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On Maximal Finite Irreducible Subgroups of $GL(n, \mathbf{Z})$ II. The Six Dimensional Case

By Wilhelm Plesken and Michael Pohst*

Abstract. All maximal finite absolutely irreducible subgroups of $GL(6, \mathbf{Z})$ are determined up to Z-equivalence.

1. Introduction. As promised in Part I [7], we determine all maximal finite irreducible subgroups of $GL(6, \mathbf{Z})$ up to Z-equivalence. There are 17 Z-classes. A set of representatives of these classes is described in Section 4, Theorem (4.1). Here, and in the following, irreducibility always means absolute irreducibility unless otherwise stated. A detailed description of the methods and definitions can be found in Part I [7].

In Section 2 we compute the minimal irreducible finite subgroups of $GL(6, \mathbf{Z})$ up to Q-equivalence. This turns out to be more complicated than in the seven dimensional case, which is due to the fact that six is no prime number. We get 33 groups whereas there were only two minimal irreducible finite groups in the five and seven dimensional cases. The Z-classes of the natural representations of the 33 groups, respectively the $\prec$-maximal centerings of the corresponding representation modules, were electronically computed on the IBM 370/158 at the California Institute of Technology. They and the quadratic forms fixed by these groups under the computed representations can be found in Section 3. The Z-automorphs of these forms form a full set of representatives of the Z-classes of the maximal finite irreducible subgroups of $GL(6, \mathbf{Z})$. They are described in Section 4.

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2. The Minimal Irreducible Finite Subgroups of $GL(6, \mathbf{Z})$. Let G be a minimal irreducible subgroup of $GL(6, \mathbf{Z})$, Δ the natural representation of G, and N a maximal abelian normal subgroup of G. Applying Theorem (3.1) in [7] (an integral version of Clifford's Theorem), we may assume that the restriction $\Delta|_N$ is equal to $\Gamma_1 \dotplus \cdots \dotplus \Gamma_r$, where $\Gamma_1, \ldots, \Gamma_r$ are integral representations of N satisfying $\Gamma_i \sim_{\mathbf{Q}} k\Delta_i$ ($i = 1, \ldots, r$; $k \in \mathbf{N}$) and $\Gamma_1(N) = \cdots = \Gamma_r(N)$. (Transform Δ rationally if necessary.) The Δ_i ($i = 1, \ldots, r$) are inequivalent Q-irreducible integral representations of N all of the same degree m.

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Therefore, we have to consider all possible solutions of $6 = krm$ ($k, r, m \in \mathbf{N}$). They are:

	k	m	r
(i)	1	1	6
(ii)	1	6	1
(iii)	1	3	2
(iv)	1	2	3
(v)	2	1	3
(vi)	2	3	1
(vii)	3	1	2
(viii)	3	2	1
(ix)	6	1	1.

But $m = 3$ is impossible because of

(2.1) LEMMA. *$GL(n, \mathbf{Z})$ has no finite abelian $\mathbf{Q}$-irreducible subgroups if n is odd and greater than one.*

Proof. (Compare the proof of Theorem (4.1) in [7].) The enveloping algebra of such a group would be a cyclotomic field of degree n over $\mathbf{Q}$. But then n had to be even. Q.E.D.

Therefore, cases (iii) and (vi) do not occur. We discuss the remaining cases in succession.

Case (i). G is a subgroup of the full monomial group H_6. By Theorem (3.2) in [7] the image of the associated permutation representation φ: $G \longrightarrow \widetilde{S}_6$: $(g_{ij}) \mapsto (|g_{ij}|)$ is a minimal transitive permutation group of degree 6. An elementary computation shows that there exist three such groups up to equivalence:

$$P_1 = \langle D((123456)) \rangle \quad (\cong C_6),$$

$$P_2 = \langle D((135)(246)), D((14)(32)(56)) \rangle \quad (\cong S_3),$$

$$P_3 = \langle D((135)(246)), D((12)(34)) \rangle \quad (\cong A_4),$$

where D denotes the natural permutation representation of S_6, D: $S_6 \longrightarrow GL(6, \mathbf{Z})$ with $D(\pi)e_i = e_{\pi(i)}$ for $\pi \in S_6$, $i = 1, \ldots, 6$, $e_1, \ldots, e_6$ the standard basis of $\mathbf{Z}^{6\times 1}$. In each of the corresponding three subcases we have to determine the kernel of φ. The action of $\varphi(G)$ on the diagonal matrices in H_6 is the same as its natural action on $\mathbf{Z}_2^{6\times 1}$. The subspace V in $\mathbf{Z}_2^{6\times 1}$, which corresponds to $\ker \varphi$, has the following properties: the coordinate projections π_i: $V \longrightarrow \mathbf{Z}_2$: $(a_1, \ldots, a_6)^T \longrightarrow a_i$ are pairwise different ($i = 1, \ldots, 6$), and V is invariant under $\varphi(G)$. All the subspaces, which are invariant under $\varphi(G) = P_1, P_2, P_3$, respectively, can easily be obtained from the lattice of centerings of G_1, G_4, and G_9 in Chapter 3. We find the following possibilities for $N := \ker \varphi$:

(α) $\varphi(G) = P_1$,

$$N_1 = \langle g \operatorname{diag}(-1, 1, \ldots, 1) g^{-1} \mid g \in P_2 \rangle,$$

$$N_2 = \langle g \operatorname{diag}(-1,-1,-1,1,1,1)g^{-1} \mid g \in P_2\rangle,$$
$$N_3 = \langle g \operatorname{diag}(-1,1,-1,1,1,1)g^{-1} \mid g \in P_2\rangle,$$
$$N_4 = \langle g \operatorname{diag}(-1,-1,1,1,1,1)g^{-1} \mid g \in P_2\rangle.$$

Note: $N_3 \subset N_4 \subset N_1$ and $N_2 \subset N_1$.

(β) $\varphi(G) = P_2$,

$$N_5 = \langle g \operatorname{diag}(-1,1,\ldots,1)g^{-1} \mid g \in P_2\rangle,$$
$$N_6 = \langle g \operatorname{diag}(1,1,-1,-1,1,-1)g^{-1} \mid g \in P_2\rangle,$$
$$N_7 = \langle g \operatorname{diag}(-1,1,1,-1,1,-1)g^{-1} \mid g \in P_2\rangle,$$
$$N_8 = \langle g \operatorname{diag}(1,1,-1,-1,-1,1)g^{-1} \mid g \in P_2\rangle,$$
$$N_9 = \langle g \operatorname{diag}(-1,1,-1,1,1,1)g^{-1} \mid g \in P_2\rangle,$$
$$N_{10} = \langle g \operatorname{diag}(-1,-1,1,-1,1,-1)g^{-1} \mid g \in P_2\rangle.$$

Note: N_6, N_7, N_8 are contained in N_5, and $N_9 \subset N_{10} \subset N_5$.

(γ) $\varphi(G) = P_3$,

$$N_{11} = \langle g \operatorname{diag}(-1,1,1,1,1,1)g^{-1} \mid g \in P_2\rangle,$$
$$N_{12} = \langle g \operatorname{diag}(-1,1,1,-1,1,1)g^{-1} \mid g \in P_2\rangle,$$
$$N_{13} = \langle g \operatorname{diag}(-1,1,-1,1,-1,1)g^{-1} \mid g \in P_2\rangle,$$
$$N_{14} = \langle g \operatorname{diag}(1,-1,-1,1,-1,1)g^{-1} \mid g \in P_2\rangle,$$
$$N_{15} = \langle g \operatorname{diag}(-1,-1,1,1,1,1)g^{-1} \mid g \in P_2\rangle.$$

Note: $N_{12} \subset N_{11}$, $N_{13} \subset N_{15} \subset N_{11}$, $N_{14} \subset N_{15} \subset N_{11}$.

We now determine the groups G. If the extension of $\ker \varphi$ by $\varphi(G)$ splits, it follows from Theorem (3.2) in [7], that $\ker \varphi$ has to be minimal with the properties listed above. N_1, N_5, N_{11} consist of all diagonal matrices implying, that G splits over $\ker \varphi$. Since they are not minimal, they cannot occur as $\ker \varphi$. In the remaining cases we must look for the inverse images of the generators of $P_i = \varphi(G)$ ($i = 1, 2, 3$) under φ. They are of the form $g = dd_0\varphi(g)$ with a fixed diagonal matrix d_0 and d an arbitrary element in $\ker \varphi$.

In order to find a set of generators for G let $\varphi(G) = \langle \bar{g}_1, \ldots, \bar{g}_k\rangle$ together with defining relations $r_l(\bar{g}_1, \ldots, \bar{g}_k)$ ($l = 1, \ldots, s$) be a presentation of $\varphi(G)$. Choose one inverse image $g_i = d_i\bar{g}_i$ of $\bar{g}_i$, where d_i is taken from a fixed set of representatives of the group of all diagonal matrices over $\ker \varphi$ ($i = 1, \ldots, k$). The choice of the d_i is restricted by the conditions $r_l(g_1, \ldots, g_k) \in \ker \varphi$ ($l = 1, \ldots, s$). Then G is given by $\langle g_1, \ldots, g_k, n\rangle$ with a diagonal matrix n such that $\ker \varphi = \langle gng^{-1} \mid g \in \varphi(G)\rangle$. Of course, this procedure usually yields many **Q**-equivalent groups G. In general, this can be avoided by a conjugation of the g_i ($i = 1, \ldots, k$) by elements, which normalize $\ker \varphi$.

