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Recurrence Formula of the Taylor Series Expansion Coefficients of the Jacobian Elliptic Functions

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Abstract. A general recurrence formula permitting calculation of the Taylor series expansion coefficients of the Jacobian elliptic functions and the number of permutations of n natural numbers with a given run up or peak is given and its application is demonstrated.

In [1] we studied properties of the Taylor series expansion coefficients A_n comprising those of the Jacobian elliptic functions and tabulated them up to $n = 15$. Further tabulations of these coefficients for $n = 16$ to $n = 50$ are published in [2]. In the present paper we are giving a recurrence formula for the coefficients A_n.

We recapitulate briefly for later use the properties of A_n studied in [1]:

1. A_n are triangle matrices with $(n_{\text{even}} + 2)/2$ or $(n_{\text{odd}} + 1)/2$ columns and rows.

2. $A_n = A_n^T$.

3. The sum of the elements of A_n is equal to $n!$.

4. The sum of the elements of a row i of A_n is the number of permutations of n natural numbers with $i - 1$ runs up.

5. The sum of the elements $a_{i,j}$ of A_n with $i + j$ constant $> m$ (m maximal rows) is the number of permutations of n natural numbers with $k = n - (i + j) - 1$ peaks.

6. For n even and $i + j = (n + 2)/2 + 1$, $(a_{i,j})_n = (a_{i,j})_{n+1}$. For n odd $a_{i,(n+1)/2} = a_{i,(n+3)/2}$.

7. $a_{(n+2)/2,(n+2)/2} = 0$ for n even. $a_{(n+1)/2,(n+1)/2} = 2^{n-1}$ for n odd.

8. The elements $(a_{i,j})_n$, $i + j = (n + 2)/2 + 1$, and $(a_{i,(n+2)/2})_n$ (n even) are the Taylor series expansion coefficients of the Jacobian elliptic functions $\operatorname{sn}(u, k)$ and $\operatorname{cn}(u, k)$, $\operatorname{dn}(u, k)$, respectively.

The formal recurrence formula for A_n reads $A_{n+1} = T_n A_n$. We have to find T_n and to define its application on A_n. By means of mathematical induction we obtained the following results.

T_n are triangle matrices with the elements

$$(t_{i,j;1}, t_{i,j;2}, t_{i,j;3})_n = (n - 2(j - 1), 3 + 2(j - 1) - n + 2i, n + 2 - 2i)_n,$$

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$$(t_{i,j;1}, t_{i,j;2}, t_{i,j;3})_n = (0, 0, 0)_n \quad \text{for } i + j < n/2 + 1, n \text{ even and}$$
$$i + j < (n-1)/2 + 1, n \text{ odd},$$
$$i; j = 1, 2, 3, \ldots, n/2 \quad \text{for } n \text{ even and } (n-1)/2 \text{ for } n \text{ odd}.$$

The symmetry

$$(t_{i,j;1}, t_{i,j;2}, t_{i,j;3}) = (t_{j,i;3}, t_{j,i;2}, t_{j,i;1})$$

and the relation

$$t_{i,j;1} + t_{i,j;2} + t_{i,j;3} = n + 5$$

are valid.

Applying $(t_{i,j;1}, t_{i,j;2}, t_{i,j;3})_n$ on $(a_{h,k})_n$ according to the relation

$$(a_{i,j})_{n+1} = (a_{i,j-1} \cdot t_{i-1,j-1;1} + a_{i,j} \cdot t_{i-1,j-1;2} + a_{i-1,j} \cdot t_{i-1,j-1;3})_n$$

and using the properties 2. and 6. of A_n mentioned above permits calculation of the elements of A_{n+1}.

