Roots of Two Transcendental Equations Involving Spherical Bessel Functions*

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Abstract. Roots of the transcendental equations $j_l(\alpha\lambda)y_l(\lambda) = j_l(\lambda)y_l(\alpha\lambda)$ and

$$[xj_l(x)]'_{x=\alpha\eta}[xy_l(x)]'_{x=\eta} = [xj_l(x)]'_{x=\eta}[xy_l(x)]'_{x=\alpha\eta}$$

for the spherical Bessel functions of the first and second kind, $j_l(z)$ and $y_l(z)$, have been computed. The ranges for the parameter α, the order l and the root index n are: $\alpha = 0.1(0.1)0.7$, $l = 1(1)15$, $n = 1(1)30$.

To determine the electromagnetic eigenfrequencies of a cavity resonator bounded by two perfectly conducting concentric spheres ($r = \alpha R$ and $r = R$, $0 < \alpha < 1$), it is necessary to solve the transcendental equations

(1)
$$j_l(\alpha\lambda)y_l(\lambda) = j_l(\lambda)y_l(\alpha\lambda),$$

and

(2)
$$[xj_l(x)]'_{x=\alpha\eta}[xy_l(x)]'_{x=\eta} = [xj_l(x)]'_{x=\eta}[xy_l(x)]'_{x=\alpha\eta},$$

for $l = 1, 2, 3, \ldots$. The spherical Bessel functions of the first and the second kind, $j_l(z)$ and $y_l(z)$, are defined in [1, p. 437]. The nth root, $\lambda_{l,n}$, of Eq. (1) is proportional to the nth characteristic frequency of the transverse electric 2^l-pole field. The nth root, $\eta_{l,n}$, of Eq. (2) is proportional to the nth characteristic frequency of the transverse magnetic 2^l-pole field.

By virtue of the relation [1, p. 439, Eq. (10.1.21)]

$$\frac{l+1}{z} f_l(z) + \frac{d}{dz} f_l(z) = f_{l-1}(z) \quad \text{where } f_l(z) = \begin{cases} j_l(z), \\ y_l(z), \end{cases}$$

the transcendental equation (2) becomes

$$(\alpha \eta j_{l-1}(\alpha \eta) - lj_l(\alpha \eta))(\eta y_{l-1}(\eta) - ly_l(\eta))$$

(2')

$$= (\alpha \eta y_{l-1}(\alpha \eta) - ly_l(\alpha \eta))(\eta j_{l-1}(\eta) - lj_l(\eta)).$$

Express Eqs. (1) and (2') each in the form $F(u) = 0$. For a given order l and a fixed parameter α the function $F(u)$ is then evaluated at a sequence of points u_j, for which $u_j - u_{j-1} = $ constant, until a sign change occurs. In this last interval a root is computed to near machine-word accuracy by using a modified Muller technique [2, p. 51]. The spherical Bessel functions are computed by means of Mechel's recurrence method [3, p. 202]. This straightforward procedure is fast, accurate and reliable.

Tables of numerical values for the roots of the two transcendental equations are to be found in the microfiche supplement of this issue. The ranges for the parameter α, the order l and the root index n are

$$\alpha = 0.1(0.1)0.7,$$
$$l = 1(1)15,$$
$$n = 1(1)30.$$

The calculations were performed on a CDC 7600 computer. When extending considerably the ranges for l and n, we encountered no difficulties with our program.

The first six roots of Eq. (1) for $l = 0, 1, 2$ and for $\alpha = 1.2, 1.5, 2.0$ are contained in Table 34 of [4]. Nomograms allowing the approximate determination of a few roots of Eq. (1) are presented in [5].

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