High Order Local Approximations
to Derivatives in the Finite Element Method

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Abstract. Consider the approximation of the solution u of an elliptic boundary value problem by means of a finite element Galerkin method of order r, so that the approximate solution u_h satisfies $u_h - u = O(h^r)$. Bramble and Schatz (*Math. Comp.*, v. 31, 1977, pp. 94–111) have constructed, for elements satisfying certain uniformity conditions, a simple function K_h such that $K_h * u_h - u = O(h^{2r-2})$ in the interior. Their result is generalized here to obtain similar superconvergent order interior approximations also for derivatives of u.

1. **Introduction.** Assume that u is a smooth function defined in $\Omega \subset R^N$. Assume also that one is given an approximation u_h which belongs to a subspace S_h (depending on a small parameter h) of $L_2(\Omega)$ such that (for notation, see Section 2)

$$\| u - u_h \|_\Omega \leqslant C(u)h^r.$$

Such a situation arises when one solves a $2m$th order elliptic problem in a finite element space of order r by Galerkin's method. In certain cases it is possible to derive estimates for $u - u_h$ in negative norms with a higher power of h, e.g.

$$\| u - u_h \|_{-l,\Omega} \leqslant C(u)h^{r+l}.$$

In the elliptic case referred to, this can sometimes be done by a duality argument, with $l = r - 2m$ if $r > 2m$.

In [2], Bramble and Schatz were able to show (this was carried out for second order equations, i.e. with $l = r - 2$) that under certain uniformity assumptions on $\{S_h\}$ in the interior, it is possible to find an approximation of u based on u_h which is of this higher order also in $L_2(\Omega_0)$ where $\Omega_0 \subset\subset \Omega$. Their approximation is a convolution $K_h * u_h$ where K_h is a linear combination of translates of B-splines, and their result thus takes the form

$$\| u - K_h * u_h \|_{\Omega_0} \leqslant C(u)h^{r+l}.$$

The interior uniformity of $\{S_h\}$ is reflected in the fact that for any α,

(1)
$$\| \partial_h^\alpha(u - u_h) \|_{-l,\Omega_0} \leqslant C(u)h^{r+l},$$

where ∂_h^α denote difference quotients. Under the assumption that in addition to (1), maximum norm estimates of the form

(2) $|\partial_h^\alpha(u - u_h)|_{\Omega_0} \leqslant C(u)h^r$

are valid (such estimates hold in the elliptic case, cf. Bramble, Nitsche and Schatz
[1]), Bramble and Schatz also showed that

$$|u - K_h * u_h|_{\Omega_0} \leqslant C(u)h^{r+l}.$$

The purpose of this note is to derive similar approximations also for arbitrary
derivatives of u, and in fact for any linear functional of u which is continuous on
some Sobolev space $W_2^n(\Omega_0)$. In deriving these results we also give alternative proofs
of the results by Bramble and Schatz.

Our first main result, Theorem 1 below, shows that an arbitrary derivative of
the smooth function u can be approximated in $L_2(\Omega_0)$ by a local average of the
corresponding difference quotient of the approximating function u_h. Hereby the
error can be bounded by $O(h^{2p})$ plus a finite sum of terms of the form of the left-
hand side in (1), where p and l are arbitrary given positive integers. Using this result
we show in Theorem 2 that the value at u of any bounded linear functional Λ on
some $W_2^n(\Omega_0)$ can be approximated to the same degree as in Theorem 1 by a linear
functional Λ_h acting on u_h. Finally, in Theorem 3, we show the analogue of Theorem
1 for the maximum norm. Now interior maximum norms of certain difference
quotients of the error have to be included in the error bound.

Our results are not restricted to the solution of elliptic problems by Galerkin's
method but may be applied in any situation where negative norm (and for Theorem
3, maximum norm) interior estimates are available for difference quotients of the
error. Such estimates were obtained recently for second order parabolic equations in
Bramble, Schatz, Thomée and Wahlbin [3]. The conclusion is therefore that in the
context of that paper, $O(h^{2r-2})$ approximations can be obtained in the interior by
local averaging, not only for the solution itself, which was shown in [3], but also for
any derivative of the solution.

