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On the l^2 Convergence of an Algorithm for Solving Finite Element Equations

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Abstract. An iterative method of multiple grid type is proposed for solving general finite element systems. It is proved that the method can produce a solution to the equations in $O(N)$ arithmetical operations where N is the number of unknowns.

1. Introduction. It is well known that the systems of linear equations arising from application of the finite element method to various boundary value problems are most often solved by some variation of the elimination method. Much progress has been made in improving the efficiency of these techniques. By contrast, the iterative methods used successfully in the finite difference case have so far not found much acceptance in the finite element field. In this paper a method which is iterative in character is proposed and its convergence properties elucidated. The problem considered is the minimization of the positive definite quadratic form $a(u, u) - 2(u, f)$ by means of the finite element method. This approach requires the solution of an $N \times N$ linear system, and it is to this linear system that the algorithm and its analysis apply. We shall prove that the system can be solved (in a definite sense) in $O(N)$ machine operations. This result shows a considerable improvement over what can be achieved by elimination—at least as far as orders of magnitude in N are concerned. The proof of the result will be carried out for quite general problems. Thus, no serious restrictions are placed on the region Ω, boundary value problems of many types for $2m$th order elliptic equations are accommodated, and there are no additional restrictions to be placed on the trial functions, other than those normally required by the finite element method.

The method to be used is of the multiple grid type. This type of method was introduced in [3] for the finite difference case and significantly extended by N. S. Bakhvalov [1] in a paper of very noteworthy technical accomplishment. The general ideas of the multiple grid approach, along with further general references, are sketched in [6]. References [4] and [7] are also relevant here.

The subsequent contents of the paper are as follows: Section 2 contains a brief discussion of the variational problem, while Section 3 contains a statement of the hypotheses under which the subsequent work is carried out. Sections 4 and 5 introduce an algorithm which is analyzed in Sections 6 and 7. This algorithm is used as a building block for another algorithm considered in Section 8. In the latter section we prove that the algorithm produces an $O(h^{2m})$ accurate solution to a $2m$th order elliptic problem in $O(N)$ operations where N is the dimension of the trial space. Finally, in Section

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9 the extension to numerically integrated finite element systems is very briefly considered.

2. Variational Problem. The problem to be solved is that of minimizing the quadratic functional

$$I(u) = a(u, u) - 2(u, f) \tag{2.1}$$

over a class of functions $H_E^m(\Omega) \subset H^m(\Omega)$, where the symbols have the following meanings. Ω is a bounded open set of R^d. $H^m(\Omega)$ is a Banach space obtained by completing $C^m(\Omega)$ in the norm $\| \ \|_m$,

$$\|u\|_m^2 = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_0^2,$$

where $\partial^\alpha u$ denotes a distribution derivative

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial p_1^{\alpha_1} \partial p_2^{\alpha_2} \cdots \partial p_d^{\alpha_d}}, \qquad p \equiv (p_1, p_2, \ldots, p_d) \in \Omega$$

for a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and where

$$\|v\|_0^2 = \int_\Omega v^2 \, d\Omega.$$

$H^m(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_0$$

with $(\ , \ )_0$ the usual $L^2(\Omega)$ inner product. As is known, $H^0(\Omega) \equiv L^2(\Omega)$. $H_E^m(\Omega)$ is a subset of $H^m(\Omega)$ whose elements satisfy certain auxiliary conditions, the essential boundary conditions of the problem. It will be assumed that $f \in H^0(\Omega)$ and that the expression (u, f) in (2.1) means $(u, f)_0$. $a(u, v)$ is a real symmetric bilinear form, assumed to satisfy the conditions

$$\begin{aligned} &a(u, v) \leq B_1 \|u\|_m \|v\|_m, \quad u, v \in H_E^m(\Omega), \\ &a(u, u) \geq b_1 \|u\|_m^2, \quad u \in H_E^m(\Omega), \quad b_1 > 0. \end{aligned} \tag{2.2}$$

The minimization problem has a unique solution for reasonable regions Ω and certain well-known types of essential boundary conditions. The Neumann problem is excluded from consideration by virtue of (2.2).

We refer to [2] for a more precise formulation of the variational problem. The above is sufficient for our purpose here.

3. Hypotheses. For the minimization problem stated in Section 2 we assume to begin with that a finite element method which is conforming in every respect is to be used. This means that the trial functions used are admissible in the variational integral, essential boundary conditions are satisfied exactly, there is no approximation of Ω or its boundary $\partial\Omega$, and all integrations are carried out exactly. These restrictions are made in order to simplify the analysis. In addition, we shall (temporarily) assume that the essential conditions are homogeneous. We envisage a sequence of trial spaces $\{S^h\}$, linear because of the homogeneous data, parametrized by h such that for all h sufficiently small $\{S^h\} \subset H_E^m(\Omega)$ and such that for a given sequence $\{h_i\}_{i=1}^{\infty} \downarrow 0$

$$S^{h_j} \subset S^{h_{j+1}}, \quad j = 1, 2, \ldots . \tag{3.1}$$

S^h for $h > 0$ are assumed finite dimensional and we write $\dim(S^{h_i}) = N_i$ and $\dim(S^h) = N$. In each S^h a basis is given, denoted by $\{\bar{\phi}_i^h\}_{i=1}^N$. Functions in the trial spaces will always be denoted with an overbar. We shall assume a normalization of the basis slightly different from the usual one; in fact, we shall suppose $\|\bar{\phi}_i^h\|_0 = 1, i = 1, 2, \ldots, N$. This normalization does not affect the applicability of the results to the usual finite element method. It is introduced to avoid the occurrence of factors involving annoying powers of h in our formulas. If $\bar{u}^h \in S^h$, then there exists u_i^h, $i = 1, 2, \ldots, N$, such that

$$\bar{u}^h = \sum_{i=1}^{N} u_i^h \bar{\phi}_i^h \in H^0(\Omega) \supset H_E^m(\Omega) \supset S^h. \tag{3.2}$$

The convention of using an overbar to denote an element of S^h and removing the overbar to denote the corresponding element of R^N will be adhered to throughout. It implies of course that an ordering is assigned to the trial functions for each value of h.

