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On the Imaginary Bicyclic Biquadratic Fields With Class-Number 2

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Abstract. Assuming that the list of imaginary quadratic number fields of class-number 4 is complete, a determination is made of all imaginary bicyclic biquadratic number fields of class-number 2.

1. Introduction. Recently, Brown and Parry [2] have determined all imaginary bicyclic biquadratic fields K with class-number $H = 1$, using results of Stark [11], [12] and Montgomery and Weinberger [10] giving all imaginary quadratic fields with class-numbers 1 and 2. Assuming that the list of imaginary quadratic fields with class-number 4 given by the first author [3], [4] is complete, we determine all imaginary bicyclic biquadratic fields with class-number 2. Available evidence suggests that this list is indeed complete, for if there were an imaginary quadratic field with class-number 4 and discriminant D with $-D > 4 \times 10^6$, then, by Dirichlet's class-number formula, we would have

$$0 < L(1, \chi_D) < \frac{4\pi}{2000} < 0.0065.$$

However, the observed minimum of $L(1, \chi_D)$ for $0 < -D < 4 \times 10^6$ is 0.1988 (see [4]).

We let k_1, k_2 and k be the three quadratic subfields of K, where we take k to be the real field. We write h for the class-number of k and h_i for the class-number of k_i $(i = 1, 2)$. The fundamental unit of k is denoted by ϵ. From the work of Herglotz [6] we have

$$H = \frac{\sigma h_1 h_2 h}{\lambda_0},$$

where $\sigma = 2$ or 1 according as $K = Q(\sqrt{-1}, \sqrt{-2})$ or not, and λ_0 is defined by $N_{K/k}(E) = \epsilon^{\lambda_0}$, where E denotes a fundamental unit of K. Herglotz [6] has noted that $\lambda_0 = 1$ or 2, and Brown and Parry [2] have remarked that if the norm of ϵ is -1, then $\lambda_0 = 2$. If $K = Q(\sqrt{-1}, \sqrt{-2})$, then $h_1 = h_2 = h = 1$, $\sigma = 2$, $\epsilon = 1 + \sqrt{2}$, $\lambda_0 = 2$; and we have $H = 1$. This field can thus be omitted from all future considerations, and we take $\sigma = 1$ from this point on.

The determination of those fields with $H = 2$ falls naturally into 4 cases:

I. $h_1 = h_2 = 1$,

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II. $h_1 = 1, h_2 = 2$,
III. $h_1 = 1, h_2 = 4$,
IV. $h_1 = h_2 = 2$.
For $H = 2$ in case I, we must have $h = 2$, $\lambda_0 = 1$, or $h = 4$, $\lambda_0 = 2$; in case II, $h = 1$, $\lambda_0 = 1$, or $h = 2$, $\lambda_0 = 2$; in case III, $h = 1$, $\lambda_0 = 2$; and in case IV, $h = 1$, $\lambda_0 = 2$.

Stark [11] has shown that the only imaginary quadratic fields with class-number 1 are the nine fields

$$Q(\sqrt{-n}): n = 1,2,3,7,11,19,43,67,163.$$

In case I, direct verification, using tables of class-numbers of real quadratic fields, shows that neither $h = 2$ nor $h = 4$ ever occurs; case I yields no fields K with $H = 2$.

2. Determination of λ_0. In this section we develop a criterion for determining the value of λ_0 in the case when $N(\epsilon) = +1$. Our first lemma, giving the roots of unity in K, is well known.

LEMMA 1. *Let m and n be positive squarefree integers with $m > 1$. Let ϵ be the fundamental unit of $Q(\sqrt{m})$.*

(a) *If $\sqrt{-1} \in Q(\sqrt{m}, \sqrt{-n})$, then the only roots of unity in $Q(\sqrt{m}, \sqrt{-n})$ are ± 1, $\pm\sqrt{-1}$, with the additional roots $\frac{1}{2}(\pm\sqrt{2} \pm \sqrt{-2})$, if $m = 2$, and $\frac{1}{2}(\pm 1 \pm \sqrt{-3})$, $\frac{1}{2}(\pm\sqrt{3} \pm \sqrt{-1})$, if $m = 3$.*

(b) *If $\sqrt{-2} \in Q(\sqrt{m}, \sqrt{-n})$, then the only roots of unity in $Q(\sqrt{m}, \sqrt{-n})$ are ± 1, with the additional roots $\pm\sqrt{-1}$, $\frac{1}{2}(\pm\sqrt{2} \pm\sqrt{-2})$, if $m = 2$, and $\frac{1}{2}(\pm 1 \pm \sqrt{-3})$, if $m = 6$.*

(c) *If $\sqrt{-3} \in Q(\sqrt{m}, \sqrt{-n})$, then the only roots of unity in $Q(\sqrt{m}, \sqrt{-n})$ are ± 1, $\frac{1}{2}(\pm 1 \pm \sqrt{-3})$, with the additional roots $\pm\sqrt{-1}$, $\frac{1}{2}(\pm\sqrt{3} \pm\sqrt{-1})$, if $m = 3$.*

(d) *If none of $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}$ belongs to $Q(\sqrt{m}, \sqrt{-n})$, then the only roots of unity in $Q(\sqrt{m}, \sqrt{-n})$ are ± 1.*

Our next lemma occurs in the work of Kuroda [9] and of Kubota [8].