2. **Notation.** For Ω a domain in R^N and s a nonnegative integer we set for
$v \in W_2^s(\Omega)$, with $\|\cdot\|_\Omega$ the norm in $L_2(\Omega)$,

$$\|v\|_{s,\Omega} = \left(\sum_{|\alpha| \leqslant s} \|D^\alpha v\|_\Omega^2 \right)^{\frac{1}{2}},$$

and for $v \in L_2(\Omega)$,

$$\|v\|_{-s,\Omega} = \sup_{w \in C_0^\infty(\Omega)} \frac{(v, w)}{\|w\|_{s,\Omega}}.$$

Note that for any integer s, $\|\cdot\|_{s,\Omega}$ increases with Ω and that $\|\cdot\|_{s,R^N}$ is equivalent to

$$\|v\|_s = \left(\int_{R^N} (1 + |\xi|^2)^s |\hat{v}|^2 \, d\xi \right)^{\frac{1}{2}}.$$

Notice also that for supp $v \subset \Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$ we have for positive s, $\|v\|_{-s,\Omega_2} \leqslant$
$C\|v\|_{-s,\Omega_1}$, so that the two norms are then equivalent. For if $\varphi \in C_0^\infty(\Omega_1)$ and
$\varphi \equiv 1$ on Ω_0 we have

$$\|v\|_{-s,\Omega_2} = \sup_{w \in C_0^\infty(\Omega_2)} \frac{(\varphi v,\, w)}{\|w\|_{s,\Omega_2}} \leqslant C \sup_{w \in C_0^\infty(\Omega_2)} \frac{(v,\, \varphi w)}{\|\varphi w\|_{s,\Omega_2}} \leqslant C\|v\|_{-s,\Omega_1}.$$

In addition to these L_2 based norms we shall use the maximum norm

$$|v|_\Omega = \sup_{x \in \Omega} |v(x)|,$$

and for $v \in C^s(\overline{\Omega})$,

$$|v|_{s,\Omega} = \max_{|\alpha| \leqslant s} |D^\alpha v|_\Omega.$$

Let $\psi_{(l)}$ denote the B-spline of order l, that is the convolution $\chi * \cdots * \chi$ with l factors, where χ is the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$. For $\alpha = (\alpha_1, \ldots, \alpha_N)$ let

$$\psi_{(\alpha)}(x) = \prod_{j=1}^N \psi_{(\alpha_j)}(x_j)$$

denote the B-spline of order α in R^N. In particular, with $\epsilon = (1, \ldots, 1)$, $\psi_{(l\epsilon)}$ denotes the B-spline of the same order l in each variable. We set $\psi_{(\alpha),h}(x) = h^{-N}\psi_{(\alpha)}(h^{-1}x)$.

We define the difference quotient operator ∂_h^α by

$$\partial_h^\alpha = \partial_{h,1}^{\alpha_1} \cdots \partial_{h,N}^{\alpha_N} \quad \text{with } \partial_{h,j}v(x) = h^{-1}(v(x + \tfrac{1}{2}he_j) - v(x - \tfrac{1}{2}he_j)).$$

Notice that for the Fourier transforms of $\psi_{(\alpha),h}$ and $\partial_h^\alpha v$ we have

$$\hat{\psi}_{(\alpha),h}(\xi) = \hat{\psi}_{(\alpha)}(h\xi) = \prod_{j=1}^N \left(\frac{\sin(\frac{1}{2}h\xi_j)}{\frac{1}{2}h\xi_j} \right)^{\alpha_j}$$

and

$$\widehat{\partial_h^\alpha v}(\xi) = (2i)^{|\alpha|} \prod_{j=1}^N (h^{-1}\sin(\tfrac{1}{2}h\xi_j))^{\alpha_j}.$$

