{L_\infty(I_i)} + \|y\| \right\}, \quad \nu = 0, \dots, l + 1.$$

Then, a projection method is locally optimal of order l for the operator L if this is true for L_0 .

Proof. Denote the projections associated with the projection method for L, L_0 by P^π, P_0^π , respectively. We first show that there are constants $C < \infty, h > 0$ such that

$$(2.1) \quad \forall \pi \in \Pi_h \quad \forall y \in S^\pi \quad \|P_0^\pi Ky - Ky\| \leq C |\pi| \|y\|.$$

Indeed, as the projection method is locally optimal of order l for L_0 , we have with suitable $C < \infty, h > 0$ for $\pi \in \Pi_h$

$$(2.2) \quad \|P_0^\pi Ky - Ky\| \leq C \inf_{z \in S^\pi} \|z - Ky\|_{I, \pi}.$$

From Lemma 2.4 we see that

$$(2.3) \quad \inf_{z \in S^\pi} \|z - Ky\|_{I, \pi} \leq C |Ky|_{l+1, \pi}.$$

By assumption (ii) and Lemma 2.1 we obtain

$$(2.4) \quad |Ky|_{l+1, \pi} \leq C |\pi| \|y\|_{I, \pi} \leq C |\pi| \|y\|.$$

Now (2.2)–(2.4) combine to yield (2.1).

Consequently, the operator $T: S^\pi \rightarrow S^\pi$ defined by

$$T = (I + P_0^\pi K)|_{S^\pi}$$

possesses for $|\pi| < h', h' > 0$ suitably chosen, an inverse because $I + K = L_0^{-1}L$ does; and there is $C < \infty$ such that

$$(2.5) \quad \forall \pi \in \Pi_h \quad \forall y \in S^\pi \quad \|T^{-1}y\| \leq C\|y\|.$$

We now show that the operator $Q = T^{-1}P_0^\pi(I + K)$ coincides with P^π for $|\pi| \leq h'$. Using $\psi^\pi L_0 P_0^\pi = \psi^\pi L_0$ and the identity

$$P_0^\pi(I + K)Q = P_0^\pi(I + K)$$

which can be verified by direct calculation, we get

$$\begin{aligned} \psi^\pi LQ &= \psi^\pi L_0(I + K)Q = \psi^\pi L_0 P_0^\pi(I + K)Q \\ &= \psi^\pi L_0 P_0^\pi(I + K) = \psi^\pi L_0(I + K) = \psi^\pi L. \end{aligned}$$

Thus P^π exists for $|\pi| < h'$ and $P^\pi = Q$. By (2.5), Lemma 2.3 and assumption (ii) we obtain for $y \in E$

$$\begin{aligned} \|P^\pi y\| &= \|T^{-1}P_0^\pi(I + K)y\| \leq C\|P_0^\pi(I + K)y\| \\ &\leq C\|(I + K)y\|_{I,\pi} \leq C\|y\|_{I,\pi} \end{aligned}$$

with C independent of π, y . The lemma follows by Lemma 2.3.

Example. In order to solve $y' = f, y(0) = 0$ by collocation with piecewise linears at Gaussian points, we put $r = 1, k = 0$ and

$$E = \{v \in C(0, 1): v' \text{ piecewise continuous, } v(0) = 0\},$$

$$F = \{g: g \text{ piecewise continuous in } [0, 1]\},$$

$$\forall g \in F \quad (\psi^\pi g)_i = g(x_{i+1/2} - 0), \quad i = 0, \dots, n - 1,$$

where $x_{i+1/2} = (x_i + x_{i+1})/2$. We obtain

$$(P^\pi y)(x_i) = \sum_{j=0}^{i-1} h_j y'(x_{j+1/2} - 0), \quad i = 0, \dots, n - 1.$$

For a uniform mesh, $h_i = h, i = 0, \dots, n - 1, n$ even, define $y \in E$ by

$$y(x) = \begin{cases} x/h, & 0 \leq x \leq h, \\ 1, & h \leq x \leq 5h/3, \\ 6 - 3x/h, & 5h/3 \leq x \leq 2h, \end{cases} \quad y(x + 2h) = y(x).$$

Then

$$(P^\pi y)(x_{2i}) = i, \quad \|y\| = 1, \quad h\|y'\| = 3,$$

hence an estimate of the form $\|P^\pi y\| \leq C\|y\|_{1,\pi}$ with C independent of y, π cannot hold. Thus, collocation is not locally optimal of order 1. Using a smoothing procedure, it is seen that it is not locally optimal of any order.