LEMMA 2. *Suppose $N(\epsilon) = +1$.*

(a) *If either $\sqrt{-1}$ or $\sqrt{-2}$ belongs to $Q(\sqrt{m}, \sqrt{-n})$, then ϵ is a fundamental unit of $Q(\sqrt{m}, \sqrt{-n})$, equivalently $\lambda_0 = 2$, if and only if there do NOT exist rational integers A and B such that*

$$2\epsilon = (A + B\sqrt{m})^2.$$

(This condition is equivalent to the condition that $2\epsilon = \mu^2$, for some algebraic integer μ of $Q(\sqrt{m})$.)

(b) *If $\sqrt{-3}$ belongs to $Q(\sqrt{m}, \sqrt{-n})$, and $Q(\sqrt{m}, \sqrt{-n}) \neq Q(\sqrt{3}, \sqrt{-1})$, then ϵ is a fundamental unit of $Q(\sqrt{m}, \sqrt{-n})$, equivalently $\lambda_0 = 2$, if and only if there does NOT exist an integer μ of $Q(\sqrt{m})$ such that*

$$3\epsilon = \mu^2.$$

(c) *If none of $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$ belongs to $Q(\sqrt{m}, \sqrt{-n})$, then ϵ is a* *fundamental unit of* $Q(\sqrt{m}, \sqrt{-n})$, *equivalently* $\lambda_0 = 2$, *if and only if there does NOT exist an integer* μ *of* $Q(\sqrt{m})$ *such that*

$$n\epsilon = \mu^2.$$

Lemma 2 provides us with a criterion for determining λ_0. We next develop an effective method for applying it.

We define rational integers x and y by setting

$$\epsilon = \begin{cases} \tfrac{1}{2}(x + y\sqrt{m}), & x \equiv y \pmod 2, \text{ if } m \equiv 1 \pmod 4, \\ x + y\sqrt{m}, & \text{if } m \equiv 2, 3 \pmod 4). \end{cases}$$

Further, in order to simplify the statements of the following lemmas, we shall use the term "representable" to mean "representable as the square of an integer of $Q(\sqrt{m})$".

For the next two lemmas, see [1, Theorem 3.1].

LEMMA 3. *Let* r *be a squarefree rational integer such that* $r\epsilon$ *is representable. Then*

(a) *if* $m \equiv 1$ *or* 2 (mod 4), *or* $m \equiv 3$ (mod 4) *and* y *even, we have* $r | m$, *and* $m\epsilon/r$ *representable, and*

(b) *if* $m \equiv 3$ (mod 4) *and* y *odd, we have* r *even,* $r/2 | m$, *and* $4m\epsilon/r$ *representable.*

LEMMA 4. *Let* r *and* s *be squarefree rational integers such that* $r\epsilon$ *and* $s\epsilon$ *are both representable. By Lemma* 3, *in case* (a) *we have* $r | m$ *and* $s | m$, *and, in case* (b) r *and* s *are even and* $r/2 | m$ *and* $s/2 | m$. *Then in case* (a) *either* $r = s$ *or* $rs = m$, *and in case* (b) *either* $r = s$ *or* $rs = 4m$.

Lemmas 3 and 4 imply that if there are any squarefree integers r such that $r\epsilon$ is representable, then there are exactly two such integers, both of which must be factors of m or $2m$. We now specify these factors.

Let the principal cycle of binary quadratic forms of discriminant d ($d = m$ if $m \equiv 1$ (mod 4), and $d = 4m$ if $m \equiv 2, 3$ (mod 4)) be $(a_0, b_0, -a_1) \sim \cdots \sim (\pm a_i, b_i, \mp a_{i+1}) \sim \cdots \sim (-a_{2k-1}, b_{2k-1}, a_{2k})$, where $a_0 = a_{2k} = 1$. Halfway through this cycle we find

$$(\mp a_{k-1}, b_{k-1}, \pm a_k) \sim (\pm a_k, b_k, \mp a_{k-1}),$$

with $a_k | b_k$, $b_{k-1} = b_k$, $a_{k-1} = a_{k+1}$.

The form $(\pm a_k, b_k, \mp a_{k+1})$ is ambiguous, and the second half of the cycle contains the opposites of the forms of the first half in reverse order.