Let $a \in C^\infty(R^N)$. We recall that a is said to be a Fourier multiplier on L_∞ or $a \in M = M_\infty$ if

$$M(a) = \sup\{|F^{-1}(a\hat{v})|_{R^N}; v \in C_0^\infty(R^N), |v|_{R^N} \leqslant 1\} < \infty,$$

where F^{-1} denotes the inverse Fourier transform (for related material, cf., e.g. [4]). We shall use below that $a(h\xi) \in M$ if $a \in M$, with $M(a(h\cdot))$ independent of h so that

$$|F^{-1}(a(h\xi)\hat{v})|_{R^N} \leqslant M(a)|v|_{R^N}.$$

We shall also use the facts that M is closed under multiplication, that any trigonometric polynomial and $\psi_{(\alpha)}$ belong to M (in fact, $M(\psi_{(\alpha)}) = 1$), that $C_0^\infty \subset M$ and further that if φ is a C^∞ function on the real line with $\varphi^{(l)}(\xi_j) = O(|\xi_j|^{-b-l})$ $(b > 0, l = 0, 1)$, for large $|\xi_j|$, then φ belongs to M in one and, as a consequence, also in N dimensions.

3. The Local Averaging Operators. We shall now turn to the construction of the kernels of the averaging operators used in our main results. The first step is the definition of certain one-dimensional trigonometric polynomials.

LEMMA 1. *Let $q \geq 1, p \geq 1$. Then there exists a unique trigonometric polynomial $k_{q,p}$ of order $p - 1$ such that*

$$(3) \qquad k_{q,p}(\sigma)\hat{\chi}(\sigma)^q = 1 + O(\sigma^{2p}) \quad \text{as } \sigma \to 0.$$

Proof. Setting $\tau = \sin \tfrac{1}{2}\sigma$ we obtain for small τ or σ,

$$\hat{\chi}(\sigma)^{-q} = \left(\frac{\arcsin \tau}{\tau}\right)^q = \sum_{j=0}^{\infty} \gamma_{q,j}\tau^{2j}.$$

We may therefore choose

$$k_{q,p}(\sigma) = \sum_{j=0}^{p-1} \gamma_{q,j}\left(\sin \frac{1}{2}\sigma\right)^{2j} = \sum_{j=0}^{p-1} \gamma_{q,j}2^{-j}(1 - \cos \sigma)^j.$$

The uniqueness follows at once since if k_1 and k_2 both satisfy (3) then $k_1 - k_2$ is a trigonometric polynomial of order $p - 1$ which vanishes of order $2p$ at the origin and hence has to vanish identically.

For $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_j > 0$ and $p \geq 1$ we now define the N-dimensional trigonometric polynomial

$$k^{(\alpha,p)}(\xi) = \prod_{j=1}^{N} k_{\alpha_j,p}(\xi_j) = \sum_{\gamma} k_{\gamma}^{(\alpha,p)}e^{-i\langle\xi,\gamma\rangle}, \qquad \xi, \gamma \in R^N.$$

Using the coefficients of this polynomial and the B-splines of order α we set

$$K^{(\alpha,p)}(x) = \sum_{\gamma} k_{\gamma}^{(\alpha,p)}\psi_{(\alpha)}(x - \gamma).$$

Notice that by (3),

$$(4) \qquad \begin{aligned} \hat{K}^{(\alpha,p)}(\xi) &= k^{(\alpha,p)}(\xi)\hat{\psi}_{(\alpha)}(\xi) = \prod_{1}^{N}(k_{\alpha_j,p}(\xi_j)\hat{\chi}(\xi_j)^{\alpha_j}) \\ &= 1 + O(|\xi|^{2p}) \quad \text{as } \xi \to 0. \end{aligned}$$

We also set

$$K_h^{(\alpha,p)}(x) = h^{-N}K^{(\alpha,p)}(h^{-1}x).$$

The order relation near the origin in (4) is the basis for proving the following approximation estimate for convolution by $K_h^{(\alpha,p)}$.