As an example, we consider an inverse image g of $D((135)(246))$ under φ in the cases $\varphi(G) = P_2$ and $\varphi(G) = P_3$. By the Schur-Zassenhaus Theorem $\langle g, \ker \varphi\rangle$ splits over $\ker \varphi$. Hence $g = \operatorname{diag}(a_1, \ldots, a_6)D((135)(246))$ can be chosen to be of order 3. The conjugation by $\operatorname{diag}(1, 1, a_1, a_2, a_1a_3, a_2a_4)$ yields $g = D((135)(246))$.

Similar computations lead to the following complete list of groups.

(2.2) LEMMA. *The minimal irreducible finite subgroups G_i of $GL(6, \mathbf{Z})$ in Case* (i) *up to* **Q**-*equivalence are*

$$(\alpha)\quad \varphi(G_i) = \langle g_1 = D((123456))\rangle \quad (i = 1, 2, 3),$$

$$G_1 = \langle g_1, \operatorname{diag}(-1, -1, -1, 1, 1, 1)\rangle \quad (\ker\varphi = N_2),$$

$$G_2 = \langle g_1, \operatorname{diag}(-1, 1, -1, 1, 1, 1)\rangle \quad (\ker\varphi = N_3),$$

$$G_3 = \langle \operatorname{diag}(-1, 1, 1, 1, 1, 1)g_1, \operatorname{diag}(-1, -1, 1, 1, 1, 1)\rangle \quad (\ker\varphi = N_4).$$

$$(\beta)\quad \varphi(G_i) = \langle g_1 = D((135)(246)), g_2 = D((14)(23)(56))\rangle \quad (i = 4, \ldots, 8),$$

$$G_4 = \langle g_1, g_2, \operatorname{diag}(1, 1, -1, -1, 1, -1)\rangle \quad (\ker\varphi = N_6),$$

$$G_5 = \langle g_1, g_2, \operatorname{diag}(-1, 1, 1, -1, 1, -1)\rangle \quad (\ker\varphi = N_7),$$

$$G_6 = \langle g_1, g_2, \operatorname{diag}(1, 1, -1, -1, -1, 1)\rangle \quad (\ker\varphi = N_8),$$

$$G_7 = \langle g_1, g_2, \operatorname{diag}(-1, 1, -1, 1, 1, 1)\rangle \quad (\ker\varphi = N_9),$$

$$G_8 = \langle g_1, \operatorname{diag}(-1, 1, 1, 1, 1, 1)g_2, \operatorname{diag}(-1, -1, 1, -1, 1, -1)\rangle \quad (\ker\varphi = N_{10}).$$

$$(\gamma)\quad \varphi(G_i) = \langle g_1 = D((135)(246)), g_2 = D((12)(34))\rangle \quad (i = 9, \ldots, 15),$$

$$G_9 = \langle g_1, g_2, \operatorname{diag}(-1, 1, 1, -1, 1, 1)\rangle \quad (\ker\varphi = N_{12}),$$

$$G_{10} = \langle g_1, g_2, \operatorname{diag}(-1, 1, -1, 1, -1, 1)\rangle \quad (\ker\varphi = N_{13}),$$

$$G_{11} = \langle g_1, \operatorname{diag}(1, 1, 1, 1, 1, -1)g_2, \operatorname{diag}(-1, 1, -1, 1, -1, 1)\rangle \quad (\ker\varphi = N_{13}),$$

$$G_{12} = \langle g_1, g_2, \operatorname{diag}(1, -1, -1, 1, -1, 1)\rangle \quad (\ker\varphi = N_{14}),$$

$$G_{13} = \langle g_1, \operatorname{diag}(1, 1, 1, 1, -1, 1)g_2, \operatorname{diag}(1, -1, -1, 1, -1, 1)\rangle \quad (\ker\varphi = N_{14}),$$

$$G_{14} = \langle g_1, \operatorname{diag}(1, 1, 1, -1, 1, 1)g_2, \operatorname{diag}(1, -1, -1, 1, -1, 1)\rangle \quad (\ker\varphi = N_{14}),$$

$$G_{15} = \langle g_1, \operatorname{diag}(1, 1, 1, -1, 1, -1)g_2, \operatorname{diag}(1, -1, -1, 1, -1, 1)\rangle \quad (\ker\varphi = N_{14}).$$

We remark that some of these groups may still be **Q**-equivalent.

Case (ii). In this case our normal subgroup N is **Q**-irreducible. Hence the enveloping algebra $E(N)$ of N in $\mathbf{Q}^{6\times 6}$ is simple and commutative, i.e. a field by Wedderburn's Theorem. Then N must be cyclic and $E(N)$ a cyclotomic field. Its degree is six, since $E(N) \subseteq \mathbf{Q}^{6\times 6}$ is irreducible. The degrees of cyclotomic fields are given by the Euler φ-function. So we must find all solutions of $\varphi(|N|) = 6$. They are $|N| = 7, 14, 9, 18$.

Let N be generated by an element g. There exists a matrix $x \in GL(6, \mathbf{C})$ with $xgx^{-1} = \operatorname{diag}(\zeta_1, \ldots, \zeta_6)$, where the ζ_i $(i = 1, \ldots, 6)$ are primitive rth roots of unity, $r \in \{7, 14, 9, 18\}$. Therefore, xGx^{-1} is an imprimitive group. Because of the irreducibility of G the ζ_i must be permuted transitively under conjugation by the

elements of xGx^{-1}. Hence G is isomorphic to an extension of the cyclic group N by the full automorphism group of N, which is cyclic of order 6. But there exist only the splitting extensions. Each of them has exactly one representation of degree 6.

(2.3). LEMMA. *The minimal irreducible finite subgroups of* $GL(6, \mathbf{Z})$ *in Case* (ii) *up to* **Q***-equivalence are*

$$G_{16} = \left\langle g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, D((132645)) \right\rangle \quad \text{of order } 7 \cdot 6,$$

$$G_{17} = \left\langle g_2 = \begin{pmatrix} 0 & 0 & B_1 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} B_2 & 0 & 0 \\ 0 & 0 & B_3 \\ 0 & B_2 & 0 \end{pmatrix} \right\rangle \quad \text{of order } 9 \cdot 6,$$

where

$$B_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. The maximal abelian normal subgroup N of G_{16} is generated by g_1 and is of order 7. $D((132645))$ conjugates g_1 into g_1^3 and corresponds to a generator of the automorphism group of N. Similar relations hold for G_{17}, where N has order 9. Clearly both groups are irreducible.

If N is of order 14 or 18, then G can be chosen as $\langle G_{16}, -I_6\rangle$, $\langle G_{17}, -I_6\rangle$, respectively. These groups are not minimal irreducible. Q.E.D.

Case (iv). We shall need

(2.4) LEMMA. *The* **Q***-irreducible abelian finite subgroups of* $GL(2, \mathbf{Z})$ *are up to* **Z***-equivalence*

$$J_1 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle (\cong C_4), \quad J_2 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle \ (\cong C_3), \text{ and}$$

$$J_3 = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle (\cong C_6).$$

Proof. Let N be such a group. Similar arguments as in Case (ii) show that N is cyclic of order $r = 4, 3$, or 6, which are all solutions of the equation $\varphi(r) = 2$. As generating matrices, we can choose the accompanying matrices of the corresponding cyclotomic polynomials. Because the class number in the rth cyclotomic field is one, a theorem of Latimer and McDuffee [11] proves that there is only one **Z**-class for these cyclic groups. Q.E.D.

The normalizers of J_1, J_2, J_3 in $GL(2, \mathbf{Z})$ are

$$\bar{J}_1 = \left\langle J_1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad \bar{J}_2 = \bar{J}_3 = \left\langle J_3, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

respectively. An easy computation shows, that the normalizers of these groups $\bar{J}_i$ in $GL(2, \mathbf{Q})$ already induce the full automorphism groups of the $\bar{J}_i$.

Next we describe our group G in terms of the maximal abelian normal subgroup N and the factor group G/N.

(2.5) LEMMA. *G/N is isomorphic to one of the groups $\bar{P}_1 = \langle D((145236))\rangle$, $\bar{P}_2 = \langle D((135)(246)), D((14)(32)(56))\rangle$, $\bar{P}_3 = \langle D((135)(246)), D((12)(34))\rangle$. N is contained in $J_i \dot{+} J_i \dot{+} J_i$ $(i \in \{1, 2, 3\})$, and G is a subgroup of $(J_i \dot{+} J_i \dot{+} J_i) \cdot P_j$ $(j \in \{1, 2, 3\})$. Moreover N satisfies:*

(1) *N is invariant under conjugation with $\bar{P}_j$,*

(2) *the projections Δ_ν: $N \longrightarrow J_i$: $\operatorname{diag}(A_1, A_2, A_3) \mapsto A_\nu$ $(\nu = 1, 2, 3)$ are surjective and pairwise $\mathbf{Q}$-inequivalent.*

If G splits over N, then N does not contain a proper subgroup with properties (1) and (2). (If $i = 3$, it may be changed to $i = 2$.)