3. Local Optimality of Galerkin's Method in the Case $k = m - 1$. In order to apply the results of Section 2 to the Galerkin method for (1.1), we put

$$\begin{aligned} E &= H_0^m(a, b), \quad F = H^{-m}(a, b), \\ L_0 y &= y^{(2m)}, \quad L_1 y = \sum_{\nu=0}^{2m-1} a_\nu y^{(\nu)}. \end{aligned}$$

If $k = m - 1$, then $S^\pi = \{v \in S(r, k, \pi) : v^{(\nu)}(a) = v^{(\nu)}(b) = 0, \nu \leq m - 1\}$. For each $g \in F$ we define $(\psi^\pi g)_i = (s_i, g), i = 1, \dots, d(\pi)$, where $(,)$ denotes the pairing between E, F and $\{s_1, \dots, s_{d(\pi)}\}$ is a basis for S^π .

Assumption (ii) of Lemma 2.4 for $l = m - 1$ is an immediate consequence of the following lemma, the proof of which is left to the reader.

LEMMA 3.1. For $y \in H_0^m(a, b)$ let $z \in H_0^m(a, b)$ be the solution of $L_0 z = L_1 y$. Then there are functions $c_{\nu\mu}, c_\nu \in C(a, b)$ such that for $\nu \leq m + 1$

$$z^{(\nu)} = \sum_{\mu=0}^{\nu-1} c_{\nu\mu} y^{(\mu)} + c_\nu,$$

where $c_{\nu\mu}$ independent of y and $\|c_\nu\| \leq C\|y\|$ with C independent of y .

THEOREM 3.1. Assume that the homogeneous problem (1.1) has only the trivial solution. Then, the Galerkin's method for the solution of (1.1) is locally optimal of order $m - 1$ for $k = m - 1, r \geq 2m - 1$.

Proof. By Lemma 2.4 and Lemma 3.1 it suffices to consider the equation $L_0 y = f$. Generalizing an idea of Wheeler [8], we construct the Galerkin approximation y^π to y locally. For each $z \in H^m(I_i)$ we define $Q_i z \in P_r$ by

$$(3.2) \quad z - Q_i z \in H_0^m(I_i),$$

$$(3.3) \quad \forall v \in P_r \cap H_0^m(I_i) \int_{I_i} (z - Q_i z)^{(m)} v^{(m)} dx = 0.$$

$Q_i z$ is well defined. Indeed, as $r \geq 2m - 1$, the dimension $r + 1$ of P_r coincides with the number of conditions in (3.2), (3.3); and if $z = 0$, then the choice $v = Q_i z$ in (3.3) yields $(Q_i z)^{(m)} = 0$, hence $Q_i z = 0$. Furthermore, there is a constant C independent on I_i, z such that

$$(3.4) \quad \|Q_i z\|_{L_\infty(I_i)} \leq C \sum_{\nu=0}^{m-1} h_i^\nu \|z^{(\nu)}\|_{L_\infty(I_i)}.$$

If we can prove this estimate for $I_i = (0, 1)$, it follows for arbitrary intervals by homogeneity. For $I_i = (0, 1)$ it is seen from (3.2) and from (3.3) by integrating by parts m times that

$$Q_i z = \sum_{\nu=0}^{m-1} (\alpha_\nu z^{(\nu)}(0) + \beta_\nu z^{(\nu)}(1)) + \int_0^1 \gamma z(x) dx,$$

where $\alpha_\nu, \beta_\nu, \gamma \in P_r$ are independent of z . This proves (3.4).