LEMMA 2. *Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_j > 0$ and $p \geq 1$ be given. Then for $\Omega_1 \subset\subset \Omega_2$ there is a constant C such that for small h,*

$$(5) \qquad \|v - K_h^{(\alpha,p)} * v\|_{\Omega_1} \leq Ch^{2p}\|v\|_{2p,\Omega_2},$$

$$(6) \qquad |v - K_h^{(\alpha,p)} * v|_{\Omega_1} \leq Ch^{2p}|v|_{2p,\Omega_2}.$$

Proof. Since by (4),

$$|1 - \hat{K}^{(\alpha,p)}(h\xi)| \leqslant Ch^{2p}|\xi|^{2p} \quad \text{for } \xi \in R^N,$$

we obtain at once by Parseval's relation, for $v \in C_0^\infty(R^N)$,

$$\|v - K_h^{(\alpha,p)} * v\|_0 \leqslant Ch^{2p}\|v\|_{2p}.$$

The estimate (5) now follows by applying this inequality to ωv where $\omega \in C_0^\infty(\Omega_2)$ and $\omega \equiv 1$ in a neighborhood of $\overline{\Omega}_1$.

Setting $\widetilde{k}_j(\sigma) = k_{\alpha_j,p}(\sigma)\hat{\chi}(\sigma)^{\alpha_j}$, we find since $(1 - \widetilde{k}_j(\xi_j))\xi_j^{-2p} \in M$ that (with $D_j = \partial/\partial x_j$)

$$|F^{-1}((1 - \widetilde{k}_j(h\xi_j))\hat{v})|_{R^N} = h^{2p}|F^{-1}((1 - \widetilde{k}_j(h\xi_j))(h\xi_j)^{-2p}\widehat{D_j^{2p}v})|_{R^N}$$

$$\leqslant Ch^{2p}|D_j^{2p}v|_{R^N}.$$

Writing

$$1 - \hat{K}^{(\alpha,p)}(h\xi) = \sum_{j=1}^N \prod_{l<j} \widetilde{k}_l(h\xi_l)(1 - \widetilde{k}_j(h\xi_j)),$$

and noticing that clearly also $\widetilde{k}_j \in M$, we conclude

$$|v - K_h^{(\alpha,p)} * v|_{R^N} = |F^{-1}((1 - \hat{K}^{(\alpha,p)}(h\xi))\hat{v})|_{R^N} \leqslant Ch^{2p}|v|_{2p,R^N},$$

from which (6) follows as above by application to ωv.

In addition to the function $K_h^{(\alpha,p)}$ we shall use the associated function defined for $\alpha \geqslant 0, l \geqslant 1, p \geqslant 1$ by

$$\widetilde{K}_h^{(\alpha,l,p)}(x) = \sum_\gamma k_\gamma^{(\alpha+l\epsilon,p)}\psi_{(l\epsilon),h}(x - \gamma h).$$

Recall that derivatives of splines of higher order are difference quotients of splines of lower order. In particular, more precisely,

$$D^\alpha \psi_{(\alpha+l\epsilon),h} = \partial_h^\alpha \psi_{(l\epsilon),h},$$

so that with the above notation

(7) $$D^\alpha K_h^{(\alpha+l\epsilon,p)} = \partial_h^\alpha \widetilde{K}_h^{(\alpha,l,p)}.$$

4. Main results. We can now prove the first of our main results.

THEOREM 1. *Let α be a given multi-index, let $l \geqslant 1$ and $p \geqslant 1$. Then, for $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$ there is a constant C such that for h small, $u \in W_2^{2p+|\alpha|}(\Omega_1)$ and $u_h \in L_2(\Omega_2)$,*

$$\|D^\alpha u - \widetilde{K}_h^{(\alpha,l,p)} * \partial_h^\alpha u_h\|_{\Omega_0}$$

$$\leqslant C\left\{h^{2p}\|u\|_{2p+|\alpha|,\Omega_1} + \sum_{|\beta|\leqslant l} \|\partial_h^{\alpha+\beta}(u - u_h)\|_{-l,\Omega_1}\right\}.$$

Proof. Notice that by (7),

$$\widetilde{K}_h^{(\alpha,l,p)} * \partial_h^\alpha v = \partial_h^\alpha \widetilde{K}_h^{(\alpha,l,p)} * v = D^\alpha K_h^{(\alpha+le,p)} * v = K_h^{(\alpha+le,p)} * D^\alpha v.$$

