MATHEMATICS OF COMPUTATION, VOLUME 31, NUMBER 138
APRIL 1977, PAGES 608–611

Quasi-Amicable Numbers

By Peter Hagis, Jr. and Graham Lord

Abstract. If $m = \sigma(n) - n - 1$ and $n = \sigma(m) - m - 1$, the integers m and n are said to be *quasi-amicable* numbers. This paper is devoted to a study of such numbers.

Let $\sigma(N)$ denote the sum of the positive divisors of the integer N (where $N > 1$), and let

$$L(N) = \sigma(N) - N - 1 \tag{1}$$

so that $L(N)$ is the sum of the "nontrivial" divisors of N. A qt-cycle ("q" for quasi) is a t-tuple of distinct positive integers $(m_1, m_2, \ldots, m_t)$ such that $m_i = L(m_{i-1})$ for $i \neq 1$ and $m_1 = L(m_t)$. A $q1$-cycle is usually called a quasi-perfect number; and we shall call $q2$-cycles quasi-amicable numbers. (In both [2] and [5] $q2$-cycles are referred to as "reduced" amicable pairs. Garcia, however, calls them "números casi amigos" (see the editorial note in [5]).) No quasi-perfect numbers have been found as yet; and if one exists, it exceeds 10^{20} (see [1]). They have been studied by Cattaneo [3], Abbott-Aull-Brown-Suryanarayana [1], and Jerrard-Temperley [4].

In [5] Lal and Forbes list the nine quasi-amicable pairs with smallest member less than 10^5. Beck and Najar [2] have continued the search as far as 10^6 and found six more quasi-amicable pairs. Using the CDC 6400 at the Temple University Computing Center, a search was made for *all* quasi-amicable pairs with smallest member less than 10^7. Forty-six pairs were found including two (526575–544784 and 573560–817479) with smallest member between 10^5 and 10^6 which apparently were missed by Beck and Najar [2]. These are listed at the end of this paper. (For the sake of convenience and completeness the pairs given in [2] and [5] are included here.)

It will be noticed that each pair in our list is of opposite parity, leading us to inquire: Are there any quasi-amicable pairs of the *same* parity? Now, the positive integers m and n are quasi-amicable if and only if $L(m) = n$ and $L(n) = m$ so that from (1), we have

$$\sigma(m) = \sigma(n) = m + n + 1. \tag{2}$$

Therefore, since $\sigma(N)$ is odd if and only if $N = S^2$ or $N = 2S^2$, we see that a necessary condition for m and n to be quasi-amicable numbers of the same parity is that each be a square or twice a square. Making use of this fact a search was made for quasi-amicable pairs of the same parity in the range $[10^7, 10^{10}]$. None was found so

Received March 8, 1976; revised October 18, 1976.
AMS (MOS) subject classifications (1970). Primary 10A20.

that we have:

PROPOSITION 1. *If* $m < n$ *and* (m, n) *is a quasi-amicable pair having the same parity, then* $m > 10^{10}$.

If m and n are *relatively prime* quasi-amicable numbers then, using (2), we have $\sigma(mn)/mn = \sigma(m)\sigma(n)/mn = (m + n + 1)^2/mn > (m + n)^2/mn = 2 + m/n + n/m > 4$. Since, if p is a prime, $\sigma(p^\alpha)/p^\alpha < p/(p - 1)$ and since $x/(x - 1)$ is a decreasing function, it follows that if mn has less than four prime factors then $\sigma(mn)/mn < (2/1)(3/2)(5/4) < 4$. If mn is odd and mn has fewer than twenty-one prime factors, then

$$\sigma(mn)/mn < (3/2)(5/4)(7/6) \cdots (71/70)(73/72) < 4.$$

We have proved:

PROPOSITION 2. *Let* m *and* n *be relatively prime quasi-amicable numbers. Then* mn *has at least four different prime factors. If, also,* m *and* n *have the same parity (so that* mn *is odd),* mn *has at least twenty-one different prime factors.*