Carrying out the Ritz method with trial functions of the form (3.2) in the functional (2.1) we arrive at the system of linear equations

$$K_h u^h = f^h, \tag{3.3}$$

where K_h is the system matrix, whose (ij)th element is $a(\bar{\phi}_i^h, \bar{\phi}_j^h)$ and where the ith component of f^h is

$$f_i^h = \int_\Omega f \bar{\phi}_i^h \, d\Omega, \quad i = 1, 2, \ldots, N. \tag{3.4}$$

Let $\kappa_{ij}^h = [K_h^{-1}]_{i,j}$, $i, j = 1, 2, \ldots, N$. Then by simple rearrangements of (3.2)–(3.4) it follows that the finite element approximation to u, the minimizing element for (2.1) is

$$\bar{u}^h = G_h f, \tag{3.5}$$

where G_h is the integral operator on $H^0(\Omega)$ defined by

$$\begin{aligned} G_h g &= \int_\Omega \Gamma_h(p, q) g(q) \, d\Omega, \quad g \in H^0(\Omega), \\ \Gamma_h(p, q) &= \sum_{i,j=1}^{N} \kappa_{ij} \bar{\phi}_i^h(p) \bar{\phi}_j^h(q). \end{aligned} \tag{3.6}$$

We shall also postulate the existence of an operator G which places into correspondence with each $f \in L^2(\Omega)$ a unique solution $u \in H_E^m(\Omega)$ to the minimization problem. G will in fact be linear and bounded both as an operator into $L^2(\Omega)$ and as an operator into $H_E^m(\Omega)$.

The following notations will be required: for $\bar{v}^h \in S^h$,

$$|\bar{v}^h|_j^2 \equiv h^{2j} \sum_{|\alpha|=j} \|\partial^\alpha \bar{v}^h\|_0^2, \quad j = 0, 1, \ldots, m,$$

$$\|v^h\|_{l^2}^2 = \sum_{i=1}^{N} |v_i^h|^2.$$

We are now in a position to state the two principal hypotheses under which the numerical solution of (3.3) will be considered.

H1: for all $f \in L^2(\Omega)$ and each $h > 0$

$$\|Gf - G_h f\|_0 \leqslant C_1 h^{2m} \|f\|_0, \qquad C_1 \neq C_1(h).$$

H2:

(a) $$|\bar{v}^h|_j^2 \leqslant \Lambda_j \|v^h\|_{l^2}^2, \qquad j = 0, 1, \ldots, m, \ \Lambda_j \neq \Lambda_j(h),$$

(b) $$|\bar{v}^h|_0^2 \geqslant \lambda_0 \|v^h\|_{l^2}^2, \qquad 0 < \lambda_0 \neq \lambda_0(h)$$

for all $\bar{v}^h \in S^h$ and for all $h > 0$.

The first of these is equivalent to the L^2 error estimate for the finite element solution $\|u - \bar{u}^h\|_0 \leqslant C_1 h^{2m} \|f\|_0$. It follows in most cases from the standard finite element error estimates. Part (b) of H2 is equivalent to the requirement that the basis functions form, for each h, what is known as a strongly minimal system [5]. H2(b) taken with the first of the inequalities of H2(a) imply that the basis functions are almost orthonormal in $L^2(\Omega)$. This term, too, is used in [5].

As immediate deductions from H2(a) and (b), we infer firstly that $|\bar{v}^h|_0$ and $\|v^h\|_{l^2}$ are equivalent norms on S^h:

(3.7) $$\lambda_0 \|v^h\|_{l^2}^2 \leqslant |\bar{v}^h|_0^2 \leqslant \Lambda_0 \|v^h\|_{l^2}^2.$$

The second deduction is an estimate for the spectral radius $\rho(K_h)$ of K_h; for by the first of inequalities (2.2)

$$a(\bar{v}^h, \bar{v}^h) \equiv (K_h v^h, v^h) \leqslant B_1 \|\bar{v}^h\|_m^2$$
$$= B_1 \sum_{j=0}^{m} h^{-2j} |\bar{v}^h|_j^2 \leqslant B_1 \sum_{j=0}^{m} h^{-2j} \Lambda_j \|v^h\|_{l^2}^2.$$

By the symmetry of K_h it now follows that

(3.8) $$\rho(K_h) \leqslant B_2 h^{-2m}, \qquad B_2 \neq B_2(h), \qquad h \leqslant h_0,$$

where B_2 depends on B_1 and the Λ_j. We shall make use of this fact later. The hypothesis H2 appears to hold for the standard finite element bases, but requires a proof in individual cases.

In addition, it will be necessary to impose a restriction on the sequence $\{h_i\}$ associated with the sequence of subspaces $\{S^{h_i}\}$. This is the following: $h_i \leqslant \rho h_{i+1}$, $i = 1, 2, \ldots$, where $\rho > 1$ is a constant independent of h.