If $z \in H_0^m(a, b)$, then we may define $Q^\pi z$ by piecing together the functions $Q_i z$. As $k = m - 1, Q^\pi$ is a projection from $H_0^m(a, b)$ into S^π . From (3.4) we conclude that there is a constant C independent of y, π such that $\|Q^\pi y\| \leq C\|y\|_{m-1, \pi}$. By Lemma 2.3 the proof is complete if we can show that $y^\pi = Q^\pi y$, i.e.

$$(3.5) \quad \forall v \in S^\pi \int_a^b (v - Q^\pi y)^{(m)} v^{(m)} dx = 0.$$

For each $v \in P_r$ we can find $v_i \in P_{2m-1}$ such that $v - v_i \in P_r \cap H_0^m(I_i)$. We now apply the definition of Q^π , integration by parts and (3.3) to get for each $v \in S^\pi$

$$\begin{aligned}
 \int_a^b (y - Q^\pi y)^{(m)} v^{(m)} dx &= \sum_{i=0}^{n-1} \int_{I_i} (y - Q_i y)^{(m)} v^{(m)} dx \\
 &= \sum_{i=0}^{n-1} (-1)^m \int_{I_i} (y - Q_i y) v^{(2m)} dx \\
 &= \sum_{i=0}^{n-1} (-1)^m \int_{I_i} (y - Q_i y) (v - v_i)^{(2m)} dx \\
 &= \sum_{i=0}^{n-1} \int_{I_i} (y - Q_i y)^{(m)} (v - v_i)^{(m)} dx \\
 &= 0.
 \end{aligned}$$

This proves the theorem.

4. Local Optimality of Galerkin's Method in the Case $k \geq m - 1$. For general r, k , we prove the following result, which is slightly milder than the preceding theorem.

THEOREM 4.1. *Assume that the homogeneous problem (1.1) has only the trivial solution. Let Π satisfy the following assumptions:*

(4.1) *For each $\epsilon > 0$ there is an integer l such that*

$$\forall \pi \in \Pi \quad \forall i, j \quad h_i / |x_i - x_j| \leq \epsilon \quad \text{if } |i - j| \geq l.$$

(4.2) *There are constants $C < \infty, \alpha < 0$ such that*

$$\forall \pi \in \Pi \quad \forall j \quad \sum_{i=0; i \neq j}^{n-1} \left(\frac{h_j}{h_i} \right)^{-2\alpha + 2m-1} \left(\frac{h_i}{|x_i - x_j|} \right)^{-2\alpha} \leq C.$$

Then, the Galerkin method for the solution of (1.1) is locally optimal of order m for $k \geq m - 1, r \geq 2m - 1$.

Examples. (1) Define a set Π of subdivisions $\Pi_n: 0 = x_0 < x_1 < \dots < x_n = 1, n = 1, 2, \dots$, as follows: Choose $\alpha \geq 1$ and put $x_i = (i/n)^\alpha, i = 0, \dots, n$. Then the hypotheses (4.1) and (4.2) of the theorem are satisfied.

(2) If we put $x_0 = 0, x_i = q^{n-i}, i = 1, \dots, n$ with $0 < q < 1$, then it is easily seen that (4.1) is not satisfied.

Theorem 4.1 will follow from certain estimates in weighted Sobolev spaces, the weight functions being defined by

$$(4.3) \quad p_\alpha(x) = (\rho^2 + (x - \bar{x})^2)^\alpha,$$

where α is a real number, $\rho > 0$ and $\bar{x} \in [a, b]$. The only estimate on p_α we need is

$$(4.4) \quad p_\alpha^{(\nu)} \leq C p_{\alpha-\nu/2}.$$

Note that the constant in this estimate depends neither on ρ nor on \bar{x} , but only on α, ν .

If π is a subdivision of $[a, b]$, we put

$$\kappa = \max_i h_i \|p_{-1/2}\|_{L_\infty(I_i)}.$$

Let j be the index defined by $x_j \leq \bar{x} < x_{j+1}$.

LEMMA 4.1. *If Π satisfies (4.1) and is locally quasi-uniform, then for each $\epsilon > 0$ there is $K > 0$ such that with $\rho = Kh_j$*

$$\forall \pi \in \Pi \quad \kappa \leq \epsilon.$$

Proof. If $\epsilon > 0$, then by (4.1) there is $l > 0$ such that for $x \in I_i$

$$h_i^2 p_{-1}(x) \leq (h_i/(\bar{x} - x))^2 \leq \epsilon^2 \quad \text{if } |i - j| > l.$$

If $|i - j| \leq l$, then $h_i \leq C^l h_j$ with C independent of l, i, j, π because Π is locally quasi-uniform; hence

$$h_i^2 p_{-1}(x) \leq K^{-2} (h_i/h_j)^2 \leq (C^l/K)^2.$$

The lemma follows by choosing $K = C^l/\epsilon$.

