MATHEMATICS OF COMPUTATION, VOLUME 31, NUMBER 140
OCTOBER 1977, PAGES 1031–1033

Corrections to Dickson's Table of Three Dimensional Division Algebras Over F_5

By F. W. Long

Abstract. In 1906 Dickson published a list of noncommutative division algebras of dimension three over the field of finite elements. Recently, Kaplansky has found a mistake in this list; and I used a computer to check all the entries in the list. I then found an algebra missing from the list and used the computer again to make a complete search for all such algebras.

In [1] Dickson published a list of 36 noncommutative division algebras of dimension three over $\mathbf{F}_5$, the field of five elements. Kaplansky has found a mistake in this table (see [2]) and has conjectured that there are only 32 such algebras.

I have run a computer programme to check Dickson's table and found that there are five mistakes. Dickson takes the algebra to have basis 1, i, j with

(1) $\quad i^2 = j, \quad ij = b + \beta i, \quad ji = a + \alpha i, \quad j^2 = d + \delta i + Dj, \quad b, \beta, a, \alpha, d, \delta, D \in \mathbf{F}_5.$

Then the following entries in his table give algebras which have zero divisors, as shown:

b	β	α	a	d	δ	D	
2	1	−1	1	0	1	2	$(-2-2i+j)(-1+2i+j) = -5i-5j$
2	1	−1	−1	0	2	−2	$(1+i+j)(2-i+j) = 5+5i$
2	1	−2	1	1	1	0	$(-1-2i+j)(-2+i+j) = -5j$
2	2	1	2	2	−1	−1	$(-2i+j)(-2+i+j) = -5j$
2	2	−1	−1	1	2	0	$(2-2i+j)(1-4i+j) = -5i+5j$

This leaves 31 algebras which are indeed division algebras. By calculating the opposite algebra of each of these 31 algebras I was able to locate a division algebra which Dickson had omitted from his table. This has the following representation in Dickson's notation:

$$b = 2, \quad \beta = 1, \quad \alpha = 2, \quad a = -2, \quad d = 2, \quad \delta = -1, \quad D = -1.$$

I used the computer to check that this is in fact a division algebra.

Hence, the computer had verified that there are at least 32 division algebras of dimension three over $\mathbf{F}_5$. The next problem was to prove that there are no more. Checking all possible cases seems, at first, to be an impossible task. However, certain reductions are possible.

Received November 9, 1976; revised March 14, 1977.
AMS (MOS) subject classifications (1970). Primary 17–04, 17A30.
Key words and phrases. Three dimensional division algebras, the field of five elements.

If the algebra is defined by relations (1) above, then the elementary transformation $i \longmapsto si$, $s \in \mathbf{F}_5$, $s \neq 0$ allows us to choose b to be some fixed element of $\mathbf{F}_5$ (it must be nonzero). So, we fix $b = 2$, as in Dickson's table.

Now Dickson has proved that if the algebra defined by the relations (1) is a division algebra, then the cubic equations $x^3 - \beta x - b$ and $x^3 - \alpha x - a$ must be irreducible over $\mathbf{F}_5$ (see [1, p. 371]). Hence, with $b = 2$, we must have $\beta = 1$ or 2. Further, we must have $a \neq 0$ and $\alpha \neq 0$.

These reductions bring the total number of algebras to be checked down to 4000. Once again I called the computer into use, and I was able to show that there are exactly 32 noncommutative division algebras of dimension 3 over $\mathbf{F}_5$ (and two commutative ones which Dickson calls I_0 and I_1). I give the complete list of algebras below, arranged so that an algebra and its opposite are on the same line.

The programmes were written in Algol 68-R and run on the ICL 1906A machine at the University of Manchester Regional Computer Centre via the link from Aberystwyth. I should like to thank the staff of the Aberystwyth Computer Unit for their help and support in using this link.

b	β	α	a	d	δ	D	b	β	α	a	d	δ	D
2	1	1	−2	0	1	−1	2	1	1	−2	0	−1	−1
2	1	−1	1	1	1	1	2	1	−1	−1	1	2	−1
2	1	−1	1	0	−1	2	2	1	−1	−1	0	−2	−2
2	1	2	2	−1	1	0	2	2	1	2	−1	1	0
2	1	2	−2	−1	−1	0	2	2	1	−2	−1	1	0
2	1	2	−2	2	2	1	2	2	1	−2	2	−2	1
2	1	2	2	2	−2	1	2	2	1	2	2	−2	1
2	1	−2	−1	1	1	0	2	2	−1	1	1	−2	0
2	1	−2	1	−2	2	2	2	2	−1	−1	−2	−1	−2
2	1	−2	−1	−2	−2	2	2	2	−1	1	−2	−1	−2
2	1	−2	1	−2	−2	−2	2	2	−1	−1	−2	1	2
2	1	−2	−1	−2	−2	−2	2	2	−1	1	−2	−1	2
2	2	1	−2	2	1	−1	2	1	2	−2	2	−1	−1
2	2	−2	1	−1	1	2	2	2	−2	−1	−1	2	−2
2	2	−2	1	0	2	−1	2	2	−2	−1	0	−1	1
2	2	−2	1	0	−2	−1	2	2	−2	−1	0	1	1

The computer also produced the pair

2	2	2	−2	0	2	−1	2	2	2	−2	0	−2	−2

These are linearly isomorphic to the algebras on the first line of the table.

Since completing this work I have learned that G. Menichetti has proved the conjecture that every three dimensional division algebra over a finite field is a twisted field. In particular, this verifies the correctness of the count of 32 over the field of five elements. See [3].

Pure Mathematics Department
University College of Wales
Penglais, Aberystwyth
Dyfed, Wales, United Kingdom

1. L. E. DICKSON, "Linear algebras in which division is always uniquely possible," *Trans. Amer. Math. Soc.*, v. 7, 1906, pp. 370–390.

2. I. KAPLANSKY, "Three dimensional division algebras. II," *Houston J. Math.*, v. I, No. 1, 1976.

3. G. MENICHETTI, *On a Kaplansky Conjecture Concerning Three-Dimensional Division Algebras over a Finite Field,* Mathematics Institute, University of Florence, Italy. (Preprint.)