4. Preliminaries. The problem (3.3) whose solution is required will be denoted by

(4.1) $$K_p u^p = f^p \qquad (h = h_p),$$

the index p having replaced the h used previously. The associated trial space will be written as S^p. Along with (4.1), it will be necessary to consider other systems of the form

$$K_q x^q = y^q, \qquad 1 \leqslant q \leqslant p \qquad (h = h_q), \tag{4.2}$$

for general right-hand sides y^q where the associated trial spaces $S^q \subset S^{q+1}$. This, of course, corresponds to considering systems with larger values of h than the one for which the given calculation is to be carried out. In addition, we shall write $R(q)$ for the space of $|S^q|$ tuples, and N_q for its dimension. For any $z^q \in R(q)$, associated with (4.2) are an error, a residual, and a residual equation defined, respectively, by

$$\epsilon^q = x^q - z^q, \qquad r^q = y^q - K_q z^q, \qquad K_q \epsilon^q = r^q.$$

Let $\bar{w}^{q-1} \in S^{q-1}$, so that $w^{q-1} \in R(q-1)$ $(q \geqslant 2)$. Then as $S^{q-1} \subset S^q$, $\bar{w}^{q-1}$ may be regarded also as an element of S^q; let $\underset{\sim}{E}^{q-1}$ denote the operator setting up this correspondence and introduce the notation $\underset{\sim}{E}^{q-1}\bar{w}^{q-1} = \bar{w}^{q-1,+}$. This "embedding" operation is clearly additive and homogeneous and corresponding to it there is an operator from $R(q-1) \longrightarrow R(q)$, also linear which will have a matrix representation relative to the bases $\{\phi_i^{q-1}\}_{i=1}^{N_{q-1}}$ and $\{\phi_i^q\}_{i=1}^{N_q}$ in $R(q-1)$ and $R(q)$, respectively. Let E_{q-1} denote this matrix (which interpolates vectors from $R(q-1)$ to $R(q)$). Then we have

$$w^{q-1,+} = E_{q-1} w^{q-1}. \tag{4.3}$$

E_{q-1} is of dimensions $N_q \times N_{q-1}$ and of rank N_{q-1}.

The matrices E_{q-1}, K_{q-1} and K_q are related to one another through the following equality:

$$K_{q-1} \equiv (E_{q-1})^T K_q E_{q-1}. \tag{4.4}$$

In order to prove this, consider the form $a(\bar{w}^{q-1}, \bar{w}^{q-1})$: then

$$\begin{aligned} a(\bar{w}^{q-1}, \bar{w}^{q-1}) &= a(\bar{w}^{q-1,+}, \bar{w}^{q-1,+}) = (K_q w^{q-1,+}, w^{q-1,+}) \\ &= ((E_{q-1})^T K_q E_{q-1} w^{q-1}, w^{q-1}), \end{aligned}$$

where we used (4.3). However, $a(\bar{w}^{q-1}, \bar{w}^{q-1}) = (K_q w^{q-1}, w^{q-1})$ and by subtraction it follows that

$$(w^{q-1})^T \{K_{q-1} - (E_{q-1})^T K_{q-1} E_{q-1}\} w^{q-1} \equiv 0.$$

w^{q-1} is arbitrary. Choosing it in succession to be the eigenvectors of the matrix in curly brackets, which is symmetric, it follows that this matrix is the zero matrix so (4.4) follows. It may similarly be proved that, for example

$$f^{q-1} = E_{q-1}^T f^q. \tag{4.5}$$

The algorithm whose convergence is to be considered consists of repeated applications of a simpler algorithm which we shall now introduce. The steps are typical for multiple grid methods, and their intuitive meaning is fully discussed in the references already mentioned. This algorithm refers to the system

$$K_q x^q = y^q, \qquad q \geqslant 2,$$

and involves relaxation iterations being carried out within another type of iteration. We require two parameters δ' and α' and two positive integers n' and ν where $\nu \geqslant 2$. The following calculations are carried out starting with a given initial approximation to x^q, $x^{q,0,0}$:

Do steps 1, 2 and 3 for $k = 0, 1, \ldots, \nu - 1$.

1. $x^{q,k,i} = x^{q,k,i-1} - \alpha'(K_q x^{q,k,i-1} - y^q)$, $i = 1, 2, \ldots, n'$.
2. With $\epsilon^{q-1,k,0}$ defined by

$$K_{q-1}\epsilon^{q-1,k,0} = E^T_{q-1} r^{q,k,n'}, \tag{4.6}$$

compute $\eta^{q-1,k,0}$ such that

$$\|\eta^{q-1,k,0} - \epsilon^{q-1,k,0}\| \leqslant \delta' \|\epsilon^{q-1,k,0}\|.$$

3. Put $x^{q,k+1,0} = x^{q,k,n'} + E_{q-1}\eta^{q-1,k,0}$.

The calculations of the first step are relaxation calculations. Those of the second constitute the computation of a solution of relative accuracy δ to the *reduced* residual equation (4.6); the third step generates a new starting vector for the first. The norm in step 2 is the l^2 norm defined earlier. The l^2 subscript on this norm symbol will be omitted from now on to simplify the writing. It follows from (4.4) and (4.6) that $\epsilon^{q-1,k,0}$ is the discrete Ritz approximation to the error whose residual is $r^{q,k,n'}$.

5. A Theorem. We will now prove the following theorem about the algorithm presented in Section 4.

THEOREM 5.1. *There exist numbers δ_0 and n_0 not depending upon q, and a number α_0, such that for any fixed $\nu \geqslant 2$, with $\alpha' = \alpha_0$, $\delta' = \delta_0$ and $n' = n_0$*

$$\|x^q - x^{q,\nu,0}\| \leqslant \delta_0 \|x^q - x^{q,0,0}\| \qquad (0 < \delta_0 < 1).$$

Proof. It is clear that

$$\epsilon^{q,0,n'} = (I - \alpha' K_q)^{n'} \epsilon^{q,0,0}. \tag{5.1}$$

For step 2 we can always write

$$K_{q-1}\eta^{q-1,0,0} = E^T_{q-1} r^{q,0,n'} - K_{q-1} C^{q-1,0}$$

for some $C^{q-1,0}$, where we shall have

$$\eta^{q-1,0,0} = \epsilon^{q-1,0,0} - C^{q-1,0}, \qquad \|C^{q-1,0}\| \leqslant \delta' \|\epsilon^{q-1,0,0}\|.$$

Also, putting

$$x^{q,1,0} = x^{q,0,n'} + E_{q-1}\eta^{q-1,0,0},$$

it follows that

$$\epsilon^{q,1,0} = \epsilon^{q,0,n'} - E_{q-1}K^{-1}_{q-1}E^T_{q-1}K_q \epsilon^{q,0,n'} + E_{q-1}C^{q-1,0}.$$

Denoting by Π_q the projection matrix

$$\Pi_q \equiv I - E_{q-1}K_{q-1}^{-1}E_{q-1}^T K_q \tag{5.2}$$

and making use of (5.1), it follows that

$$\epsilon^{q,1,0} = \Pi_q(I - \alpha' K_q)^{n'}\epsilon^{q,0,0} + E_{q-1}C^{q-1,0}. \tag{5.3}$$

The rest of the proof hinges on a detailed analysis of (5.3) for which purpose it is necessary to use a number of auxiliary results. These will be proved in Section 6, and we shall return to complete the proof in Section 7.