LEMMA 4.2. *If Π satisfies (4.1) and is locally quasi-uniform, then for each α there are positive constants K, C such that with $\rho = Kh_j$*

$$\forall \pi \in \Pi \quad \forall i \quad \max_{x \in I_i} p_\alpha(x) / \min_{x \in I_i} p_\alpha(x) \leq C.$$

Proof. Let p_α assume its maximal (minimal) value in I_i at $t_0(t_1)$. Then by the mean value theorem and by (4.4),

$$\begin{aligned} (4.6) \quad & |p_\alpha(t_0)p_\alpha(t_1)^{-1} - 1| \leq h_i p_\alpha(t_1)^{-1} \|p'_\alpha\|_{L_\infty(I_i)} \\ & \leq C p_\alpha(t_0)p_\alpha(t_1)^{-1} h_i \|p_{-1/2}\|_{L_\infty(I_i)}. \end{aligned}$$

By the preceding lemma we can choose K such that $Ch_i \|p_{-1/2}\|_{L_\infty(I_i)} \leq 1/2$, where C is the constant in (4.6). Then from (4.6) we get $p_\alpha(t_0)/p_\alpha(t_1) \leq 2$. This proves the lemma.

We now introduce in the Sobolev spaces $H^\nu(I), I$ an interval, the seminorms

$$|u|_{\nu, \alpha, I} = \left(\int_I p_\alpha \{u^{(\nu)}\}^2 dx \right)^{1/2}.$$

If $I = [a, b]$, we drop the index I . These seminorms are not to be confused with $|u|_{\nu, \pi}$ defined in Section 2, where π always denotes a subdivision. We also use the notation $|u|_{\nu, \alpha, I}$ if $u \in H^\nu(I_i \cap I)$ for $I_i \cap I \neq \emptyset$ in an obvious way.

LEMMA 4.3. *Let $r > k \geq m - 1$. If Π is locally quasi-uniform then there are constants C, t such that for all $g \in E \cap C^k(a, b) \cap C_\pi^{r+1}, \pi \in \Pi$ there is $z \in S^\pi$ satisfying*

$$\forall i \quad |g - z|_{m, 0, I_i} \leq Ch_i^{r+1-m} |g|_{r+1, 0, I_i'}$$

where I_i' is the union of I_i and at most t adjacent intervals.

Proof. The inequality follows immediately from Lemma 2.4 if $g \in H^{r+1}(a, b)$. The case considered here requires some extra work.

Approximating $g^{(k+1)}$ by $w \in S(r - k - 1, -1, \pi)$ and solving $v^{(k+1)} = w$, we find $v \in S(r, k, \pi)$ such that

$$\forall i \quad |g - v|_{k+1,0,I_i} \leq Ch_i^{r-k} |g|_{r+1,0,I_i'}$$

Since $g - v \in H^{k+1}(a, b)$, we can use Lemma 2.4 to obtain $u \in S(r, k, \pi)$ such that

$$\begin{aligned} \forall i \quad |g - v - u|_{m,0,I_i} &\leq Ch_i^{k+1-m} |g - v|_{k+1,0,I_i'} \\ &\leq Ch_i^{r+1-m} |g|_{r+1,0,I_i'} \end{aligned}$$

Putting $z = u + v$ proves the lemma.