Proof. Since we are in Case (iv), Lemma (2.4) implies that N is a subgroup of $J_i \dot{+} J_i \dot{+} J_i$ $(i \in \{1, 2, 3\})$ with property (2) and that the normalizer of N in $GL(6, \mathbf{Z})$ is contained in $\bar{J}_i \sim S_3$. Therefore, G is a subgroup of $\bar{J}_i \sim S_3$. Let M be the $\mathbf{C}G$-module belonging to the natural representation of G. As $\mathbf{C}N$-module M decomposes into the direct sum of six one dimensional inequivalent $\mathbf{C}N$-modules M_i $(i = 1, \ldots, 6)$. These are permuted by the elements of G. G is irreducible if and only if it permutes the M_i transitively. The kernel of this permutation representation is N and because G is minimal irreducible G/N has to be isomorphic to one of the groups C_6, S_3, A_4 listed as P_1, P_2, P_3 in Case (i).

For $g \in G \leqslant \bar{J}_i \sim S_3$ we have the unique factorization $g = g\bar{g}$ with

$$g \in J_i \dot{+} J_i \dot{+} J_i \ (i \in \{1, 3\}) \quad \text{and} \quad \bar{g} \in \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \sim S_3.$$

The first two assertions and property (1) of N follow easily. If G splits over N, then N is certainly minimal with properties (1) and (2). Q.E.D.

Since $D((135)(246))$ is contained in $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ we first determine all subgroups N of $J_i \dot{+} J_i \dot{+} J_i$ $(i = 1, 2, 3)$ which are invariant under conjugation by $\langle D((135)(246))\rangle$. They are:

(2.6) (α) $N \leqslant J_1 \dot{+} J_1 \dot{+} J_1$,

$$N_1 = \langle \operatorname{diag}(A, A, -I_2), \operatorname{diag}(A, -I_2, A)\rangle \quad \text{of order } 16,$$

$$N_2 = \langle \operatorname{diag}(\pm A, \pm A, \pm A)\rangle \quad \text{of order } 16,$$

$$N_3 = \langle \operatorname{diag}(I_2, A, A), \operatorname{diag}(A, I_2, A), \operatorname{diag}(A, A, I_2)\rangle \quad \text{of order } 32,$$

$$N_4 = J_1 \dot{+} J_1 \dot{+} J_1 \quad \text{of order } 64,$$

where $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ generates J_1.

(β) $N \leqslant J_2 \dot{+} J_2 \dot{+} J_2$,

$$N_5 = \langle \mathrm{diag}(B, B, B), \mathrm{diag}(B, B^2, I_2)\rangle \quad \text{of order } 9,$$
$$N_6 = J_2 \dot{+} J_2 \dot{+} J_2 \quad \text{of order } 27,$$

where $B = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ generates J_2.

(γ) $N \leqslant J_3 \dot{+} J_3 \dot{+} J_3$,

$$N_7 = \langle \mathrm{diag}(B, B, B)\rangle \cdot U_1 \quad \text{of order } 3 \cdot 4,$$
$$N_8 = \langle \mathrm{diag}(B, B, B)\rangle \cdot U_2 \quad \text{of order } 3 \cdot 8,$$
$$N_9 = N_5 \cdot \langle -I_6\rangle \quad \text{of order } 3^2 \cdot 2,$$
$$N_{10} = N_5 \cdot U_1 \quad \text{of order } 3^2 \cdot 4,$$
$$N_{11} = N_5 \cdot U_2 \quad \text{of order } 3^2 \cdot 8,$$
$$N_{12} = N_6 \cdot \langle -I_6\rangle \quad \text{of order } 3^3 \cdot 2,$$
$$N_{13} = N_6 \cdot U_1 \quad \text{of order } 3^3 \cdot 4,$$
$$N_{14} = J_3 \dot{+} J_3 \dot{+} J_3 \quad \text{of order } 3^3 \cdot 8,$$

where $U_1 = \langle \mathrm{diag}(-I_2, -I_2, I_2), \mathrm{diag}(-I_2, I_2, -I_2)\rangle$ and $U_2 = \langle \mathrm{diag}(\pm I_2, \pm I_2, \pm I_2)\rangle$.

Note $N_1 \subset N_3 \subset N_4$, $N_2 \subset N_4$, $N_5 \subset N_i$ ($i \in \{6, 9, 10, \ldots, 14\}$), $N_7 \subset N_i$ ($i \in \{8, 10, 11, 13, 14\}$).

All these groups N_i ($i = 1, \ldots, 14$) turn out to be invariant under $\bar{P}_1$ and $\bar{P}_2$. Under $\bar{P}_3$ only N_2, N_3, N_4, N_6, N_{12}, N_{13}, and N_{14} stay invariant.

Using (2.5) and (2.6), the groups G can be determined in the same way as in Case (i). ($J_1 \dot{+} J_1 \dot{+} J_1$, respectively, $J_3 \dot{+} J_3 \dot{+} J_3$ take the place of the group of all diagonal matrices.) Our remark about the normalizers of $\bar{J}_i$ ($i = 1, 2, 3$) in $GL(2, \mathbf{Q})$ enables us to detect some $\mathbf{Q}$-equivalent groups. We also eliminate groups, which are obviously $\mathbf{Q}$-equivalent to one of the groups $G_1, \ldots, G_{17}$. We end up with the following list:

(2.7) LEMMA. *The minimal irreducible finite subgroups of* $GL(6, \mathbf{Z})$ *in Case* (iv) *are* $\mathbf{Q}$*-equivalent to one of the groups* G_i ($i = 1, \ldots, 30$; $i \neq 16$). $G_{18}, \ldots, G_{30}$ *are given by*

(β_1) $G/N \cong \bar{P}_2$ *and* $N \leqslant J_1 \dot{+} J_1 \dot{+} J_1$,

$$G_{18} = \langle g_1, g_2, \mathrm{diag}(A, A, -I_2)\rangle \quad (\ker \varphi = N_1),$$
$$G_{19} = \langle g_1, \mathrm{diag}(A, I_2, I_2)g_2, \mathrm{diag}(A, A, -I_2)\rangle \quad (\ker \varphi = N_1),$$
$$G_{20} = \langle g_1, g_2, \mathrm{diag}(A, -A, A)\rangle \quad (\ker \varphi = N_2),$$

where $g_1 = D((135)(246))$, $g_2 = D((14)(32)(56))$, *and* $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *generates* J_1.

(γ_1) $G/N \cong \bar{P}_3$ *and* $N \leqslant J_1 \dot{+} J_1 \dot{+} J_1$,

$$G_{21} = \langle g_1, g_3, \mathrm{diag}(A, -A, A)\rangle \quad (\ker \varphi = N_2),$$
$$G_{22} = \langle \mathrm{diag}(A, I_2, I_2)g_1, g_3, \mathrm{diag}(A, -A, A)\rangle \quad (\ker \varphi = N_2),$$

where $g_3 = \mathrm{diag}(A, I_2, I_2)D((12)(34))$ *and* g_1, A *as in* (β_1).

(α_2) $G/N \cong \bar{P}_1$ *and* $N \leqslant J_2 \dot{+} J_2 \dot{+} J_2$,

$G_{23} = \langle D((145236)), \operatorname{diag}(B, B^2, I_2)\rangle$ $(\ker \varphi = N_5)$,

where $B = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ *generates* J_2.

(α_3) $G/N \cong \bar{P}_1$ *and* $N \leqslant J_3 \dot{+} J_3 \dot{+} J_3$,

$G_{24} = \langle D((145236)), \operatorname{diag}(-B, -B, B)\rangle$ $(\ker \varphi = N_7)$.

(β_2) $G/N \cong \bar{P}_2$ *and* $N \leqslant J_2 \dot{+} J_2 \dot{+} J_2$,

$G_{25} = \langle g_1, g_2, \operatorname{diag}(B, B^2, I_2)\rangle$ $(\ker \varphi = N_5)$,

$G_{26} = \langle g_1, \operatorname{diag}(B, I_2, I_2)g_2, \operatorname{diag}(B, B^2, I_2)\rangle$ $(\ker \varphi = N_5)$,

with g_1, g_2 *as defined in* (β_1).

(β_3) $G/N \cong \bar{P}_2$ *and* $N \leqslant J_3 \dot{+} J_3 \dot{+} J_3$,

$G_{27} = \langle g_1, g_2, \operatorname{diag}(-B, -B, B)\rangle$ $(\ker \varphi = N_7)$,

$G_{28} = \langle \operatorname{diag}(I_2, I_2, B)g_1, \operatorname{diag}(I_2, I_2, B)g_2, \operatorname{diag}(-B, -B, B)\rangle$ $(\ker \varphi = N_7)$.