We have by the triangle inequality

$$\| D^\alpha u - \widetilde{K}_h^{(\alpha,l,p)} * \partial_h^\alpha u_h \|_{\Omega_0}$$

$$\leqslant \| D^\alpha u - K_h^{(\alpha+le,p)} * D^\alpha u \|_{\Omega_0} + \| \widetilde{K}_h^{(\alpha,l,p)} * \partial_h^\alpha (u - u_h) \|_{\Omega_0} = I + II.$$

By Lemma 2,

$$I \leqslant Ch^{2p} \| D^\alpha u \|_{2p,\Omega_1} \leqslant Ch^{2p} \| u \|_{2p+|\alpha|,\Omega_1}.$$

The proof is therefore completed by the following lemma.

LEMMA 3. *Let $l \geqslant 1$ and $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$. Then there exists a constant C such that for small h and $v \in L_2(\Omega_2)$,*

$$\| \psi_{(le),h} * v \|_{\Omega_0} \leqslant C \sum_{|\beta| \leqslant l} \| \partial_h^\beta v \|_{-l,\Omega_1}.$$

Proof. We notice that for each j,

$$\left(\frac{\sin(\tfrac{1}{2}h\xi_j)}{\tfrac{1}{2}h\xi_j} \right)^{2l} \leqslant C \frac{1 + (h^{-1}\sin(\tfrac{1}{2}h\xi_j))^{2l}}{(1+\xi_j^2)^l}, \quad \xi_j \in R, h < 1.$$

This is obvious for small $h\xi_j$ since then the left side is bounded above and the right side below. For large $h\xi_j$ and hence ξ_j large the result is also immediate. Letting ξ_0 denote the component of ξ with largest modulus, we conclude, since $|\hat{\chi}| \leqslant 1$,

$$(8) \qquad \hat{\psi}_{(le)}(h\xi)^2 \leqslant \left(\frac{\sin(\tfrac{1}{2}h\xi_0)}{\tfrac{1}{2}h\xi_0} \right)^{2l} \leqslant C \frac{1 + (h^{-1}\sin(\tfrac{1}{2}h\xi_0))^{2l}}{(1+\xi_0^2)^l}$$

$$\leqslant C \frac{1 + \sum_j (h^{-1}\sin(\tfrac{1}{2}h\xi_j))^{2l}}{(1+|\xi|^2)^l}.$$

Parseval's relation therefore yields

$$\| \psi_{(le),h} * v \|_{R^N} \leqslant C \left\{ \| v \|_{-l,R^N} + \sum_j \| \partial_h^{le_j} v \|_{-l,R^N} \right\}$$

$$\leqslant C \sum_{|\beta| \leqslant l} \| \partial_h^\beta v \|_{-l,R^N}.$$

An application of this inequality to $v\omega$, where $\omega \in C_0^\infty(\Omega_1)$ and $\omega \equiv 1$ in a neighborhood of Ω_0 completes the proof.

Theorem 1 may now be used to produce approximations to Λu for any bounded linear functional on a Sobolev space on Ω_0.

THEOREM 2. *Let $l \geqslant 1$, $p \geqslant 1$ and $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$. Let Λ be a bounded linear functional on $W_2^n(\Omega_0)$ for some $n \geqslant 0$. Then there exists a constant C and*

for sufficiently small h a bounded linear functional Λ_h *on* $L_2(\Omega_1)$ *such that for* $u \in W_2^{2p+n}(\Omega_2), u_h \in L_2(\Omega_2)$,

$$|\Lambda u - \Lambda_h u_h| \leqslant C \left\{ h^{2p} \|u\|_{2p,\Omega_1} + \sum_{|\alpha| \leqslant n; |\beta| \leqslant l} \|\partial_h^{\alpha+\beta}(u - u_h)\|_{-l,\Omega_1} \right\}.$$

Proof. By the Riesz representation theorem there is a $\varphi \in W_2^n(\Omega_0)$ such that with $(\cdot, \cdot)_{\Omega_0}$ the inner product in $L_2(\Omega_0)$,

$$\Lambda u = (u, \varphi)_{W_2^n(\Omega_0)} = \sum_{|\alpha| \leqslant n} (D^\alpha u, D^\alpha \varphi)_{\Omega_0}.$$

