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Optimizing the Arrangement of Points on the Unit Sphere

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Abstract. This paper is concerned with the problem of placing N points on the unit sphere in E^3 so as to maximize the sum of their mutual distances. A necessary condition is proved which led to a computer algorithm. This in turn led to the apparent best arrangements for values of N from 5 to 10 inclusive.

How does one place N points on the surface of the unit sphere in E^3 so as to maximize the sum $S(N)$ of their mutual distances? This problem and its generalizations to E^m have a rich and extensive literature. A sample is given in the references. For $m = 2$ the solution is the regular N-gon with sum $N \cot(\pi/2N)$ [6]. It is also known that the regular $N + 1$ simplex inscribed in the unit sphere in E^N is optimal. Thus, the regular tetrahedron gives $S(4) = 9.79796$. For $N > 4$ in E^3 the problem is still open.

In [3] a similar problem was considered: Place N points on the surface of the sphere in E^3 so as to maximize the volume of their convex hull. The technique used was to find the best position for a point, relative to the other points, so as to maximize the volume function. This allowed for a complete solution of the problem up to $N = 8$; and in particular, the solution for 8 points agreed with a candidate obtained by a computer search in [8]. An attempt to use this approach for the mutual distances problem leads to the following.

LEMMA. *Let $p_1, \ldots, p_n$ be points on the unit sphere S in E^3. Let $f: S \longrightarrow R$ be defined by $f(x) = \Sigma_{i=1}^n |x - p_i|$. If f has a maximum at p, then $p = q/|q|$ where $q = \Sigma_{i=1}^n (p - p_i)/|p - p_i|$.*

Proof. Since $\operatorname{grad} f(x) = \Sigma_{i=1}^n (x - p_i)/|x - p_i|$, then the normal to S at p, that is, p itself, must be a positive multiple of $\operatorname{grad} f(p)$, and the conclusion follows.

This lemma gives the position of the point p as a function of p itself. Nonetheless we have used this lemma as a basis for an iterative program whereby each point is repeatedly moved in an attempt to improve its position so as to satisfy the condition of the lemma. Naturally p_j should be moved in the direction of

$$q_j = \sum_{i \neq j} (p_j - p_i)/|p_j - p_i|.$$

We found that simply replacing p_j by $q_j/|q_j|$ is adequate. Our results are summarized in Table 1.

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TABLE 1

Configuration	5	6	7	8	9	10
regular N-gon	15.3884	22.3923	30.6690	40.2187	51.0415	63.1375
regular pyramid	15.6748	22.8237	31.2447	40.9381	51.9041	64.1429
regular bipyramid	15.6814	22.9706	31.5306	41.3629	52.4680	64.8461
prism	–	22.9128	–	41.4523	–	65.2747
skew prism	–	22.9706	–	41.4731	–	65.2817
computer search	15.6814	22.9706	31.5309	41.4731	52.7436	65.3497
$(2/3)N^2 - 1/2$	16.1667	23.5	32.1667	42.1667	53.5	66.1667

In the first row are the numerical values of the formula of Fejes Tóth mentioned above. The regular N-pyramid is a right pyramid with a regular $(N-1)$-gon base and a height chosen to optimize $S(N)$. The regular N-bipyramid has a point at the North and South poles and a regular $(N-2)$-gon in the equatorial plane. The prism listed in the fourth row has N vertices, with regular polygons for its bases, and optimal height. The skew prism is a prism with one of its bases rotated through $2\pi/N$ radians, and height optimal. The final row represents the upper bound for $S(N)$ given in [1].

For $N = 5, 6$ our computer search gave the regular bipyramid as the optimal configuration. For $N = 8$ a skew square-based prism was obtained (i.e. an inscribed square-based prism with the top square twisted through 45°). The bases have edge length 1.1633, and the planes of the bases are at a distance of 1.1373. With $N = 9$ the configuration is an equilateral triangle in the equatorial plane and equilateral triangles above and below this plane at a height of .7031. These smaller triangles are skewed 60° with respect to the equatorial triangle. For $N = 10$ the algorithm produced a point at each pole and a skew square-based prism in between.

TABLE 2

	1	2	3	4	5	6	7
1	0.00000	1.56030	1.31347	1.43770	1.42931	1.31681	1.99914
2	1.56030	0.00000	1.18070	1.88124	1.88478	1.18164	1.29635
3	1.31347	1.18070	0.00000	1.19471	1.91488	1.86137	1.52339
4	1.43770	1.88124	1.19471	0.00000	1.20434	1.91690	1.35798
5	1.42931	1.88478	1.91488	1.20434	0.00000	1.19618	1.36356
6	1.31681	1.18164	1.86137	1.91690	1.19618	0.00000	1.51618
7	1.99914	1.29635	1.52339	1.35798	1.36356	1.51618	0.00000

In the case $N = 7$ the computer search produced a configuration close to the regular 7-bipyramid. It consists of two points almost antipodal and the remaining five points sprinkled around an equatorial band. In the table of mutual distances above, points 1 and 7 are the almost antipodal points. The sum of the mutual distances of the points of this figure improve by .001% that of the regular 7-bipyramid. Moreover, the vector sum of the 7 points is not at the origin.

We note that with the exception of $N = 7$, the configurations which arose in our computer search closely agree with the known best solutions for minimizing the total potential of unit charges on the sphere, i.e. minimizing $\Sigma 1/|p_i - p_j|$. Consult [5] or [7] for details. Also, [10] contains a detailed discussion of these and related problems.

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