(γ_2) $G/N \cong \bar{P}_3$ *and* $N \leqslant J_2 \dot{+} J_2 \dot{+} J_2$,

$G_{29} = \langle g_1, D((12)(34)), \operatorname{diag}(B, I_2, I_2)\rangle$ $(\ker \varphi = N_6)$,

$G_{30} = \langle g_1, \operatorname{diag}(I_2, -I_2, -I_2)D((12)(34)), \operatorname{diag}(B, I_2, I_2)\rangle$ $(\ker \varphi = N_6)$.

Some of these groups may still be **Q***-equivalent.*

Case (v). Clearly there are only two possibilities for our normal subgroup N:

$$N_1 = \langle \operatorname{diag}(\pm I_2, \pm I_2, \pm I_2)\rangle,$$

$$N_2 = \langle \operatorname{diag}(-I_2, -I_2, I_2), \operatorname{diag}(I_2, -I_2, -I_2)\rangle.$$

It is well known that there exist only two irreducible maximal finite subgroups of $GL(2, \mathbf{Z})$ up to $\mathbf{Z}$-equivalence:

$$\bar{J}_1 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \quad \text{and} \quad \bar{J}_2 = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

So G has to be $\mathbf{Z}$-equivalent to a subgroup of either $\bar{J}_1 \sim S_3$ or $\bar{J}_2 \sim S_3$. First, we show that $G \leqslant J_2 \sim S_3$ cannot occur. We consider all (2×2)-block diagonal matrices in $G \leqslant \bar{J}_2 \sim S_3$. The inertia group of Δ_1 (defined at the beginning of this chapter) contains the group of all block diagonal matrices as a subgroup of index 1 or 2. On the other hand 3 has to divide the order of the inertia group ($|\bar{J}_2| = 12$). Hence there is a diagonal matrix or order 3. So

$$(J_3 \dot{+} J_3 \dot{+} J_3) \cap G \quad \text{with } J_3 = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle$$

contains $N = N_1$ or $N = N_2$ properly and is an abelian normal subgroup of G_1 since J_3 is a characteristic subgroup of $\bar{J}_2$. This is a contradiction. In case $G \leqslant \bar{J}_1 \sim S_3$

after a suitable conjugation G must contain the matrix $D((135)(246))$. For G certainly contains a matrix $h = \mathrm{diag}(a, b, c)D((135)(246))$ of order 3 with $a, b, c \in \bar{J}_1$. Conjugation by $\mathrm{diag}(I_2, b^{-1}, a^{-1})$ leaves N invariant and transforms h into $D((135)(246))$.

Next we assume that there is no element $\mathrm{diag}(A, x, y)$ in G with $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $x, y \in \bar{J}_1$. Then the inertia group of Δ_1 (see above) must contain two elements

$$h_1 = \mathrm{diag}(A, u, v)D((35)(46)) \quad \text{and} \quad h_2 = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, u', v'\right)D((35)(46))$$

with $u, v, u', v' \in \bar{J}_1$. The product h_1h_2 is the block diagonal matrix

$$\mathrm{diag}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, uv', vu'\right).$$

According to our assumption uv' or vu' cannot be equal to A or A^{-1}. So uv' and vu' must be contained in

$$\left\{\pm I_2, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

Since all diagonal matrices in G form an abelian normal subgroup, uv' and vu' are certainly not in

$$\left\{\pm I_2, \pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

Inspecting the remaining possibilities, one easily gets a contradiction to the maximality of $N = N_1$ or $N = N_2$.

So there must be an element $\mathrm{diag}(A, x, y)$ in G with $x, y \in \bar{J}_1$. Similar considerations as above lead to one group G.

(2.8) LEMMA. *The minimal irreducible finite subgroups of* $GL(6, \mathbf{Z})$ *in case* (v) *are* $\mathbf{Q}$*-equivalent to one of the groups* $G_1, \ldots, G_{30}$ *or to*

$$G_{31} = \left\langle D((135)(246)),\ \mathrm{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\right\rangle.$$

Case (vii). N must be the group $\langle \mathrm{diag}(\pm I_3, \pm I_3)\rangle$. Then G is $\mathbf{Q}$-equivalent to a subgroup of $H_3 \sim C_2$, because there is only one maximal finite irreducible subgroup of $GL(3, \mathbf{Z})$ up to $\mathbf{Q}$-equivalence. (Note H_3 is the full monomial group of degree 3.) The inertia group I of Δ_1 (defined at the beginning of this section) consists of matrices $\mathrm{diag}(A_1, A_2)$ with A_1, A_2 in an irreducible subgroup S of H_3. I is a subdirect product of S with itself. So there exists an automorphism τ of S such that I contains $\mathrm{diag}(A, \tau(A))$ for all $A \in S$. The irreducible subgroups of $H_3 \cong C_2 \times S_4$ are isomorphic to $C_2 \times S_4$, $C_2 \times A_4$, S_4, or A_4. Therefore each automorphism of S maps the diagonal matrices of S on diagonal matrices. S certainly contains a diagonal matrix $d \neq \pm I_3$, but then $\mathrm{diag}(d, \tau(d)) \notin \langle \mathrm{diag}(\pm I_3, \pm I_3)\rangle$, which is a contradiction.

(2.9) LEMMA. *In case* (vii) *there are no minimal irreducible finite subgroups of* $GL(6, \mathbf{Z})$.

Case (viii). By Lemma (2.4) N must be $\mathbf{Z}$-equivalent to one of the three groups

$$N_1 = \langle \operatorname{diag}(A, A, A) \rangle \quad \text{with } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$N_2 = \langle \operatorname{diag}(B, B, B) \rangle \quad \text{with } B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

$$N_3 = \langle \operatorname{diag}(-B, -B, -B) \rangle.$$

At first we consider the centralizer $C(N)$ of N in $GL(6, \mathbf{Z})$. An easy computation yields that $C(N)$ consists of all matrices $(a_{ij})_{i,j=1,2,3}$ where the a_{ij} are in the enveloping algebra $E_{\mathbf{Z}}(N)$ of N in $\mathbf{Z}^{2\times 2}$ and $\det(a_{ij}) = \pm 1$. (Note $E_{\mathbf{Z}}(N)$ are the algebraic integers in the fourth respectively sixth cyclotomic field.) We conclude that the centralizer $C_G(N)$ of N in G has a faithful representation Ω: $C_G(N) \longrightarrow GL(3, \mathbf{C})$. This representation has to be irreducible, because G contains $C_G(N)$ of index 1 or 2 and the natural representation of $C_G(N)$ is $\mathbf{C}$-equivalent to $\Omega \dotplus \bar{\Omega}$. ($\bar{\Omega}$ is the complex conjugate representation of Ω.) The values of the character of Ω lie in the fourth respectively sixth cyclotomic field. Moreover, $C_G(N)$ is nonsolvable, since N is central in $C_G(N)$ and a maximal abelian normal subgroup of $C_G(N)$ over N. An inspection of the character tables [9], [10] of the nonsolvable complex linear groups of third degree [1] shows, that there are no groups with these conditions.

(2.10) LEMMA. *In case* (viii) *there are no minimal irreducible finite subgroups of* $GL(6, \mathbf{Z})$.

Case (ix). In this case we get $N = \langle I_6 \rangle$ or $N = \langle -I_6 \rangle$ for our maximal abelian normal subgroup N of G. Hence N is the center of G, and G is nonsolvable.

At first we assume that $SL(6, \mathbf{Z}) \cap G =: \tilde{G}$ is reducible. By Clifford's Theorem $\tilde{G}$ is $\mathbf{C}$-equivalent to a group of the form $\{\operatorname{diag}(\Omega_1(g), \Omega_2(g)) | g \in G\}$, where the Ω_i $(i = 1, 2)$ are irreducible faithful representations of degree 3, such that the natural representation of $\tilde{G}$ is $\mathbf{C}$-equivalent to $\Omega_1 \dotplus \Omega_2$. The center of $\Omega_i(G)$ $(i = 1, 2)$ is of order 1 or 2. From [1], [9], [10] we get $\tilde{G} \cong A_5$ or $\tilde{G} \cong PSL(2, 7)$. While the first possibility leads to a minimal irreducible group $G_{32} \cong S_5$, the second contains the irreducible group G_{16}, as we shall show in Section 4.