LEMMA 4.4. *If Π satisfies (4.1) and is locally quasi-uniform, then for each α there are positive constants K, C such that with $\rho = Kh_j$*

$$\forall \pi \in \Pi \quad \forall u \in S^\pi \quad \inf_{z \in S^\pi} |p_\alpha u - z|_{m,-\alpha} \leq Ck \sum_{\nu=0}^m |u|_{\nu,\alpha-m+\nu}$$

Proof. It follows from Lemma 4.3 and Lemma 4.2 that there is a $z \in S^\pi$ such that

$$(4.7) \quad \forall i \quad |p_\alpha u - z|_{m,-\alpha,I_i} \leq Ch_i^{r+1-m} |p_\alpha u|_{r+1,-\alpha,I_i'}$$

Due to Lemma 4.2 we have for $u \in S^\pi$ the inverse estimate

$$|u|_{\nu,-\alpha,I_i'} \leq Ch_i^{\mu-\nu} |u|_{\mu,-\alpha,I_i'}, \quad \nu \geq \mu.$$

By Leibnitz' rule, (4.4) and the inverse estimate we obtain

$$\begin{aligned} h_i^{r+1-m} |p_\alpha u|_{r+1,-\alpha,I_i'} &\leq Ch_i^{r+1-m} \sum_{\nu=0}^r |p_\alpha^{(r+1-\nu)} u^{(\nu)}|_{0,-\alpha,I_i'} \\ &\leq Ch_i^{r+1-m} \sum_{\nu=0}^r |u|_{\nu,\alpha-r-1+\nu,I_i'} \\ &\leq C \left\{ h_i^{r+1-m} \sum_{\nu=0}^m |u|_{\nu,\alpha-r-1+\nu,I_i'} \right. \\ &\quad \left. + \sum_{\nu=m+1}^r h_i^{r+1-\nu} |u|_{m,\alpha-r-1+\nu,I_i'} \right\} \\ &\leq C \left\{ k^{r+1-m} \sum_{\nu=0}^m |u|_{\nu,\alpha-m+\nu,I_i'} + \sum_{\nu=m+1}^r k^{r+1-\nu} |u|_{m,\alpha,I_i'} \right\} \\ &\leq Ck \sum_{\nu=0}^m |u|_{\nu,\alpha-m+\nu,I_i'} \end{aligned}$$

The lemma follows.

Now let P_0^π be the projection associated with Galerkin's method for the solution of $L_0 y = f$ where L_0 is defined as in Section 3.

LEMMA 4.5. *Let $k \geq m - 1$. If Π is locally quasi-uniform and satisfies (4.1), then for each α there are positive constants C, K such that with $\rho = Kh_j$*

$$\forall \pi \in \Pi \quad \forall u \in H_0^m(a, b) \quad |P_0^\pi u|_{m,\alpha} \leq C\{|P_0^\pi u|_{0,\alpha-m} + |u|_{m,\alpha}\}.$$

Proof. As in Nitsche [6], we start out from the identity (integrals from a to b)

$$\begin{aligned} |P_0^\pi u|_{m,\alpha}^2 &= \int (P_0^\pi u)^{(m)} (p_\alpha P_0^\pi u)^{(m)} dx \\ &\quad - \sum_{\nu=0}^{m-1} \binom{m}{\nu} \int (P_0^\pi u)^{(m)} p_\alpha^{(m-\nu)} (P_0^\pi u)^{(\nu)} dx. \end{aligned}$$

With $z \in S^\pi$ it follows from the definition of P_0^π that the first term on the right-hand side of this identity becomes

$$\begin{aligned} (4.8) \quad \int (P_0^\pi u)^{(m)} (p_\alpha P_0^\pi u)^{(m)} dx &= \int (P_0^\pi u)^{(m)} (p_\alpha P_0^\pi u - z)^{(m)} dx + \int u^{(m)} z^{(m)} dx \\ &\leq |P_0^\pi u|_{m,\alpha} |p_\alpha P_0^\pi u - z|_{m,-\alpha} + |u|_{m,\alpha} |z|_{m,-\alpha}. \end{aligned}$$

By Lemma 4.4 we can find $z \in S^\pi$ such that

$$(4.9) \quad |p_\alpha P_0^\pi u - z|_{m,-\alpha} \leq C\kappa \sum_{\nu=0}^m |P_0^\pi u|_{\nu,\alpha-m+\nu}.$$

Using Leibnitz' rule and (4.4), we see that z also satisfies

$$(4.10) \quad |z|_{m,-\alpha} \leq |p_\alpha P_0^\pi u - z|_{m,-\alpha} + |p_\alpha P_0^\pi u|_{m,-\alpha} \leq C \sum_{\nu=0}^m |P_0^\pi u|_{\nu,\alpha-m+\nu}.$$