Now, if $(m, n) = 1$ and mn is odd then, as noted earlier, m and n are squares so that from Proposition 2 we have $mn \geqslant (3 \cdot 5 \cdot 7 \cdots 73 \cdot 79)^2 > 25 \cdot 10^{59}$. If $n > m$ and $n < 2.5m$, then $2.5m^2 > mn > 25 \cdot 10^{59}$ and $m > 10^{30}$. If $n > 2.5m$, then $\sigma(m)/m = (m + n + 1)/m > 3.5$. m has at least thirteen prime factors since $(3/2)(5/4) \cdots (37/36)(41/40) < 3.5$; and $m > (3 \cdot 5 \cdot 7 \cdots 41 \cdot 43)^2 > 10^{30}$. We have established:

PROPOSITION 3. *If* m *and* n *are relatively prime quasi-amicable numbers of the same parity, then* m *and* n *each exceeds* 10^{30}.

We note that no number in our list of quasi-amicable numbers is a prime power. *If* p^α and n are quasi-amicable numbers then, of course, $\alpha > 1$. If $p = 2$, then $\sigma(n)$ $(= \sigma(p^\alpha))$ is odd so that n is of the form S^2 or $2S^2$. From (2) $2^{\alpha+1} - 1 = 2^\alpha + n + 1$ so that $n = 2(2^{\alpha-1} - 1)$. Therefore, $S^2 = 2^{\alpha-1} - 1$. But this is impossible since $S^2 \equiv 1 \pmod 8$ and $2^{\alpha-1} - 1 \equiv -1 \pmod 8$. (For $\alpha > 3$ since neither 2^2 nor 2^3 is a member of a quasi-amicable pair.) If p is odd and $2|\alpha$, then $\sigma(n)$ *and* n are both odd and $n = S^2$. From (2) $1 + p + \cdots + p^\alpha = p^\alpha + n + 1$ so that $n = p(1 + p + \cdots + p^{\alpha-2})$. Thus, $p \parallel n$ which contradicts $n = S^2$. Since we now know that p and α are both odd, it follows that $\sigma(p^\alpha)$ *and* n are both even. If $\alpha = 3$, $n = p(1 + p)$ so that

$$\sigma(n) = (1 + p)\sigma(1 + p) < (1 + p)(1 + 2 + 3 + \cdots + (p + 1))$$
$$= (1 + p)(1 + p)(2 + p)/2.$$

But $\sigma(n) = \sigma(p^3) = (1 + p)(1 + p^2)$. It follows that $2 + 2p^2 < 2 + 3p + p^2$ so that $p < 3$. This contradiction completes the proof of

PROPOSITION 4. *If* (p^α, n) *is a quasi-amicable pair, then* p *is an odd prime,* α *is an odd number greater than* 3, *and* n *is even.*

COROLLARY 4.1. *It is not possible for both members of a quasi-amicable pair to be prime powers.*