Now let G be contained in $SL(6, \mathbf{Z})$. If G has a proper normal subgroup U, such that the natural representation Δ of G splits into κ irreducible nonequivalent $\mathbf{C}$-representations: $\Delta|_U \sim_C \Omega_1 \dotplus \cdots \dotplus \Omega_\kappa$, then κ has to be 2, 3, or 6. For $\kappa = 6$ the group G must be solvable (G is minimal irreducible!). For $\kappa = 3$ the normal subgroup U cannot be solvable. But there is only one nonsolvable finite subgroup of $GL(2, \mathbf{C})$ [1], which is isomorphic to the icosahedron group. The corresponding representation of the icosahedron group has a nonrational character, which leads to a contradiction. For $\kappa = 2$ we conclude as in the first part of this case. At last we must consider the possibility that G is quasiprimitive as defined in [5]. From the

complete list of these groups in [5] we have to take only those, which are minimal irreducible and whose natural representation can be made rational. By [6] the order of G must divide $2^{10}3^45 \cdot 7$. Further the center of G must be of order 1 or 2, and G must not contain an abelian normal subgroup unequal $\langle I_6 \rangle$ or $\langle -I_6 \rangle$. There are the following groups left:

I. $G/N \cong A_5 \times V$ where $V \cong PSL(2, 7)$, A_5 or A_6,
II. $G/N \cong A_5$ or S_5, $N = \langle -I_6 \rangle$,
III. $G/N \cong A_7$ or S_7, $N = \langle I_6 \rangle$,
IV. $G/N \cong PSL(2, 7)$, $N = \langle I_6 \rangle$,
V. $G/N \cong PSU_4(2)$, $N = \langle I_6 \rangle$,
VI. $G/N \cong U_3(3)$, $N = \langle I_6 \rangle$.

We can eliminate most of the cases.

Ad I. These groups have no rational characters.

Ad II. The corresponding characters are of the second kind, hence these groups cannot be subgroups of $GL(6, \mathbf{Z})$.

Ad III. Since the group $PSL(2, 7)$ has a doubly transitive permutation representation, A_7 has a subgroup isomorphic to $PSL(2, 7)$. So $G \cong A_7$ or S_7 cannot be minimal irreducible subgroups of $GL(6, \mathbf{Z})$.

Ad V. We shall prove in Section 4 that this group contains a proper irreducible subgroup.

Ad VI. From the character table of this group [3] we see that $G \cong U_3(3)$ has no rational irreducible representation of degree six.

In the remaining cases we get new groups G.

(2.11) LEMMA. *The minimal irreducible finite subgroups of* $GL(6, \mathbf{Z})$ *in case* (ix) *are* $\mathbf{Q}$*-equivalent to one of the following groups*:

$$G_{32} = \left\langle g_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}, \; g_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

$$G_{32} \cong S_5,$$

$$G_{33} = \left\langle D((1237)(46)), g_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \right\rangle, \qquad G_{33} \cong PSL(2, 7).$$

3. Computation of the Z-Classes. The centerings of the groups G_i ($i = 1, \ldots, 33$) yield a set of representatives of the **Z**-classes of the natural representations of the G_i as we described in [7]. These representatives fix uniquely determined positive definite primitive integral quadratic forms F. At first, we list the matrices of all occuring forms. The **Z**-automorphs of these forms are the maximal finite irreducible subgroups of $GL(6, \mathbf{Z})$.

There are 17 forms F. J_n denotes the $n \times n$ matrix with all entries equal to 1.

$$F_1 = I_6, \qquad F_2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix},$$

$$F_4 = (I_3 + J_3) \dot{+} (I_3 + J_3), \qquad F_5 = (4I_3 - J_3) \dot{+} (4I_3 - J_3),$$

$$F_6 = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & -1 \\ 1 & 1 & 1 & 3 & 1 & -1 \\ 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & -1 & -1 & 1 & 3 \end{pmatrix}, \qquad F_7 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \dot{+} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \dot{+} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

$$F_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \qquad F_9 = \begin{pmatrix} 4 & 1 & -2 & -2 & 1 & 1 \\ 1 & 4 & 1 & -2 & -2 & 1 \\ -2 & 1 & 4 & 1 & -2 & -2 \\ -2 & -2 & 1 & 4 & 1 & -2 \\ 1 & -2 & -2 & 1 & 4 & 1 \\ 1 & 1 & -2 & -2 & 1 & 4 \end{pmatrix},$$

$$F_{10} = \begin{pmatrix} 4 & 2 & 2 & -2 & 1 & 1 \\ 2 & 4 & 2 & 1 & -2 & 1 \\ 2 & 2 & 4 & 1 & 1 & -2 \\ -2 & 1 & 1 & 4 & 2 & 2 \\ 1 & -2 & 1 & 2 & 4 & 2 \\ 1 & 1 & -2 & 2 & 2 & 4 \end{pmatrix}, \qquad F_{11} = \begin{pmatrix} 6 & -2 & -2 & -3 & 1 & 1 \\ -2 & 6 & -2 & 1 & -3 & 1 \\ -2 & -2 & 6 & 1 & 1 & -3 \\ -3 & 1 & 1 & 6 & -2 & -2 \\ 1 & -3 & 1 & -2 & 6 & -2 \\ 1 & 1 & -3 & -2 & -2 & 6 \end{pmatrix},$$

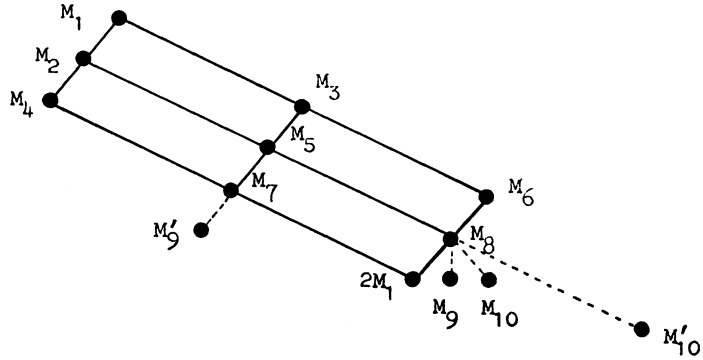
$$F_{12} = 7I_6 - J_6, \quad F_{13} = I_6 + J_6, \quad F_{14} = \begin{pmatrix} 4 & -1 & -2 & 1 & 1 & -2 \\ -1 & 4 & -1 & -2 & 1 & 1 \\ -2 & -1 & 4 & -1 & -2 & 1 \\ 1 & -2 & -1 & 4 & -1 & -2 \\ 1 & 1 & -2 & -1 & 4 & -1 \\ -2 & 1 & 1 & -2 & -1 & 4 \end{pmatrix},$$

$$F_{15} = \begin{pmatrix} 3 & -1 & -1 & 1 & 1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 1 \\ -1 & -1 & 3 & 0 & -1 & -1 \\ 1 & -1 & 0 & 3 & -1 & 1 \\ 1 & 0 & -1 & -1 & 3 & -1 \\ 0 & 1 & -1 & 1 & -1 & 3 \end{pmatrix}, \quad F_{16} = \begin{pmatrix} 4 & 1 & 2 & 2 & 1 & 0 \\ 1 & 4 & 1 & 2 & 2 & 1 \\ 2 & 1 & 4 & 1 & 2 & -1 \\ 2 & 2 & 1 & 4 & 1 & -1 \\ 1 & 2 & 2 & 1 & 4 & 1 \\ 0 & 1 & -1 & -1 & 1 & 4 \end{pmatrix},$$

$$F_{17} = \begin{pmatrix} 5 & 1 & -1 & -1 & 1 & 2 \\ 1 & 5 & 1 & -1 & -1 & 2 \\ -1 & 1 & 5 & 1 & -1 & 2 \\ -1 & -1 & 1 & 5 & 1 & 2 \\ 1 & -1 & -1 & 1 & 5 & 2 \\ 2 & 2 & 2 & 2 & 2 & 5 \end{pmatrix}.$$

The determinants of $F_1, \ldots, F_{17}$ are $1, 2^2, 2^4, 2^4, 2^8, 2^6, 3^3, 3, 3^5, 2^2 3^3, 2^4 3^3, 7^5, 7, 7^3, 5^3, 2^2 5^3, 2^4 5^3$, respectively.

Now we give a description of the $\prec$-maximal centerings of the minimal irreducible groups G_i ($i = 1, \ldots, 33$). We start with the centerings of G_1, G_2, G_3. The numbers in brackets in front of the bases of the centerings are the numbers of the corresponding quadratic form F_i.



G_1, G_2, G_3 have the $\prec$-maximal centerings $M_1, \ldots, M_8$ in common. Beyond that the lattice of centerings of G_1 contains M'_9, M'_{10}, that of G_2 contains M_9, M_{10}. All occurring indices are powers of 2.