The second term in the identity can be estimated by (4.4):

$$(4.11) \quad \sum_{\nu=0}^{m-1} \binom{m}{\nu} \int (P_0^\pi u)^{(m)} p_\alpha^{(m-\nu)} (P_0^\pi u)^{(\nu)} dx \leq C|P_0^\pi u|_{m,\alpha} \sum_{\nu=0}^{m-1} |P_0^\pi u|_{\nu,\alpha-m+\nu}.$$

Using (4.8)–(4.11) in the identity yields

$$\begin{aligned} |P_0^\pi u|_{m,\alpha}^2 &\leq C \left\{ \kappa |P_0^\pi u|_{m,\alpha}^2 + |u|_{m,\alpha} |P_0^\pi u|_{m,\alpha} \right. \\ &\quad \left. + (|u|_{m,\alpha} + |P_0^\pi u|_{m,\alpha}) \sum_{\nu=0}^{m-1} |P_0^\pi u|_{\nu,\alpha-m+\nu} \right\}. \end{aligned}$$

From this inequality we remove the derivatives of order ν , $1 \leq \nu \leq m - 1$, by using repeatedly the estimate

$$|v|_{1,\beta+1} \leq C\{\epsilon|v|_{2,\beta+2} + \epsilon^{-1}|v|_{0,\beta}\},$$

which is easily established by (4.4) and integration by parts for all $v \in H_0^1(a, b) \cap H^2(a, b)$ and $0 < \epsilon < 1$, with C independent of ν, ϵ . We obtain for each $\epsilon > 0$,

$$\sum_{\nu=0}^{m-1} |P_0^\pi u|_{\nu,\alpha-m+\nu} \leq C\{\epsilon|P_0^\pi u|_{m,\alpha} + \epsilon^{1-m}|P_0^\pi u|_{0,\alpha-m}\};$$

hence

$$\begin{aligned} |P_0^\pi u|_{m,\alpha}^2 &\leq C\{(\kappa + \epsilon)|P_0^\pi u|_{m,\alpha}^2 + |u|_{m,\alpha} |P_0^\pi u|_{m,\alpha} \\ &\quad + \epsilon^{1-m}(|u|_{m,\alpha} + |P_0^\pi u|_{m,\alpha})|P_0^\pi u|_{0,\alpha-m}\}. \end{aligned}$$

By Lemma 4.1 we can choose K in such a way that $C(\kappa + \epsilon) \leq 1/2$ in this inequality. Applying the inequality $|ab| \leq \delta a^2/2 + b^2/(2\delta)$ in an appropriate manner completes the proof.

LEMMA 4.6. *Let $k \geq m - 1$ and $r \geq 2m - 1$. If Π is locally quasi-uniform and satisfies (4.1), then for each α there are positive constants C, K such that*

$$\forall \pi \in \Pi \quad \forall u \in H_0^m(a, b) \quad |P_0^\pi u|_{m, \alpha} + |P_0^\pi u|_{0, \alpha-m} \leq C\{|u|_{m, \alpha} + |u|_{0, \alpha-m}\}.$$

Proof. Let $w \in H_0^m(a, b)$ be the solution of

$$(-1)^m w^{(2m)} = p_{\alpha-m} P_0^\pi u.$$

Then, for each $z \in S^\pi$ we have

$$\begin{aligned} |P_0^\pi u|_{0, \alpha-m}^2 &= \int P_0^\pi u (-1)^m w^{(2m)} dx = \int (P_0^\pi u)^{(m)} w^{(m)} dx \\ &= \int (P_0^\pi u - u)^{(m)} (w - z)^{(m)} dx + \int u^{(m)} w^{(m)} dx \\ (4.12) \quad &= \int (P_0^\pi u - u)^{(m)} (w - z)^{(m)} dx + \int p_{\alpha-m} u P_0^\pi u dx \\ &\leq |P_0^\pi u - u|_{m, \alpha} |w - z|_{m, -\alpha} + |u|_{0, \alpha-m} |P_0^\pi u|_{0, \alpha-m}. \end{aligned}$$