6. Auxiliary Results. Let M_h denote the $N \times N$ matrix whose (ij)th entry is $(\bar{\phi}_i^h, \bar{\phi}_j^h)_0$. This matrix is positive definite, because of H2(b). For if $\bar{v}^h \in S^h$, then

$$|\bar{v}^h|_0^2 = (M_h v^h, v^h) \geqslant \lambda_0 \|v^h\|^2. \tag{6.1}$$

M_h is actually uniformly positive definite with respect to h since by hypothesis $\lambda_0 \neq \lambda_0(h)$.

We shall make some use of the following observation; let $y^q \in R(q)$, and define $y_0^q = M_q^{-1}y^q$. Then the finite element system on S^q for the functional $a(x, x) - 2(x, \bar{y}_0^q)$ is $K_q x^q = y^q$. This follows from (3.4) since $\bar{y}_0^q$ is evidently in $H^0(\Omega)$. Let x_0 denote the element $G\bar{y}_0^q \in H^m(\Omega)$.

LEMMA 6.1. *Let* $\bar{x}^{q-1} = G_{q-1}\bar{y}_0^q$; *then*

$$\|x^q - E_{q-1}x^{q-1}\| \leqslant B_1' h_q^{2m}\|y^q\|, \qquad B_1' = B_1'(C_1, \Lambda_0, \lambda_0, \rho).$$

Proof. By H1,

$$\|x_0 - \bar{x}^q\|_0 \leqslant C_1 h_q^{2m}\|\bar{y}_0^q\|_0, \qquad \|x_0 - \bar{x}^{q-1}\|_0 \leqslant C_1 h_{q-1}^{2m}\|\bar{y}_0^q\|_0$$

so that by the hypothesis $h_{q-1} \leqslant \rho h_q$,

$$\|\bar{x}^q - \bar{x}^{q-1}\|_0 \leqslant (1 + \rho^{2m})C_1 h_q^{2m}\|\bar{y}_0^q\|_0.$$

On the other hand, by the equivalence of the norms, specified in (3.7), and deduced from H2 we have

$$\lambda_0^{1/2}\|x^q - E_{q-1}x^{q-1}\| \leqslant (1 + \rho^{2m})C_1 h_q^{2m}\Lambda_0^{1/2}\|M_q^{-1}y^q\|;$$

and making use of (6.1) and rearranging,

$$\|x^q - E_{q-1}x^{q-1}\| \leqslant (1 + \rho^{2m})C_1(\Lambda_0\lambda_0)^{1/2}h_q^{2m}\|y^q\|,$$

which is equivalent to the stated result.

The set of elements $z^q \in R(q)$ satisfying the equation $E_{q-1}^T z^q = 0$ will be denoted by $\{E_{q-1}\}^\perp$.

LEMMA 6.2. *Let* $w^q \in \{E_{q-1}\}^\perp$. *Then*

$$\|w^q\| \leqslant B_1' h_q^{2m}\|K_q w^q\|.$$

Proof. Let $K_q v^q = w^q$. This is the finite element system for a certain free term

$\widetilde{w}_0^q \in H^0(\Omega)$, on S^q. The corresponding system on S^{q-1} will be

$$K_{q-1}v^{q-1} = E_{q-1}^T w^q = 0$$

and so $v^{q-1} = 0$. Applying Lemma 6.1 with $x^q = v^q$, it follows that $\|v^q\| \leqslant B_1' h_q^{2m} \|w^q\|$; and therefore, from

$$(w^q, w^q) = (w^q, K_q v^q) = (K_q w^q, v^q) \leqslant \|K_q w^q\| \, \|v^q\|$$

we get, after cancelling out a factor $\|w^q\|$ from each side, that

$$\|w^q\| \leqslant B_1' h_q^{2m} \|K_q w^q\|$$

as desired.

LEMMA 6.3. *With Π_q as defined in* (5.2), *and $h_q \leqslant h_0$*

$$\|\Pi_q x^q\| \leqslant B_2' \|x^q\| \quad \textit{for all } x^q \in R(q), \qquad B_2' = B_2'(B_1', B_2).$$

Proof. Consider the equation $K_q x^q = y^q$. As above, it is the finite element system on S^q for a certain continuous problem. The finite element system on S^{q-1} for this continuous problem will be $K_{q-1}x^{q-1} = E_{q-1}^T y^q$ so that

$$E_{q-1}x^{q-1} = E_{q-1}K_{q-1}^{-1}E_{q-1}^T y^q.$$

But then

$$\|\Pi_q x^q\| = \|x^q - E_{q-1}K_{q-1}^{-1}E_{q-1}^T K_q x^q\| = \|x^q - E_{q-1}x^{q-1}\|$$

and by Lemma 6.1

$$\|\Pi_q x^q\| \leqslant B_1' h_q^{2m} \|K_q x^q\|.$$

But we saw in (3.8) that $\rho(K_q) \leqslant B_2 h_q^{-2m}$ for h_q sufficiently small and the lemma follows, with $B_2' = B_1' B_2$.

For the next result some additional notations are needed.