Examples. We illustrate these formulas on n even and n odd.

$$A_7 = T_6 A_6,$$

$$A_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 135 & 44 \\ 0 & 135 & 328 & 16 \\ 1 & 44 & 16 & 0 \end{pmatrix}, \qquad T_6 = \begin{pmatrix} 0,0,0, & 0,0,0 & 2,3,6 \\ 0,0,0, & 4,3,4 & 2,5,4 \\ 6,3,2 & 4,5,2 & 2,7,2 \end{pmatrix},$$

$$\begin{aligned}
(a_{2,4})_7 &= (a_{2,3} \cdot t_{1,3;1} + a_{2,4} \cdot t_{1,3;2} + a_{1,4} \cdot t_{1,3;3})_6 \\
&= 135 \cdot 2 + 44 \cdot 3 + 1 \cdot 6 = 408, \\
(a_{3,4})_7 &= (a_{3,3} \cdot t_{2,3;1} + a_{3,4} \cdot t_{2,3;2} + a_{2,4} \cdot t_{2,3;3})_6 \\
&= 328 \cdot 2 + 16 \cdot 5 + 44 \cdot 4 = 912, \\
(a_{4,4})_7 &= (a_{4,3} \cdot t_{3,3;1} + a_{4,4} \cdot t_{3,3;2} + a_{3,4} \cdot t_{3,3;3})_6 \\
&= 16 \cdot 2 + 0 \cdot 7 + 16 \cdot 2 = 64, \\
(a_{3,3})_7 &= (a_{3,2} \cdot t_{2,2;1} + a_{3,3} \cdot t_{2,2;2} + a_{2,3} \cdot t_{2,2;3})_6 \\
&= 135 \cdot 4 + 328 \cdot 3 + 135 \cdot 4 = 2064.
\end{aligned}$$

According to property 6 we obtain $a_{1,4} = 1, a_{2,3} = 135$, and since A_n is symmetric all elements of A_7 are known.

$$A_8 = T_7 A_7,$$

$$A_7 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 135 & 408 \\ 0 & 135 & 2064 & 912 \\ 1 & 408 & 912 & 64 \end{pmatrix}, \qquad T_7 = \begin{pmatrix} 0,0,0 & 0,0,0 & 3,2,7 \\ 0,0,0 & 5,2,5 & 3,4,5 \\ 7,2,3 & 5,4,3 & 3,6,3 \end{pmatrix},$$

$$\begin{aligned}
(a_{2,4})_8 &= (a_{2,3} \cdot t_{1,3;1} + a_{2,4} \cdot t_{1,3;2} + a_{1,4} \cdot t_{1,3;3})_7 \\
&= 135 \cdot 3 + 408 \cdot 2 + 1 \cdot 7 = 1228, \\
(a_{3,4})_8 &= (a_{3,3} \cdot t_{2,3;1} + a_{3,4} \cdot t_{2,3;2} + a_{2,4} \cdot t_{2,3;3})_7 \\
&= 2064 \cdot 3 + 912 \cdot 4 + 408 \cdot 5 = 11880, \\
(a_{4,4})_8 &= (a_{4,3} \cdot t_{3,3;1} + a_{4,4} \cdot t_{3,3;2} + a_{3,4} \cdot t_{3,3;3})_7 \\
&= 912 \cdot 3 + 64 \cdot 6 + 912 \cdot 3 = 5856, \\
(a_{3,3})_8 &= (a_{3,2} \cdot t_{2,2;1} + a_{3,3} \cdot t_{2,2;2} + a_{2,3} \cdot t_{2,2;3})_7 \\
&= 135 \cdot 5 + 2064 \cdot 2 + 135 \cdot 5 = 5478.
\end{aligned}$$

Using the properties 6. and 2. of A_n, we obtain the remaining elements of A_8.

$$A_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1228 & 408 \\ 0 & 0 & 5478 & 11880 & 912 \\ 0 & 1228 & 11880 & 5856 & 64 \\ 1 & 408 & 912 & 64 & 0 \end{pmatrix}.$$

In conclusion, the present recurrence formula permits calculation of the Taylor series expansion coefficients of the Jacobian elliptic functions and the number of permutations of n natural numbers with a given run up or peak.

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1. A. SCHETT, "Properties of the Taylor series expansion coefficients of the Jacobian elliptic functions," *Math. Comp.*, v. 30, 1976, pp. 143–147. MR 52 #12298.

2. A. SCHETT, Addendum to "Properties of the Taylor series expansion coefficients of the Jacobian elliptic functions," Microfiche supplement, *Math. Comp.*, v. 31, 1977, no. 137.