Setting

$$\Lambda_h v = \sum_{|\alpha| \leqslant n} (\widetilde{K}_h^{(\alpha,l,p)} * \partial_h^\alpha v, D^\alpha \varphi)_{\Omega_0},$$

we obtain

$$|\Lambda u - \Lambda_h u_h| \leqslant \|\varphi\|_{n,\Omega_0} \sum_{|\alpha| \leqslant n} \|D^\alpha u - \widetilde{K}_h^{(\alpha,l,p)} * \partial_h^\alpha u_h\|_{\Omega_0},$$

and the result follows by Theorem 1.

In particular, in the case of the elliptic problem treated by Bramble and Schatz, we conclude by Theorem 2 and (1) that the approximating function u_h contains enough information to produce a $O(h^{r+l})$ approximation of $D^\alpha u(x)$ for any α and any $x \in \Omega$. The corresponding functional Λ_h is of course not practical, and since also (2) holds, we therefore have use for the following direct interior maximum norm estimate.

THEOREM 3. *Let* α *be a given multi-index, let* $l \geqslant 1$ *and* $p \geqslant 1$. *Then for* $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$ *there is a constant* C *such that for* h *small and* $u \in C^{2p+|\alpha|}(\Omega_2), u_h \in C(\Omega_2)$ *(with* $N_0 = [N/2] + 1$),

$$|D^\alpha u - \widetilde{K}^{(\alpha,l,p)} * \partial_h^\alpha u_h|_{\Omega_0}$$

$$\leqslant C \left\{ h^{2p} |u|_{2p+|\alpha|,\Omega_1} + \sum_{|\beta| \leqslant l+N_0} \|\partial_h^{\alpha+\beta}(u - u_h)\|_{-l,\Omega_1} \right.$$

$$\left. + h^l \sum_{|\beta| \leqslant l} |\partial_h^{\alpha+\beta}(u - u_h)|_{\Omega_1} \right\}.$$

Proof. In the same way as in the proof of Theorem 1, using now the second part of Lemma 2, we obtain

$$|D^\alpha u - \widetilde{K}_h^{(\alpha,l,p)} * \partial_h^\alpha u_h|_{\Omega_0} \leqslant Ch^{2p} |u|_{2p+|\alpha|,\Omega_1} + |\widetilde{K}_h^{(\alpha,l,p)} * \partial_h^\alpha (u - u_h)|_{\Omega_0}.$$

The result therefore follows as above by the following analogue of Lemma 3.

LEMMA 4. *Let* $l \geqslant 1$ *and* $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$. *Then there exists a constant* C *such that for* h *small and* $v \in C(\Omega_2)$,

$$|\psi_{(l\epsilon),h} * v|_{\Omega_0} \leqslant C \left\{ \sum_{|\beta| \leqslant l+N_0} \|\partial_h^\beta v\|_{-l,\Omega_1} + h^l \sum_{|\beta| \leqslant l} |\partial_h^\beta v|_{\Omega_1} \right\}.$$

Proof. As previously, it is sufficient to prove the result for $v \in C_0^\infty(\Omega_1)$. Let $\varphi \in C_0^\infty(-\pi, \pi)$ with $\varphi \equiv 1$ near the origin and set $\Phi(\xi) = \Pi_1^N \varphi(\xi_j)$. We may write, at $x \in \Omega_0$,

$$\psi_{(l\epsilon),h} * v = F^{-1}(\hat{\psi}_{(l\epsilon)}(h\xi)\hat{v})$$

$$= F^{-1}(\hat{\psi}_{(l\epsilon)}(h\xi)\Phi(h\xi)\hat{v}) + F^{-1}(\hat{\psi}_{(l\epsilon)}(h\xi)(1 - \Phi(h\xi))\hat{v}) = I + II.$$