We shall denote by $V_{1,\mu}^q$ that invariant subspace of K_q spanned by eigenvectors $\Phi^{q,i}$ of K_q with corresponding eigenvalues $\lambda_{q,i}$ satisfying

$$\lambda_{q,i} \leqslant \mu(B_1' h_q^{2m})^{-1}, \qquad \mu > 0.$$

In addition, we shall denote by $V_{2,\mu}^q$ the orthogonal complement of $V_{1,\mu}^q$ in $R(q)$, and by P_q the orthogonal projector of $R(q)$ onto $\{E_{q-1}\} \equiv \operatorname{span}(E_{q-1})$.

LEMMA 6.4. *Let $x^q \in V_{1,\mu}^q$; then*

$$\|(I - P_q)x^q\| \leqslant \mu \|x^q\|.$$

Proof. By definition,

$$\|(I - P_q)x^q\| \leqslant \|x^q - g^q\| \quad \text{for all } g^q \in \{E_{q-1}\}.$$

We shall take for g^q a vector constructed thus: if $K_q x^q = y^q$, then as done several times before, form the vector x^{q-1} associated with the continuous problem solved on S^{q-1}. Clearly, $E_{q-1}x^{q-1} \in \{E_{q-1}\}$; and we set $g^q = E_{q-1}x^{q-1}$. Then using Lemma 6.1,

$$\|(I - P_q)x^q\| \leqslant \|x^q - E_{q-1}x^{q-1}\| \leqslant B_1' h_q^{2m} \|K_q x^q\|.$$

On the other hand, since $x^q \in V_{1,\mu}^q$ it has an expansion in eigenvectors spanning the latter subspace,

$$x^q = \sum_{V_{1,\mu}^q} (x^q, \Phi^{q,i})\Phi^{q,i}$$

so that

$$K_q x^q = \sum_{V_{1,\mu}^q} (x^q, \Phi^{q,i})\lambda_{q,i}\Phi^{q,i}$$

and

$$\|K_q x^q\| \leqslant \mu(B_1' h_q^{2m})^{-1} \|x^q\|.$$

Inserting this in the above proved inequality, it follows that

$$\|(I - P_q)x^q\| \leqslant \mu \|x^q\|$$

which we wanted to prove.

The final lemma which is needed is the following:

LEMMA 6.5. *For all $h_q \leqslant h_0$ the inequality*

$$\|E_{q-1} C^{q-1,0}\| \leqslant B_3' \delta' \|\epsilon^{q,0,n'}\|, \qquad B_3' = B_3'(B_2', \Lambda_0, \lambda_0),$$

is valid.

Proof. By the definition of $C^{q-1,0}$ we have

$$\|C^{q-1,0}\| \leqslant \delta' \|\epsilon^{q-1,0,0}\|. \tag{6.2}$$

Also, the following inequalities hold: for all $v^{q-1} \in R(q-1)$

$$\lambda_0/\Lambda_0 \|v^{q-1}\|^2 \leqslant \|E_{q-1}v^{q-1}\|^2 \leqslant \Lambda_0/\lambda_0 \|v^{q-1}\|^2. \tag{6.3}$$

To prove these, consider for example the left-hand one. Then from $(\bar{v}^{q-1}, \bar{v}^{q-1})_0 = (\bar{v}^{q-1,+}, \bar{v}^{q-1,+})_0$ (these are $L^2(\Omega)$ inner products) and using H2(a) and (b) it follows that

$$\lambda_0 \|v^{q-1}\|^2 \leqslant (\bar{v}^{q-1}, \bar{v}^{q-1})_0 = (\bar{v}^{q-1,+}, \bar{v}^{q-1,+})_0 \leqslant \Lambda_0 \|E_{q-1}v^{q-1}\|^2,$$

and the left inequality is proved. The other one may be proved similarly. Applying (6.3) to (6.2) with $v^{q-1} = C^{q-1,0}$ and $v^{q-1} = \epsilon^{q-1,0,0}$ gives

$$\lambda_0/\Lambda_0 \|E_{q-1} C^{q-1,0}\|^2 \leqslant (\delta')^2 \Lambda_0/\lambda_0 \|E_{q-1}\epsilon^{q-1,0,0}\|^2. \tag{6.4}$$

Since

$$E_{q-1}\epsilon^{q-1,0,0} = (I - \Pi_q)\epsilon^{q,0,n'}$$

by Lemma 6.3

$$\|E_{q-1}\epsilon^{q-1,0,0}\| \leqslant (1 + B_2')\|\epsilon^{q,0,n'}\|$$

and substituting this into (6.4) shows that

$$\|E_{q-1}C^{q-1,0}\| \leqslant \delta' \cdot (\Lambda_0/\lambda_0)(1 + B_2')^{1/2}\|\epsilon^{q,0,n'}\|,$$

which is equivalent to the stated result.

This concludes the auxiliary results required for the proof of Theorem 5.1. In the next section the proof of this latter result is completed.

7. **Proof of Theorem and Further Deductions.** Returning to (5.3) we may decompose the initial error $\epsilon^{q,0,0}$ as

$$\epsilon^{q,0,0} = \epsilon_{1,\mu}^{q,0,0} + \epsilon_{2,\mu}^{q,0,0}, \qquad \epsilon_{i,\mu}^{q,0,0} \in V_{i,\mu}^q, \qquad i = 1, 2,$$

from which it follows that

$$(I - \alpha' K_q)^{n'} \epsilon^{q,0,0} = \epsilon_{1,\mu}^{q,0,n'} + \epsilon_{2,\mu}^{q,0,n'},$$

$$\epsilon_{i,\mu}^{q,0,n'} = (I - \alpha' K_q)^{n'} \epsilon_{i,\mu}^{q,0,0}, \qquad i = 1, 2.$$

Now as is easily verified $\Pi_q P_q \equiv 0$, so that

$$\text{(7.1)} \quad \begin{aligned} \|\Pi_q(I - \alpha' K_q)^{n'} \epsilon^{q,0,0}\| &= \|\Pi_q(I - P_q)\epsilon_{1,\mu}^{q,0,n'} + \Pi_q(I - P_q)\epsilon_{2,\mu}^{q,0,n'}\| \\ &\leqslant \|\Pi_q(I - P_q)\epsilon_{1,\mu}^{q,0,n'}\| + \|\Pi_q(I - P_q)\epsilon_{2,\mu}^{q,0,n'}\|. \end{aligned}$$