Bases of the centerings:

(1) $B(M_1) = I_6$,

(2) $B(M_2) = (x_1, h_1x_1, \ldots, h_1^4x_1, y_1)$ with $x_1^T = (1, 1, 0, 0, 0, 0), y_1^T = (0, 0, 0, 0, 1, -1)$, and $h_1 = D((123456))$,

(4) $B(M_4) = (x_2, h_2x_2, h_2^2x_2, y_2, h_2y_2, h_2^2y_2)$ with $x_2^T = (1, 0, 1, 0, 0, 0)$, $y_2^T = (0, 1, 0, 1, 0, 0)$, and $h_2 = D((135)(246))$,

(1) $B(M_5) = (x_3, h_2x_3, h_2^2x_3, y_3, h_2y_3, h_2^2y_3)$ with $x_3^T = (1, 0, 0, 1, 0, 0)$, $y_3^T = (-1, 0, 0, 1, 0, 0)$,

(3) $B(M_8) = (x_4, h_1x_4, \ldots, h_1^4x_4, y_4)$ with $x_4^T = (2, 0, 0, 0, 0, 0), y_4^T = (1, 1, 1, 1, 1, 1)$,

(3) $B(M_3) = \frac{1}{2}B(M_5)B(M_8)$,

(2) $B(M_7) = B(M_5)B(M_2)$,

(5) $B(M_6) = (x_5, h_2x_5, h_2^2x_5, y_5, h_2y_5, h_2^2y_5)$ with $x_5^T = (-1, 0, 1, 0, 1, 0)$, $y_5^T = (0, -1, 0, 1, 0, 1)$,

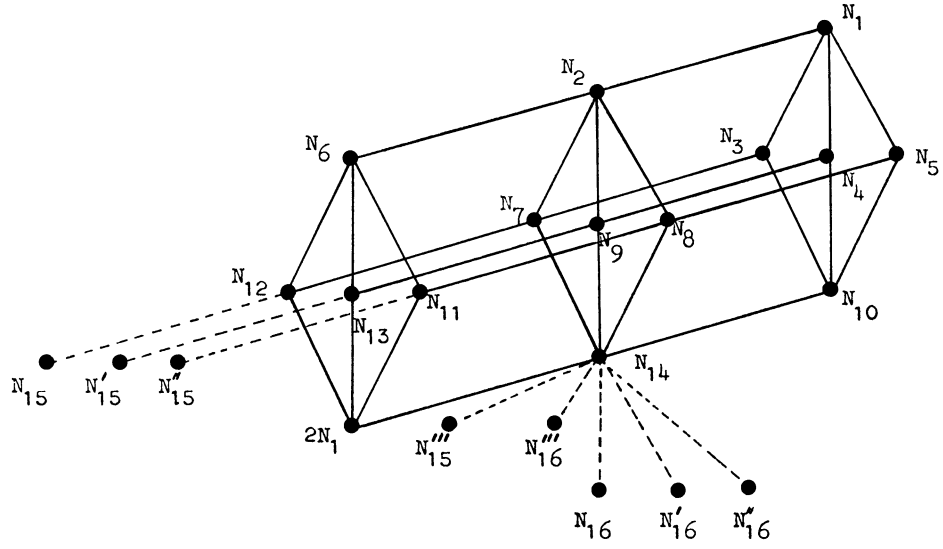
(4) $B(M'_9) = (x_6, h_1x_6, h_1^2x_6, y_6, h_1y_6, z_6)$ with $x_6^T = (1, 1, 0, 1, 1, 0), y_6^T = (-1, -1, 0, 1, 1, 0)$, and $z_6^T = (-1, 0, -1, 1, 0, 1)$,

(5) $B(M'_{10}) = (x_7, h_1x_7, h_1^2x_7, y_7, h_1^2y_7, z_7)$ with $x_7^T = (-1, 1, 1, -1, 1, 1)$, $y_7^T = (-1, -1, 1, 1, 1, -1)$ and $z_7^T = (1, -1, -1, -1, 1, -1)$,

(6)
$$B(M_9) = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

(6) $B(M_{10}) = \mathrm{diag}(-1, 1, 1, 1, 1, 1)B(M_9)$.

We proceed to the $\prec$-maximal centerings of $G_4, \ldots, G_8$.



$G_4, \ldots, G_8$ have the $\prec$-maximal centerings $N_1, \ldots, N_{14}$ in common. Furthermore, the lattice of centerings of G_4 contains N_{15}, N_{16}, that of G_5 contains N'_{15}, N'_{16}, that of G_6 contains N''_{15}, N''_{16}, and that of G_7 contains N'''_{15}, N'''_{16}. All occurring indices are powers of 2. Bases of the centerings:

(1) $B(N_1) = I_6$, $B(N_8) = B(M_5)$, $B(N_7) = D((12)(34)(56))B(N_8)$, $B(N_9) = D((165432))B(N_7)$,

(2) $B(N_2) = B(M_2)$, $B(N_{12}) = B(N_7)B(N_2)$, $B(N_{13}) = B(N_9)B(N_2)$, $B(N_{11}) = B(N_8)B(N_2)$,

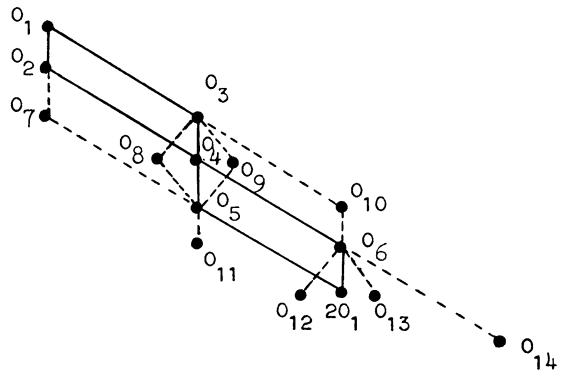
(4) $B(N_6) = B(M_4)$, $B(N''_{15}) = B(M'_9)$, $B(N_{15}) = D((12)(34)(56))B(N''_{15})$, $B(N'_{15}) = D((165432))B(N_{15})$,

(3) $B(N_{14}) = B(M_8)$, $B(N_3) = \frac{1}{2}B(N_7)B(N_{14})$, $B(N_4) = \frac{1}{2}B(N_9)B(N_{14})$, $B(N_5) = \frac{1}{2}B(N_8)B(N_{14})$,

(5) $B(N_{10}) = B(M_6)$, $B(N''_{16}) = B(M'_{10})$, $B(N_{16}) = D((12)(34)(56))B(N''_{16})$, $B(N'_{16}) = D((165432))B(N_{16})$,

(6) $B(N'''_{15}) = B(M_9)$, $B(N'''_{16}) = B(M_{10})$.

We continue with the $\prec$-maximal centerings of $G_9, \ldots, G_{15}, G_{18}, \ldots, G_{22}, G_{31}$.



$G_9, \ldots, G_{15}, G_{18}, \ldots, G_{22}, G_{31}$ all have $O_1, \ldots, O_6$ as $\prec$-maximal centerings. Moreover, the lattices of centerings of $G_9, \ldots, G_{15}, G_{18}, G_{19}, G_{31}$ contain O_8, O_9, those of G_9, G_{18}, G_{31} contain O_{12}, O_{13}, those of G_{10}, G_{12}, G_{20} contain O_{11}, O_{14}. In addition, the lattice of centerings of G_{20} contains O_7 and O_{10}. All occurring indices are powers of 2.

Bases of the centerings:

(1) $B(O_1) = I_6$, $B(O_4) = B(N_9)$,

(2) $B(O_2) = B(M_2)$, $B(O_5) = B(O_4)B(O_2)$,

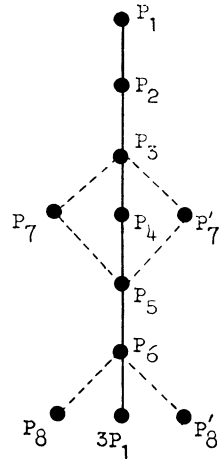
(3) $B(O_6) = B(M_8)$, $B(O_3) = \frac{1}{2}B(O_4)B(O_6)$,

(4) $B(O_7) = B(M_4)$, $B(O_{11}) = B(N'_{15})$,

(5) $B(O_{10}) = B(M_6)$, $B(O_{14}) = B(N'_{16})$,

(6) $B(O_{12}) = B(M_9)$, $B(O_{13}) = B(M_{10})$, $B(O_8) = \frac{1}{2}B(O_4)B(O_{12})$, $B(O_9) = \frac{1}{2}B(O_4)B(O_{13})$.

The $\prec$-maximal centerings of G_{17}, G_{23}, G_{25}, and G_{26} follow.



G_{17}, G_{23}, G_{25}, G_{26} have the $\prec$-maximal centerings $P_1, \ldots, P_6$ in common. Beyond that the lattices of centerings of G_{23}, G_{25} contain P_7, P_8, the lattice of centerings of G_{26} contains P_7', P_8'. All occurring indices are powers of 3.

