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Uniqueness of Padé Approximants From Series of Orthogonal Polynomials

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Abstract. It is proved that whenever a nonlinear Padé approximant, derived from a series of orthogonal polynomials, exists, it is unique.

Let $\phi_r(x)$, $r = 0, 1, 2, \ldots$, be a set of polynomials which are orthogonal on an interval $[a, b]$, finite, semi-infinite, or infinite, with weight function $w(x)$, whose integral over any subinterval of $[a, b]$ is positive; i.e.,

$$\int_a^b w(x)\phi_r(x)\phi_s(x)\,dx = 0 \quad \text{if } r \neq s. \tag{1}$$

Then it is known that $\phi_r(x)$ is a polynomial of degree exactly r.

Suppose now $f(x)$ is a function which has a formal expansion of the form

$$f(x) = \sum_{r=0}^{\infty} a_r\phi_r(x) \tag{2}$$

on $[a, b]$. The (m, n) Padé approximant to $f(x)$ is defined to be the rational function

$$S_{m,n}(x) = \frac{P(x)}{Q(x)} = \frac{\sum_{r=0}^{m} p_r\phi_r(x)}{\sum_{s=0}^{n} q_s\phi_s(x)} \tag{3}$$

having an expansion in $\phi_r(x)$, $r = 0, 1, 2, \ldots$, which agrees with that of $f(x)$ given in (2) up to and including the term $a_{m+n}\phi_{m+n}(x)$. It is assumed that the polynomials $P(x)$ and $Q(x)$ have no common factor, apart from a constant, and that $Q(x)$ does not vanish on $[a, b]$. It is worth mentioning that the approximations defined above are the ones called "nonlinear Padé approximants" in [2].

THEOREM 1. *If $g(x)$ is any continuous function on $[a, b]$ such that $\int_a^b w(x)g(x)\phi_r(x)\,dx = 0$, $r = 0, 1, \ldots, k - 1$, then $g(x)$ either changes sign at least k times in the interval (a, b) or is identically zero.*

The proof of this theorem can be found in [1, p. 110].

As a consequence of Theorem 1, it follows that if $Q(x)$ is nonzero on $[a, b]$, then $q_0 \neq 0$; hence one can normalize $Q(x)$ by taking $q_0 = 1$.

THEOREM 2. *If the (m, n)th nonlinear Padé approximant $P(x)/Q(x)$ to f exists, in the sense of* (3), *and, after dividing out common factors, if Q is of one sign on $[a, b]$, then it is unique.*

Proof. By the definition of $S_{m,n}(x) = P(x)/Q(x)$ one has

$$f(x) - S_{m,n}(x) = \sum_{r=m+n+1}^{\infty} A_r\phi_r(x). \tag{4}$$

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If $\bar{S}_{m,n}(x) = \bar{P}(x)/\bar{Q}(x)$ is another (m, n) Padé approximant to (1), then

$$f(x) - \bar{S}_{m,n}(x) = \sum_{r=m+n+1}^{\infty} \bar{A}_r \phi_r(x). \tag{5}$$

Subtracting (4) from (5) one obtains

$$S_{m,n}(x) - \bar{S}_{m,n}(x) = \sum_{r=m+n+1}^{\infty} (\bar{A}_r - A_r)\phi_r(x). \tag{6}$$

Now since $S_{m,n}(x)$ and $\bar{S}_{m,n}(x)$ are continuous on $[a, b]$ so is $D(x) \equiv S_{m,n}(x) - \bar{S}_{m,n}(x)$. Then from (6) it follows that $D(x)$ satisfies $\int_a^b w(x)D(x)\phi_r(x)\,dx = 0$, $r = 0, 1, \dots, m + n$. Hence by Theorem 1, $D(x)$ either changes sign at least $m + n + 1$ times on (a, b), or is identically zero there. But

$$D(x) = \frac{P(x)}{Q(x)} - \frac{\bar{P}(x)}{\bar{Q}(x)} = \frac{P(x)\bar{Q}(x) - \bar{P}(x)Q(x)}{Q(x)\bar{Q}(x)}, \tag{7}$$

i.e., the numerator of $D(x)$ is a polynomial of degree at most $m + n$, therefore, can have at most $m + n$ zeros on (a, b). Since $Q(x)$ and $\bar{Q}(x)$ are nonzero on $[a, b]$, $D(x)$ changes sign at most $m + n$ times on (a, b). Therefore, $D(x) \equiv 0$; hence $S_{m,n}(x) \equiv \bar{S}_{m,n}(x)$. Q.E.D.

So far Padé approximants from Legendre series [2] and Chebyshev series have been considered [3], [4]. As is explained in [2], the determination of the q_s, $s = 1, 2, \dots, n$, in general, involves the solution of n nonlinear equations, the determination of the p_r being trivial then. However, these n equations may have several solutions. But, as is mentioned in [2], only one solution with $Q(x) \neq 0$ on $[a, b]$ has been found for the examples in [2]. By Theorem 2 there is no other solution, and it is at this point that the result of Theorem 2 becomes important.

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