From Lemma 6.4 and Lemma 6.3,

$$\text{(7.2)} \quad \|\Pi_q(I - P_q)\epsilon_{1,\mu}^{q,0,n'}\| \leqslant B_2'\mu\|\epsilon_{1,\mu}^{q,0,n'}\|,$$

and because $I - P_q$ is an orthogonal projector

$$\text{(7.3)} \quad \|\Pi_q(I - P_q)\epsilon_{2,\mu}^{q,0,n'}\| \leqslant B_2'\|\epsilon_{2,\mu}^{q,0,n'}\|.$$

From (7.1)–(7.3) we have

$$\text{(7.4)} \quad \|\Pi_q(I - \alpha' K_q)^{n'} \epsilon^{q,0,0}\|^2 \leqslant 2[\mu^2(B_2')^2\|\epsilon_{1,\mu}^{q,0,n'}\|^2 + (B_2')^2\|\epsilon_{2,\mu}^{q,0,n'}\|^2].$$

Now let $T_{i,q}$, $i = 1, 2$, denote the restriction to $V_{i,\mu}^q$, $i = 1, 2$, respectively, of $(I - \alpha' K_q)$; and let $\gamma_{i,q,\mu}$ denote the bounds of these operators, $i = 1, 2$. Since $V_{i,\mu}^q$ are invariant subspaces of $T_{i,q}$, respectively,

$$\|T_{i,q}^{n'} x_i^q\| \leqslant \gamma_{i,q,\mu}^{n'}\|x_i^q\|, \qquad x_i^q \in V_{i,\mu}^q, \qquad i = 1, 2;$$

and consequently,

$$\|\epsilon_{i,\mu}^{q,0,n'}\| \leqslant \gamma_{i,q,\mu}^{n'}\|\epsilon_{i,\mu}^{q,0,0}\|, \qquad i = 1, 2,$$

so that (7.4) may be rewritten as

$$\text{(7.5)} \quad \begin{aligned} &\|\Pi_q(I - \alpha' K_q)^{n'} \epsilon^{q,0,0}\|^2 \\ &\quad \leqslant 2[\mu^2(B_2')^2\gamma_{1,q,\mu}^{2n'}\|\epsilon_{1,\mu}^{q,0,0}\|^2 + (B_2')^2\gamma_{2,q,\mu}^{2n'}\|\epsilon_{2,\mu}^{q,0,0}\|^2]. \end{aligned}$$

By (5.3), (7.5) and Lemma 6.5,

$$\text{(7.6)} \quad \begin{aligned} \|\epsilon^{q,1,0}\|^2 &\leqslant 2\|\Pi_q(I - \alpha' K_q)^{n'} \epsilon^{q,0,0}\|^2 + 2\|E_{q-1}C^{q-1,0}\|^2 \\ &\leqslant 4[\mu^2(B_2')^2\gamma_{1,q,\mu}^{2n'}\|\epsilon_{1,\mu}^{q,0,0}\|^2 + (B_2')^2\gamma_{2,q,\mu}^{2n'}\|\epsilon_{2,\mu}^{q,0,0}\|^2] \\ &\quad + 2(B_3')^2(\delta')^2\|\epsilon^{q,0,n'}\|^2. \end{aligned}$$

We may now select values for the parameters δ', μ, n' and α' as follows. First, choose δ' to be any solution of the inequalities

(7.7) $$2(B'_3)^2(\delta')^2 \leqslant \tfrac{1}{2}(\delta')^{2/\nu}, \qquad 0 < \delta' < 1,$$

say δ_0. δ_0 is independent of q. Next choose μ to be any positive solution of the inequality

(7.8) $$4(\mu)^2(B'_2)^2 \leqslant \tfrac{1}{2}\delta_0^{2/\nu},$$

say μ_0. μ_0 is also independent of q. Third, we choose $\alpha' \equiv \alpha_0$ by

(7.9) $$\alpha_0 = 2[\mu_0(B'_1 h_q^{2m})^{-1} + B_2 h_q^{-2m}]^{-1}, \qquad \alpha_0 = \alpha_0(q).$$

A standard computation based on (7.9) shows that

(7.10) $$\rho(I - \alpha_0 K_q) < 1$$

and also that

(7.11) $$\gamma_{1,q,\mu_0} < 1, \qquad \gamma_{2,q,\mu_0} \leqslant \theta < 1, \qquad \theta \neq \theta(q),$$

where θ is independent of q. These calculations make use of the positive definiteness of K_q. It remains to choose n'. Choose $n' \equiv n_0$ where n_0 is a definite, positive integer solution of the inequality

(7.12) $$4(B'_2)\theta^{2n'} \leqslant \tfrac{1}{2}\delta_0^{2/\nu}, \qquad n_0 \neq n_0(q).$$

From (7.10) it follows that

$$\|\epsilon^{q,0,n_0}\| \leqslant \|\epsilon^{q,0,0}\|.$$

Substituting this into (7.6), along with δ_0, μ_0 and n_0 and using the inequalities (7.7), (7.8), (7.11) and (7.12) we get

$$\|\epsilon^{q,1,0}\|^2 \leqslant \delta_0^{2/\nu}\|\epsilon^{q,0,0}\|^2,$$

and repeating the iteration ν times as specified in the algorithm gives finally

$$\|\epsilon^{q,\nu,0}\| \leqslant \delta_0\|\epsilon^{q,0,0}\|$$

so that the theorem is proved.