Bases of the centerings:

$$(7) \qquad B(P_1) = I_6, \qquad B(P_4) = \operatorname{diag}\left(\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\right),$$

$$B(P_7) = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \qquad B(P_7') = \begin{pmatrix} 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix},$$

$$B(P_8) = B(P_4) \cdot B(P_7), \qquad B(P_8') = B(P_4)B(P_7')$$

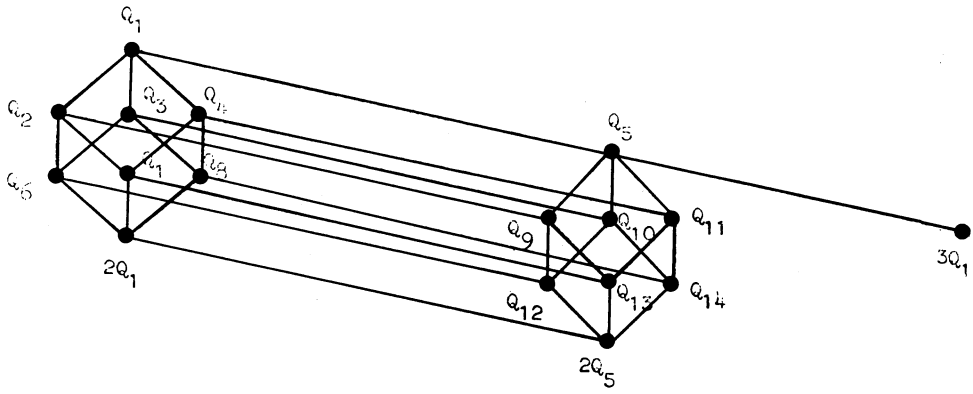
(9) $B(P_2) = (x_8, h_3x_8, \ldots, h_3^5x_8)$ with $x_8^T = (0, 1, -1, 0, 0, 0)$ and $h_3 = \operatorname{diag}(\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right), I_2, I_2)D((135)(246))$,

$B(P_5) = B(P_4)B(P_2)$,

$$(8) \qquad B(P_3) = \begin{pmatrix} 1 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}, \qquad B(P_6) = B(P_4)B(P_3).$$

The lattices of centerings of G_{28}, G_{29}, G_{30} are linearly ordered and consist of P_1, P_4

in case of G_{28}, G_{30} and of P_1, P_2, P_3, P_4 in case of G_{29}. We proceed to the $\prec$-maximal centerings of G_{24} and G_{27}.



The $\prec$-maximal centerings of G_{27} are $Q_1, \ldots, Q_{14}$, those of G_{24} are Q_1, Q_2, Q_5, Q_8, Q_9, Q_{14}.

Bases of the centerings:

(7) $B(Q_1) = I_6$, $B(Q_5) = B(P_4)$,

$$(10)\quad B(Q_2) = B(M_4), \quad B(Q_4) = \operatorname{diag}\left(\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}\right) B(Q_2),$$

$$B(Q_3) = \operatorname{diag}\left(\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}\right) B(Q_2), \quad B(Q_9) = B(Q_5)B(Q_2),$$

$$B(Q_{10}) = B(Q_5)B(Q)_3, \quad B(Q_{11}) = B(Q_5)B(Q_4),$$

$$(11)\quad B(Q_8) = B(M_6), \quad B(Q_7) = \operatorname{diag}\left(\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right) B(Q_8),$$

$$B(Q_6) = \operatorname{diag}\left(\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}\right) B(Q_8),$$

$$B(Q_{12}) = B(Q_5)B(Q_6), \quad B(Q_{13}) = B(Q_5)B(Q_7), \quad B(Q_{14}) = B(Q_5)B(Q_8).$$

The lattice of centerings of G_{16} is linearly ordered. We denote the $\prec$-maximal centerings by $R_1, \ldots, R_6$ ($R_i \subseteq R_{i-1}$ and R_{i-1}: $R_i = 7$ for $i = 2, \ldots, 6$).

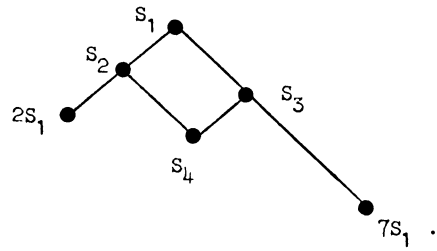
Bases of the centerings:

(12) $B(R_1) = I_6$, $B(R_4) = (x_9, h_4x_9, \ldots, h_4^5x_9)$ with $x_9^T = (1, 1, -1, 1, -1, -1)$ and h_4 is the element g_1 in the definition of G_{16} in Lemma (2.3).

(13) $B(R_2) = I_6 + J_6$, $B(R_5) = B(R_4)B(R_2)$,

$$(14)\ B(R_3) = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & -1 \\ -1 & 1 & 1 & -2 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 & 0 \end{pmatrix}, \quad B(R_6) = \begin{pmatrix} 3 & 4 & -5 & -3 & 3 & -1 \\ -1 & 1 & -3 & 1 & -1 & -2 \\ 2 & -2 & -1 & -2 & 2 & -3 \\ -2 & 2 & 1 & -5 & -2 & 3 \\ 1 & -1 & -4 & -1 & 1 & 2 \\ -3 & 3 & -2 & -4 & 4 & 1 \end{pmatrix}$$

The $\prec$-maximal centerings of G_{33} are $S_1, \ldots, S_4$:

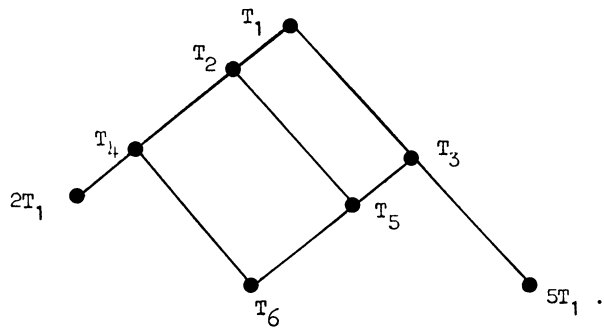


Bases of the centerings:

(12) $B(S_1) = I_6$, $B(S_2) = (x_{10}, h_5 x_{10}, \ldots, h_5^5 x_{10})$ with $x_{10}^T = (1, 0, 1, 0, 1, 0)$ and $h_5 = g_3$ in the definition of G_{33} in Lemma (2.11),

(13) $B(S_3) = B(R_2)$, $B(S_4) = B(S_2)B(S_3)$.

Finally the $\prec$-maximal centerings of G_{32} are:



Bases of the centerings:

$$(15)\qquad B(T_1) = I_6, \qquad B(T_3) = \begin{pmatrix} 3 & 1 & 1 & -1 & -1 & 0 \\ 1 & 3 & 1 & 1 & 0 & -1 \\ 1 & 1 & 3 & 0 & 1 & 1 \\ -1 & 1 & 0 & 3 & 1 & -1 \\ -1 & 0 & 1 & 1 & 3 & 1 \\ 0 & -1 & 1 & -1 & 1 & 3 \end{pmatrix},$$

$$(16)\qquad B(T_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad B(T_5) = B(T_3)B(T_2),$$

$$(17)\qquad B(T_4) = \begin{pmatrix} -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}, \qquad B(T_6) = B(T_3)B(T_4).$$

4. The Irreducible Maximal Finite Subgroups of $GL(6, \mathbf{Z})$. If L is an irreducible $\mathbf{Z}G$-representation module and M a $\prec$-maximal centering of L, then $M^{\#}$ denotes the unique $\prec$-maximal centering of L, which belongs to the inverse transposed representation coming from M [7]. By Theorem (6.1) in [7] we know that an irreducible maximal finite subgroup G of $GL(n, \mathbf{Z})$ with form X yields another maximal finite subgroup $G^{\#}$ fixing the form X^{-1}. G and $G^{\#}$ are $\mathbf{Q}$-equivalent.

For the centerings $M_i, \ldots, T_i$ in the last section the $M_i^{\#}, \ldots, T_i^{\#}$ are easily derived.

$$M_1^{\#} = M_1,\ M_2^{\#} = M_8,\ M_4^{\#} = M_6,\ M_5^{\#} = M_5,\ M_3^{\#} = M_7,\ M_9'^{\#} = M_{10}',\ M_9^{\#} = M_{10}.$$

$$N_1^{\#} = N_1,\ N_2^{\#} = N_{14},\ N_3^{\#} = N_{12},\ N_4^{\#} = N_{13},\ N_5^{\#} = N_{11},\ N_6^{\#} = N_{10},$$
$$N_7^{\#} = N_7,\ N_8^{\#} = N_8,$$

$$N_9^{\#} = N_9,\ N_{15}^{\#} = N_{16},\ N_{15}'^{\#} = N_{16}',\ N_{15}''^{\#} = N_{16}'',\ N_{15}'''^{\#} = N_{16}'''.$$

$$O_1^{\#} = O_1,\ O_2^{\#} = O_6,\ O_3^{\#} = O_5,\ O_4^{\#} = O_4,\ O_7^{\#} = O_{10},\ O_8^{\#} = O_9,$$
$$O_{11}^{\#} = O_{14},\ O_{12}^{\#} = O_{13}.$$

$$P_1^{\#} = P_4,\ P_2^{\#} = P_3,\ P_5^{\#} = P_6,\ P_7^{\#} = P_8,\ P_7'^{\#} = P_8'.$$

$$Q_1^{\#} = Q_5,\ Q_2^{\#} = Q_{14},\ Q_3^{\#} = Q_{13},\ Q_4^{\#} = Q_{12},\ Q_6^{\#} = Q_{11},\ Q_7^{\#} = Q_{10},\ Q_8^{\#} = Q_9.$$

$$R_1^{\#} = R_2,\ R_3^{\#} = R_6,\ R_4^{\#} = R_5.$$

$$S_1^{\#} = S_3,\ S_2^{\#} = S_4.$$

$$T_1^{\#} = T_3,\ T_2^{\#} = T_6,\ T_4^{\#} = T_5.$$

From this we conclude immediately that the automorphism groups of F_2 and F_3, of F_4 and F_5, of F_8 and F_9, of F_{10} and F_{11}, of F_{12} and F_{13}, and of F_{16} and F_{17} are rationally equivalent.