By Lemma 2.4 and Lemma 4.2 we can find $z \in S^\pi$ such that

$$\begin{aligned} |w - z|_{m, -\alpha, I_i} &\leq Ch_i^m |w|_{2m, -\alpha, I_i'} \leq Ch_i^m |P_0^\pi u|_{0, \alpha-2m, I_i'} \\ &\leq Ck^m |P_0^\pi u|_{0, \alpha-m, I_i'}. \end{aligned}$$

Thus, cancelling one factor $|P_0^\pi u|_{0, \alpha-m}$ we obtain from (4.12)

$$|P_0^\pi u|_{0, \alpha-m} \leq Ck |P_0^\pi u - u|_{m, \alpha} + |u|_{0, \alpha-m}.$$

Now we estimate $|P_0^\pi u|_{m, \alpha}$ on the right-hand side of this inequality by means of Lemma 4.5 to get

$$|P_0^\pi u|_{0, \alpha-m} \leq C\{|u|_{m, \alpha} + |u|_{0, \alpha-m} + \kappa |P_0^\pi u|_{0, \alpha-m}\}.$$

By Lemma 4.1 we can choose K such that $Ck \leq 1/2$ in this inequality. It follows that

$$|P_0^\pi u|_{0, \alpha-m} \leq C\{|u|_{m, \alpha} + |u|_{0, \alpha-m}\}.$$

The result follows by Lemma 4.5.

Proof of Theorem 4.1. The above estimates in weighted L_2 -norms are shifted to the uniform norm by means of the obvious inequalities

$$\begin{aligned} p_\alpha(\bar{x}) ((P_0^\pi u)(\bar{x}))^2 &\leq Ch_j^{-1} |P_0^\pi u|_{0, \alpha, I_j}^2 \leq Ch_j^{2m-1} |P_0^\pi u|_{0, \alpha-m, I_j}^2 \\ &\leq Ch_j^{2m-1} |P_0^\pi u|_{0, \alpha-m}^2. \end{aligned}$$

We apply Lemma 4.6 to obtain

$$(4.13) \quad p_\alpha(\bar{x})((P_0^\pi u)(\bar{x}))^2 \leq Ch_j^{2m-1} \sum_{\nu=0}^m |u|_{\nu, \alpha-m+\nu}^2.$$

Observing that $p_\alpha(\bar{x}) = \rho^{2\alpha} = (Kh_i)^{2\alpha} \geq Ch_i^{2\alpha}$ and for $\alpha < 0$

$$\begin{aligned} |u|_{\nu, \alpha-m+\nu, I_i}^2 &\leq Ch_i \|p_{\alpha-m+\nu}\|_{L_\infty(I_i)} \|u^{(\nu)}\|_{L_\infty(I_i)}^2 \\ &= Ch_i^{1-2\nu} \|p_{\alpha-m+\nu}\|_{L_\infty(I_i)} |u|_{\nu, \pi}^2 \\ &\leq Ch_i^{1-2\nu} (h_j + |x_{i'} - x_{j'}|)^{2(\alpha-m+\nu)} |u|_{\nu, \pi}^2 \end{aligned}$$

with $|i - i'| \leq 1$, $|j - j'| \leq 1$, we obtain from (4.13)

$$|(P_0^\pi u)(\bar{x})|^2 \leq C \max_{\nu=0}^m \sum_{i=0}^{n-1} h_j^{-2\alpha+2m-1} h_i^{1-2\nu} (h_j + |x_{i'} - x_{j'}|)^{2(\alpha-m+\nu)} \|u\|_{m, \pi}^2.$$

By (4.2) the factor of $\|u\|_{m, \pi}^2$ on the right-hand side is bounded independently of u , π , \bar{x} if α is chosen such that the series (4.2) converges. Thus, we have obtained the final estimate

$$\|P_0^\pi u\| \leq C \|u\|_{m, \pi},$$

which shows that Galerkin's method is locally optimal of order m for L_0 . As in Section 3 we use Lemma 3.1 to verify assumption (ii) of Lemma 2.5 in the case $l = m$. Now Lemma 2.5 shows that Galerkin's method is locally optimal of order m for L , too.

Angewandte Mathematik und Informatik
Universität des Saarlandes
6600 Saarbrücken, West Germany

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