Quasi-Amicable Pairs

$48 = 2^4 \cdot 3$	$75 = 3 \cdot 5^2$
$140 = 2^2 \cdot 5 \cdot 7$	$195 = 3 \cdot 5 \cdot 13$
$1050 = 2 \cdot 3 \cdot 5^2 \cdot 7$	$1925 = 5^2 \cdot 7 \cdot 11$
$1575 = 3^2 \cdot 5^2 \cdot 7$	$1638 = 2^4 \cdot 103$
$2024 = 2^3 \cdot 11 \cdot 23$	$2295 = 3^3 \cdot 5 \cdot 17$
$5775 = 3 \cdot 5^2 \cdot 7 \cdot 11$	$6128 = 2^4 \cdot 383$
$8892 = 2^2 \cdot 3^2 \cdot 13 \cdot 19$	$16587 = 3^2 \cdot 19 \cdot 97$
$9504 = 2^5 \cdot 3^3 \cdot 11$	$20735 = 5 \cdot 11 \cdot 13 \cdot 29$
$62744 = 2^3 \cdot 11 \cdot 23 \cdot 31$	$75495 = 3 \cdot 5 \cdot 7 \cdot 719$
$186615 = 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 29$	$206504 = 2^3 \cdot 83 \cdot 311$
$196664 = 2^3 \cdot 13 \cdot 31 \cdot 61$	$219975 = 3 \cdot 5^2 \cdot 7 \cdot 419$
$199760 = 2^4 \cdot 5 \cdot 11 \cdot 227$	$309135 = 3 \cdot 5 \cdot 37 \cdot 557$
$266000 = 2^4 \cdot 5^3 \cdot 7 \cdot 19$	$507759 = 3 \cdot 7 \cdot 24179$
$312620 = 2^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 29$	$549219 = 3 \cdot 11^2 \cdot 17 \cdot 89$
$526575 = 3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 59$	$544784 = 2^4 \cdot 79 \cdot 431$
$573560 = 2^3 \cdot 5 \cdot 13 \cdot 1103$	$817479 = 3^3 \cdot 13 \cdot 17 \cdot 137$
$587460 = 2^2 \cdot 3 \cdot 5 \cdot 9791$	$1057595 = 5 \cdot 7 \cdot 11 \cdot 41 \cdot 67$
$1000824 = 2^3 \cdot 3 \cdot 11 \cdot 17 \cdot 223$	$1902215 = 5 \cdot 7 \cdot 17 \cdot 23 \cdot 139$
$1081184 = 2^5 \cdot 13 \cdot 23 \cdot 113$	$1331967 = 3 \cdot 7^2 \cdot 13 \cdot 17 \cdot 41$
$1139144 = 2^3 \cdot 23 \cdot 41 \cdot 151$	$1159095 = 3 \cdot 5 \cdot 7^2 \cdot 19 \cdot 83$
$1140020 = 2^2 \cdot 5 \cdot 7 \cdot 17 \cdot 479$	$1763019 = 3^3 \cdot 17 \cdot 23 \cdot 167$
$1173704 = 2^3 \cdot 7 \cdot 20959$	$1341495 = 3^3 \cdot 5 \cdot 19 \cdot 523$
$1208504 = 2^3 \cdot 11 \cdot 31 \cdot 443$	$1348935 = 3 \cdot 5 \cdot 7 \cdot 29 \cdot 443$
$1233056 = 2^5 \cdot 11 \cdot 31 \cdot 113$	$1524831 = 3 \cdot 7^2 \cdot 11 \cdot 23 \cdot 41$
$1236536 = 2^3 \cdot 7 \cdot 71 \cdot 311$	$1459143 = 3^2 \cdot 7 \cdot 19 \cdot 23 \cdot 53$
$1279950 = 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 23 \cdot 53$	$2576945 = 5 \cdot 7 \cdot 17 \cdot 61 \cdot 71$
$1921185 = 3^3 \cdot 5 \cdot 7 \cdot 19 \cdot 107$	$2226014 = 2 \cdot 7 \cdot 17 \cdot 47 \cdot 199$
$2036420 = 2^2 \cdot 5 \cdot 19 \cdot 23 \cdot 233$	$2681019 = 3^5 \cdot 11 \cdot 17 \cdot 59$
$2102750 = 2 \cdot 5^3 \cdot 13 \cdot 647$	$2142945 = 3^2 \cdot 5 \cdot 7 \cdot 6803$
$2140215 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 109$	$2421704 = 2^3 \cdot 263 \cdot 1151$
$2171240 = 2^3 \cdot 5 \cdot 17 \cdot 31 \cdot 103$	$3220119 = 3^2 \cdot 7 \cdot 79 \cdot 647$
$2198504 = 2^3 \cdot 7 \cdot 11 \cdot 43 \cdot 83$	$3123735 = 3 \cdot 5 \cdot 29 \cdot 43 \cdot 167$
$2312024 = 2^3 \cdot 11 \cdot 13 \cdot 43 \cdot 47$	$3010215 = 3 \cdot 5 \cdot 13 \cdot 43 \cdot 359$
$2580864 = 2^7 \cdot 3 \cdot 11 \cdot 13 \cdot 47$	$5644415 = 5 \cdot 7 \cdot 29 \cdot 67 \cdot 83$
$2958500 = 2^2 \cdot 5^3 \cdot 61 \cdot 97$	$3676491 = 3^2 \cdot 7 \cdot 13 \cdot 67^2$
$4012184 = 2^3 \cdot 11 \cdot 127 \cdot 359$	$4282215 = 3 \cdot 5 \cdot 7 \cdot 17 \cdot 2399$
$4311024 = 2^4 \cdot 3 \cdot 19 \cdot 29 \cdot 163$	$7890575 = 5^2 \cdot 7 \cdot 11 \cdot 4099$