(4.1) THEOREM. *The* **Z***-automorphs of the forms* $F_1, \ldots, F_{17}$ *defined at the beginning of Section* 3 *are a full set of representatives for the* **Z***-classes of the maximal finite subgroups of* $GL(6, \mathbf{Z})$.

(i) $\mathrm{Aut}(F_1)$, $\mathrm{Aut}(F_2)$, *and* $\mathrm{Aut}(F_3)$ *are* **Q***-equivalent. They are isomorphic to the wreath product* $C_2 \sim S_6$ *of order* $2^6 6!$ $(\mathrm{Aut}(F_1) = H_6)$.

(ii) $\mathrm{Aut}(F_4) \sim_{\mathbf{Q}} \mathrm{Aut}\ F_5$. *Both are isomorphic to the wreath product* $(C_2 \sim S_3) \sim C_2$ $(\cong (C_2 \times S_4) \sim C_2)$ *of order* $(2^3 3!)^2 2$.

(iii) $\mathrm{Aut}(F_6)$ *is* **Q***-equivalent to the subgroup of the full monomial group* H_6 *generated by the diagonal matrices of even trace and all permutation matrices. Its order is* $2^5 6!$.

(iv) $\mathrm{Aut}(F_7)$ *is isomorphic to the wreath product* $(C_2 \times S_3) \sim S_3$ *of order* $(2 \cdot 3!)^3 3!$.

(v) $\mathrm{Aut}(F_8) \sim_{\mathbf{Q}} \mathrm{Aut}(F_9)$. $\mathrm{Aut}(F_8)$ *is the direct product of* $\langle -I_6 \rangle$ *and the Weyl group of the root system* E_6. *Its order is* $2^8 3^4 5$.

(vi) $\mathrm{Aut}(F_{10}) \sim_{\mathbf{Q}} \mathrm{Aut}(F_{11})$. *They are isomorphic to the direct product* $S_3 \times S_4 \times C_2$ *of order* $3!4!2$.

(vii) $\mathrm{Aut}(F_{12}) \sim_{\mathbf{Q}} \mathrm{Aut}(F_{13})$. *Both are isomorphic to* $C_2 \times S_7$ *of order* $2 \cdot 7!$.

(viii) $\mathrm{Aut}(F_{14})$ *is isomorphic to* $C_2 \times PGL(2, 7)$ *of order* $2 \cdot 336$.

(ix) $\mathrm{Aut}(F_{15})$, $\mathrm{Aut}(F_{16})$ and $\mathrm{Aut}(F_{17})$ *are* **Q***-equivalent. They are isomorphic to* $C_2 \times S_5$ *of order* $2 \cdot 5!$.

Proof.

Ad (i). Compare Theorem (6.1) in [7] or, for a more general result, (III. 6) in [8].

Ad (ii). This follows from the main theorem in [2] and the well-known fact that the **Z**-automorphs of $I_3 + J_3$ and $4I_3 - J_3$ are **Q**-equivalent to the **Z**-automorph H_3 of I_3.

Ad (iii). We consider all vectors of shortest length in the centering M_9 (compare Section 3), which has this form F_6. Their coordinates are all equal to ± 1 with an even number of minus ones. The subgroup of H_6 described in the theorem operates on the set of these vectors. This already yields the whole automorph, since there are no other permutations of the vectors of shortest length respecting the scalar product, which follows from elementary considerations.

Ad (iv). The same argument as in (ii) holds.

Ad (v). The form F_8 can also be derived from the root system E_6 [4, p. 66], and the automorph of F_8 is equal to the automorphism group of the root system, in this case $W(E_6) \times (-I_6)$.

Ad (vi). We have $F_{11} = (3I_2 - J_2) \otimes (4I_3 - J_3)$. So $\mathrm{Aut}(F_{11})$ has a subgroup H consisting of the elements $g \otimes h$ with $g \in \mathrm{Aut}(3I_2 - J_2)$ and $h \in \mathrm{Aut}(4I_3 - J_3)$. H is isomorphic to $(S_3 \times S_4) \times C_2$. (Note $\mathrm{Aut}(3I_2 - J_2) \cong S_3 \times C_2$ and $\mathrm{Aut}(4I_3 - J_3) \cong S_4 \times C_2$.) That H is already equal to $\mathrm{Aut}(F_{11})$ can be proved as follows. The lattice corresponding to F_{11} has 12 vectors of shortest length (up to sign) and H acts transitively on these 24 vectors. Let x be one of these shortest vectors. Then the stabilizer of x in $\mathrm{Aut}(F_{11})$ has a faithful permutation representation on the vectors of

shortest length whose scalar product with x is equal to 1, because these are six linearly independent vectors. Considering the scalar products one easily sees that the order of the stabilizer is $6 \cdot 2$.

Ad (vii). This proof is similar to the one of Theorem (6.3) in [7]. For a more general result see (III. 3) in [8]. Note the vectors of shortest length in R_1 are up to sign the unit vectors and the negative sum of these.

Ad (viii). $\mathrm{Aut}(F_{14})$ contains a subgroup which is $\mathbf{Q}$-equivalent to G_{16}. Let s be the number of the 7-Sylow groups in $\mathrm{Aut}(F_{14})$. From the definition of G_{16} one sees easily that the normalizer of the subgroup corresponding to G_{16} in $\mathrm{Aut}(F_{14})$ is of order $7 \cdot 6 \cdot 2$. Let $|\mathrm{Aut}(F_{14})| = 2^{\alpha}3^{\beta}5^{\gamma}7^{\delta}$ ($\alpha, \beta, \gamma, \delta \in \mathbf{Z}^{\geqslant 0}$; $\alpha \leqslant 10, \beta \leqslant 4$, $\gamma \leqslant 1, \delta \leqslant 1$ by [6]). Then $\delta = 1, \alpha \geqslant 2, \beta \geqslant 1$ and $2^{\alpha-2}3^{\beta-1}5^{\gamma} \equiv 1 \bmod 7$ by Sylow's Theorem. On the other hand there are up to sign 21 vectors of shortest length in the centering R_3 belonging to the form F_{14}. The subgroup of $\mathrm{Aut}(F_{14})$ which is $\mathbf{Q}$-equivalent to G_{16} already operates transitively on these vectors. Let x be one of the vectors of shortest length. Then there are exactly four vectors of shortest length (up to sign) which are orthogonal to x. These are linearly independent. Hence the stabilizer of x in $\mathrm{Aut}(F_{14})$ has a monomial representation of degree 4, where the kernel is at most of order 2; hence $\gamma = 0$. Considering the values of the scalar products of the four vectors one sees, that the order of the stabilizer of x cannot be divisible by 3 and that the order must divide 2^4. So the order of $\mathrm{Aut}(F_{14})$ is $2 \cdot 21 \cdot 2^{\alpha-1}$ with $\alpha \leqslant 5$. Considering the congruence $2^{\alpha-2} \equiv 1 \bmod 7$ we get the solutions $\alpha_1 = 2$ and $\alpha_2 = 5$. One easily finds a permutation of the vectors of shortest length fixing x and respecting the scalar product which is not induced by an element of G_{16} and therefore $\alpha = 5$ holds. Also the action is primitive. But 21 is no prime power. So $\mathrm{Aut}(F_{14})$ has to be nonsolvable and our assertion follows, as the center of $\mathrm{Aut}(F_{14})$ must have order 2.

Ad (ix). There are 10 vectors of shortest length in T_1 (up to sign). An easy computation yields that $\mathrm{Aut}(F_{15})$ is isomorphic to $C_2 \times S_5$. (Note G_{32} is already isomorphic to S_5.) To complete the proof it suffices to show $\mathrm{Aut}(F_{17}) \sim_{\mathbf{Q}} \mathrm{Aut}(F_{15})$. This can be done as in the proof of Theorem (6.2) in [7] by observing that the vectors of length 12 (respectively 6 if the primitive form belonging to T_4 is taken) are already contained in $2T_1$. Q.E.D.

In the discussion of case (ix) in Section 2 we did not prove the following two assertions:

If $G \leqslant GL(6, \mathbf{Z})$ is irreducible and

(1) $G \cap SL(6, \mathbf{Z}) \cong PSL(2, 7)$ or

(2) $G \cong PSU_4(2)$,

then G is not minimal irreducible.

This follows now easily from the proof of Theorem (4.1) (viii) or (4.1) (v), respectively.

RWTH Aachen
Lehrstuhl D für Mathematik
51 Aachen, Templergraben 64, West Germany

Mathematisches Institut der Universität zu Köln
Weyertal 86–90
5 Köln 41, West Germany

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