In order to reduce any initial approximation to the solution of (4.1) by a factor δ_0^k we have only to apply the algorithm of Section 4 k times over, where k is any positive integer. Each of these k iterations will involve the computation of a solution with relative accuracy δ_0 to a reduced residual equation of the form (4.2). This may be done by applying the same algorithm to the latter problem, starting with initial approximation zero. Then we shall have to solve a problem with $q = p - 2$, and so on. Eventually, a problem with a coefficient matrix of size $N_1 \times N_1$ will be arrived at. We shall assume that N_1 is sufficiently small that the system can be solved directly, e.g. by elimination. In this way a solution of (4.1) with any prescribed accuracy may be found.

The following observations may be made. First, the choice of the parameters given in the theorem and hence the conclusion of the theorem are independent of the right-hand side of the linear system. From this it follows that the system (4.1) can have its initial error reduced by the factor δ_0^k independently of its right-hand term. This observation enables us to see that any aspect of the finite element method which involves modifications to the right member of the assembled linear system leaves the latter amenable to the method of solution we have proposed above. In particular, nonhomogeneous boundary data of various common types are allowed. Secondly, concerning the algorithm itself it may be observed that if, as we suggested above, the various $N_1 \times N_1$ systems are solved exactly, then the parameter δ_0 actually makes no explicit appearance in the algorithm. It will be determined implicitly instead by the values of α_0, n_0 and ν that are used. The choice of the parameter ν will be considered in more detail below; there is no difficulty either practical or theoretical in choosing it. Therefore, only the two parameters α_0 and n_0 have to be chosen. Practical work shows that it is sufficient to use the Gauss-Seidel method instead of the relaxation method discussed above. Some theoretical justification for this can be given provided we restrict ourselves to model problems. Anyhow, use of the Gauss-Seidel method eliminates one of the two parameters and leaves only the number of relaxation sweeps n_0 free.

We shall now discuss the choice of ν. The selection of this number has a significant effect on the number of arithmetical operations required to carry out the algorithm of Section 4. It is necessary to express it as a function of ν and p.

Let w_1 be the number of operations required to solve exactly the $N_1 \times N_1$ linear systems, w_q the work to do the operations specified in steps 1–3 of the basic algorithm with the parameters α_0, n_0, δ_0 and ν and notice that w_q is independent of the right-hand sides y^q. It is clear that steps 1 and 3 can be carried out in at most $B_4 N_q$ operations where B_4 is independent of q. Therefore, from the relation

$$w_q \leqslant \nu(w_{q-1} + B_4 N_q), \qquad q \geqslant 2,$$

it follows that

$$w_p \leqslant \nu^{p-1} w_1 + B_4 \sum_{j=1}^{p-1} \nu^j N_{p+1-j}. \tag{7.13}$$

Putting $N_q = \beta_q N_{q-1}$ and $\overline{\beta}_{j,p} = (\beta_p \beta_{p-1} \cdots \beta_{p-j+1})^{1/j}$, (7.13) can be rewritten as

$$w_p \leqslant \nu^{p-1} w_1 + B_4 N_p \sum_{j=1}^{p-1} (\nu/\overline{\beta}_{j-1,p})^j. \tag{7.14}$$

This is a bound on work to solve the definite system (4.1). In order to bound the work as $p \longrightarrow \infty$ some hypothesis has to be introduced to ensure that the right side of (7.14) behaves reasonably. We shall assume the following: for some β,

$$2 \leqslant \nu < \beta \leqslant \beta_i, \qquad i = 2, 3, \ldots, \nu \in Z_+. \tag{7.15}$$

Then the series in (7.14) converges, $\nu^{p-1} < N_p/N_1$ and we get

$$w_p \leqslant B_4' N_p, \qquad B_4' = B_4'(B_4, w_1, \beta, \nu), \tag{7.16}$$

i.e. the work required to reduce the initial error by a factor δ_0 is bounded by a quantity proportional to the number of unknowns in the linear system. The condition (7.15) will be satisfied for finite element systems if some form of grid halving is adopted when $d \geqslant 2$. For then $\beta \sim B_5 2^d$ where B_5 is a constant dependent upon the particular finite element trial space in question.

If we want to reduce $\|\epsilon^{p,0,0}\|$ by δ_0^k, then the work count (7.16) becomes

$$w_p \leqslant kB_4' N_p. \tag{7.17}$$

On the other hand, if the only information we have about the error is that given in H1, it seems wasteful to compute solutions to (4.1) with accuracy greater than $O(h^{2m})$. If we adopt this viewpoint, then by means of an extension of the algorithm it is possible to show that $w_p = O(N_p)$ for a solution with $O(h^{2m})$ accuracy; i.e. the factor k in (7.17) is unnecessary. We shall prove this in the next section.

8. Coarse to Fine Grids.

We pose the problem of computing $\bar{U}^p$ such that

$$\|u - \bar{U}^p\|_0 \leqslant \xi h_p^{2m} \|f\|_0, \qquad \xi > C_1, \tag{8.1}$$

where ξ is a given constant independent of h_p, and f.

It will be necessary to consider with (8.1) the systems

$$K_q u^q = f^q, \qquad 1 \leqslant q \leqslant p. \tag{8.2}$$

We propose to solve a typical member of (8.2) by means of the algorithm discussed in the previous sections and to use the approximate solution $u^{q,1}$ thus obtained, in the form $E_q u^{q,1}$ as an initial approximation for the solution of

$$K_{q+1} u^{q+1} = f^{q+1}, \qquad q + 1 \leqslant p, \tag{8.3}$$

by the same algorithm. The parameters for these applications of the previous algorithm are α_0, δ_0, n_0 and ν and the algorithm will be applied $k \equiv k(\xi)$ times, where k will be defined exactly in the theorem which follows.