$5088650 = 2 \cdot 5^2 \cdot 7^2 \cdot 31 \cdot 67$	$6446325 = 3 \cdot 5^2 \cdot 23 \cdot 37 \cdot 101$
$5416820 = 2^3 \cdot 5 \cdot 43 \cdot 47 \cdot 67$	$7509159 = 3^3 \cdot 7 \cdot 67 \cdot 593$
$6081680 = 2^4 \cdot 5 \cdot 11 \cdot 6911$	$9345903 = 3 \cdot 7 \cdot 17 \cdot 47 \cdot 557$
$6618080 = 2^5 \cdot 5 \cdot 7 \cdot 19 \cdot 311$	$12251679 = 3 \cdot 11 \cdot 17 \cdot 21839$
$7460004 = 2^2 \cdot 3 \cdot 23 \cdot 151 \cdot 179$	$10925915 = 5 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 59$
$7875450 = 2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 37 \cdot 43$	$16381925 = 5^2 \cdot 7^2 \cdot 43 \cdot 311$
$8713880 = 2^3 \cdot 5 \cdot 7 \cdot 31121$	$13693959 = 3^2 \cdot 17 \cdot 37 \cdot 41 \cdot 59$
$8829792 = 2^5 \cdot 3^2 \cdot 23 \cdot 31 \cdot 43$	$18845855 = 5 \cdot 7 \cdot 23 \cdot 41 \cdot 571$
$9247095 = 3^3 \cdot 5 \cdot 11 \cdot 13 \cdot 479$	$10106504 = 2^3 \cdot 47 \cdot 26879$

Acknowledgement. The authors were informed almost simultaneously and independently by the referee and Professor R. Najar of the work of Lal and Forbes [5] on "quasi-aliquot" sequences and quasi-amicable numbers. An earlier version of the present paper contained many of the results to be found in [5], and we are grateful for being informed of this duplication of effort.

Department of Mathematics
Temple University
Philadelphia, Pennsylvania 19122

Département de Mathématiques (Actuariat)
Université Laval
Québec, P.Q. G1K 7P4, Canada

1. H. L. ABBOTT, C. E. AULL, E. BROWN & D. SURYANARAYANA, "Quasiperfect numbers," *Acta Arith.*, v. 22, 1973, pp. 439–447. MR 47 #4915.

2. W. E. BECK & R. M. NAJAR, "More reduced amicable pairs," *Fibonacci Quart.* (To appear.)

3. P. CATTANEO, "Sui numeri quasiperfetti," *Boll. Un. Mat. Ital.*, v. 6, 1951, pp. 59–62.

4. R. P. JERRARD & N. TEMPERLEY, "Almost perfect numbers," *Math. Mag.*, v. 46, 1973, pp. 84–87. MR 51 #12686.

5. M. LAL & A. FORBES, "A note on Chowla's function," *Math. Comp.*, v. 25, 1971, pp. 923–925. MR 45 #6737.