Using Schwarz's inequality, we obtain

$$|I| = \left| \int (1 + |\xi|^2)^{-N_0/2} e^{-i\langle x, \xi \rangle} (1 + |\xi|^2)^{N_0/2} \hat{\psi}_{(l\epsilon)}(h\xi)\Phi(h\xi)\hat{v}(\xi) d\xi \right|$$

$$\leqslant C \| (1 + |\xi|^2)^{N_0/2} \hat{\psi}_{(l\epsilon)}(h\xi)\Phi(h\xi)\hat{v}(\xi) \|_{R^N}.$$

For $h\xi$ in the support of $\Phi(h\xi)$ we have

$$1 + |\xi|^2 \leqslant C \left\{ 1 + \sum_j \left(h^{-1} \sin\left(\frac{1}{2} h\xi_j \right) \right)^2 \right\};$$

and hence by (8),

$$(1 + |\xi|^2)^{N_0/2} \hat{\psi}_{(l\epsilon)}(h\xi)\Phi(h\xi)$$

$$\leqslant C(1 + |\xi|^2)^{-\frac{1}{2}} \left\{ 1 + \sum_j \left(h^{-1} \sin\left(\frac{1}{2} h\xi_j \right) \right)^{2(l+N_0)} \right\}^{\frac{1}{2}}.$$

We conclude

$$|I| \leqslant C \left\{ \|v\|_{-l, R^N} + \sum_j \| \partial_h^{(l+N_0)e_j} v \|_{-l, R^N} \right\} \leqslant C \sum_{|\beta| \leqslant l + N_0} \| \partial_h^\beta v \|_{-l, \Omega_1}.$$

In order to estimate II we write

$$II = F^{-1} \left(\prod_j \hat{\chi}(h\xi_j)^l \sum_j \prod_{i<j} \varphi(h\xi_i)(1 - \varphi(h\xi_j))\hat{v} \right),$$

and hence since $\hat{\chi}$, φ and $(1 - \varphi(\xi_j))/\xi_j^l$ belong to M,

$$|II| \leqslant C \sum_j | F^{-1}((1 - \varphi(h\xi_j))\hat{\chi}(h\xi_j)^l \hat{v}) |_{R^N}$$

$$= Ch^l \sum_j | F^{-1}((1 - \varphi(h\xi_j))(h\xi_j)^{-l} \widehat{\partial_h^{le_j} v}) |_{R^N} \leqslant Ch^l \sum_j | \partial_h^{le_j} v |_{\Omega_1}.$$

Together the estimates for I and II prove the lemma.

As an illustration, consider the case when a two-point boundary value problem for a second order ordinary differential equation (or a second order parabolic problem) has been approximately solved using cubic splines on a mesh which is uniform in the interior, so that $O(h^6)$ interior estimates for the difference quotients of the error are available in norms of order -2. Then the above theory (with $l = 2, p = 3, r = 4$) shows that $O(h^6)$ approximations of for instance u, du/dx and d^2u/dx^2 can be obtained using the trigonometric polynomials k_{23}, k_{33} and k_{43}. These approximations are

(with $T_\eta v(x) = v(x - \eta)$)

$$\left(\frac{37}{30} - \frac{23}{180}\ (T_h + T_{-h}) + \frac{1}{90}\ (T_{2h} + T_{-2h})\right) \psi_{(2),h}\ * u_h,$$

$$\left(\frac{437}{320} - \frac{97}{480}\ (T_h + T_{-h}) + \frac{37}{1920}\ (T_{2h} + T_{-2h})\right) \psi_{(2),h}\ * \partial_h u_h,$$

$$\left(\frac{181}{120} - \frac{17}{60}\ (T_h + T_{-h}) + \frac{7}{240}\ (T_{2h} + T_{-2h})\right) \psi_{(2),h}\ * \partial_h^2 u_h.$$

Notice that these approximations can easily be expressed in terms of the coefficients of u_h with respect to a basis in the finite dimensional approximating space and the convolutions of $\psi_{(2),h}$ with these basis functions. In the particular case of smooth (C^2) cubic splines these convolutions are translates of $\psi_{(6),h} = \psi_{(2),h} * \psi_{(4),h}$.

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