THEOREM 8.1. *A function* $\bar{U}^p$ *satisfying* (8.1) *can be found in* w_p' *arithmetical operations, where*

$$w_p' \leqslant B_5' N_p, \qquad p \geqslant p_0.$$

Proof. Consider first the step from (8.2) to (8.3). Let $\xi' = \Lambda_0^{-1/2}\xi$ and assume that

$$\|u^q - u^{q,1}\| \leqslant \xi' h_q^{2m} \|f\|_0.$$

By (3.7)

$$\|\bar{u}^q - \bar{u}^{q,1}\|_0 \leqslant \Lambda_0^{1/2} \|u^q - u^{q,1}\| \leqslant \xi h_q^{2m} \|f\|_0. \tag{8.4}$$

Let $E_q u^{q,1}$ be used as initial approximation in (8.4). Then

$$\|u^{q+1} - E_q u^{q,1}\| \leqslant \|u^{q+1} - E_q u^q\| + \|E_q u^q - E_q u^{q,1}\|. \tag{8.5}$$

Again by (3.7)

$$\|u^{q+1} - E_q u^q\| \leqslant \lambda_0^{-1/2} \|\bar{u}^{q+1} - \overline{E_q u^q}\|_0,$$

and from the inequalities

$$\|u - \overline{E_q u^q}\| \leqslant C_1 \rho^{2m} h_{q+1}^{2m} \|f\|_0, \qquad \|u - u^{q+1}\| \leqslant C_1 h_{q+1}^{2m} \|f\|_0$$

it follows that

$$\|u^{q+1} - E_q u^q\| \leqslant \lambda_0^{-1/2}(1 + \rho^{2m}) C_1 h_{q+1}^{2m} \|f\|_0.$$

In addition, as in the proof of Lemma 6.5,

$$\|E_q(u^q - u^{q,1})\| \leqslant (\Lambda_0/\lambda_0)^{1/2} \|u^q - u^{q,1}\|$$
$$\leqslant \Lambda_0^{1/2} \lambda_0^{-1} \|\bar{u}^q - \bar{u}^{q,1}\|_0 \leqslant (\Lambda_0/\lambda_0) \xi' \rho^{2m} h_{q+1}^{2m} \|f\|_0$$

using (8.4) in the last step. Then in (8.5),

$$\|u^{q+1} - E_q u^{q,1}\| \leqslant (D_1 + D_2 \xi') h_{q+1}^{2m} \|f\|_0, \qquad D_i = D_i(\lambda_0, \Lambda_0, \rho, C_1).$$

Let k be the smallest integer such that

$$\delta_0^k \leqslant (\xi')(D_1 + D_2 \xi')^{-1}, \qquad k \neq k(q).$$

Applying the algorithm of Section 7 to (8.3) with initial approximation $E_q u^{q,1}$ and with k defined above, we can compute $u^{q+1,1}$ satisfying

$$\|u^{q+1} - u^{q+1,1}\| \leqslant \xi' h_{q+1}^{2m} \|f\|_0;$$

and hence, $\bar{u}^{q+1,1}$ so that

$$\|\bar{u}^{q+1} - \bar{u}^{q+1,1}\| \leqslant \xi h_{q+1}^{2m} \|f\|_0.$$

The total arithmetic work for this calculation, w'_{q+1}, satisfies

$$w'_{q+1} \leqslant B''_4 N_{q+1}, \qquad B''_4 = B''_4(\xi).$$

Starting with the problem (8.3) with $q = 1$ and carrying out the above operations, it follows that $\bar{u}^p$ of (8.1) can be found in

$$\sum_{j=1}^{p} w'_j \leqslant w'_1 + B''_4 \sum_{j=2}^{p} N_j \leqslant w'_1 + B''_4 N_p \sum_{j=2}^{p} \beta^{-(j-2)}, \qquad \beta > 2,$$

arithmetical operations, where w'_1 is the work to find $\bar{u}^1$ satisfying (8.1) with $p = 1$. Assuming that w'_1 is independent of p (e.g. the equations are solved exactly), it follows that

$$\sum_{j=1}^{p} w'_j \leqslant B'_5 N_p \quad \text{for all } p \geqslant p_0;$$

and so the theorem is proved.

A similar result and the algorithm of this section were introduced (for finite differences) in [1]. The theorem above provides some justification for the natural (and old) idea of using approximate solutions on coarse grids as starting values on finer grids. The possibility arises of using the approximate coarse grid solution not only as a starting value for a finer grid, but to define the finer grid itself. The development of this idea should eventually free the user from the need to specify any grid whatsoever. Even the algorithm discussed above however frees us from having to specify in advance a grid where a solution is required. Instead, the user can specify the desired accuracy; and the machine can then find a solution which achieves it. As we have seen, the entire calculation will take a number of operations proportional only to the number of grid points in the final grid.

9. Numerically Integrated Systems. All the results so far have required that the system matrices K_q and the right-hand side f^p are computed exactly. In practice this will not be the case because some numerical integration processes will have to be used. However, the earlier results remain valid provided certain conditions are satisfied. In this section we shall consider briefly the nature of these conditions. Let $\widetilde{K}_q$, $q = 1, 2, \ldots, p$, be the system matrices computed numerically. The algorithm of Section 4 can be formally implemented with $\widetilde{K}_q$ replacing K_q. We assume first of all that $\widetilde{K}_h$ is positive definite for each $h > 0$, and secondly that (3.8), which we can no longer deduce analytically, holds in the form

$$\rho(\widetilde{K}_h) \leqslant \widetilde{B}_2 h^{-2m}, \qquad \widetilde{B}_2 \neq \widetilde{B}_2(h), \qquad h > 0.$$

In addition, we shall modify H1 in the following way. Let $\widetilde{G}_h$ be the discrete solution operator corresponding to $\widetilde{K}_h$. We require

$$\|Gf - \widetilde{G}_h f\|_0 \leqslant \widetilde{C}_1 h^{2m} \|f\|_0 \quad \text{for all } f \in L^2(\Omega),\ h > 0,$$

whether the right-hand side of the finite element system is computed by numerical integration, or by exact integration. These conditions can be translated into (reasonable) conditions on the accuracy of the quadrature formulas employed. With these modifications the entire argument excluding (4.4) can be repeated with obvious verbal and notational changes all the way through, up to and including Section 9.

This concludes our analysis of the algorithms presented. It is hoped to be able to report elsewhere on implementation and other topics.

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