LEMMA 5. *The two squarefree integers* r_1 *and* r_2 *such that* $r_1\epsilon$ *and* $r_2\epsilon$ *are representable are, in case* (a), a_k *and* m/a_k *and, in case* (b), a_k *and* $4m/a_k$.

Proof. Defining recursively the integers α_i of $Q(\sqrt{m})$ by

$$\alpha_0 = u_0 + v_0\sqrt{m} = \frac{b_0 + \sqrt{m}}{2a_1},$$

$$\alpha_i = u_i + v_i\sqrt{m} = \alpha_{i-1}\left(\frac{b_i + \sqrt{m}}{2a_{i+1}}\right) \qquad (i \geq 1),$$

we have the following well-known result [7]:

$$\epsilon = \alpha_{2k-1} = \left(\frac{b_0 + \sqrt{m}}{2a_1}\right)\left(\frac{b_1 + \sqrt{m}}{2a_2}\right) \cdots \left(\frac{b_{2k-1} + \sqrt{m}}{2a_{2k}}\right).$$

But, as $b_{k+i} = b_{k-1-i}$, $a_{k+i} = a_{k-i}$, $a_{2k} = 1$, the above product is

$$\epsilon = \left(\frac{b_0 + \sqrt{m}}{2}\right)^2 \left(\frac{b_1 + \sqrt{m}}{2a_1}\right)^2 \cdots \left(\frac{b_{k-1} + \sqrt{m}}{2a_{k-1}}\right)^2 \frac{1}{a_k}.$$

Thus $a_k\epsilon$ is the square of an integer of $Q(\sqrt{m})$. Since $a_k \mid b_k$, $a_k \mid d$, and thus a_k is squarefree.

The following useful lemma is a consequence of Lemma 3.

LEMMA 6. *If n_1 and n_2 are distinct positive squarefree integers, and $g = (n_1, n_2) > 1$, then $\lambda_0 = 2$ for $Q(\sqrt{-n_1}, \sqrt{-n_2})$.*

Proof. Let $m = n_1 n_2/g^2$, so that m is a squarefree integer > 1. The real quadratic subfield of $Q(\sqrt{-n_1}, \sqrt{-n_2})$ is $Q(\sqrt{m})$. As $g > 1$, we have $n_1 \nmid m$ and $n_2 \nmid m$ so that by Lemma 3, $n_1\epsilon$ and $n_2\epsilon$ are not representable. Hence the fundamental unit ϵ of $Q(\sqrt{m})$ is not a fundamental unit of $Q(\sqrt{-n_1}, \sqrt{-n_2})$, and we must have $\lambda_0 = 2$

3. Consideration of Case II. In this case we consider those fields K for which $h_1 = 1$ and $h_2 = 2$. Montgomery and Weinberger [10] and Stark [12] have shown that there are exactly 18 imaginary quadratic fields with class-number 2, namely,

$$Q(\sqrt{-n}): n = 5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427.$$

Thus, we have $9 \times 18 = 162$ fields to consider. In order to have $H = 2$ we must have $h = \lambda_0 = 1$ or $h = \lambda_0 = 2$. Of these, nineteen have $h = 1$, of which four have $\lambda_0 = 1$

$$Q(\sqrt{-1}, \sqrt{-6}), \quad Q(\sqrt{-1}, \sqrt{-22}), \quad Q(\sqrt{-2}, \sqrt{-6}), \quad Q(\sqrt{-2}, \sqrt{-22}). \tag{3.1}$$

These fields have $H = 2$. The other 15 fields appear in the list given by Brown and Parry [2] and have $H = 1$. It should be noted that for each of these $N(\epsilon) = -1$. There were no fields with $h = 1$, $N(\epsilon) = 1$, $\lambda_0 = 2$, $H = 1$. Of the remaining 143 fields, eight have $h > 2$ and so can be excluded. Five of the remaining fields have $N(\epsilon) = -1$, $\lambda_0 = 2$, hence $H = 2$. They are

$$\begin{gathered} Q(\sqrt{-1}, \sqrt{-10}), \quad Q(\sqrt{-1}, \sqrt{-58}), \quad Q(\sqrt{-2}, \sqrt{-5}), \\ Q(\sqrt{-2}, \sqrt{-13}), \quad Q(\sqrt{-2}, \sqrt{-37}). \end{gathered} \tag{3.2}$$

Finally, using the criterion given in Lemma 5 we determine whether $\lambda_0 = 1$ or 2 for the remaining 130 fields. We find that there are exactly 85 with $\lambda_0 = 2$, so that $H = 2$:

$$
(3.3)\quad
\begin{array}{ll}
Q(\sqrt{-1},\sqrt{-n}), & n = 15,35,91,115,403,\\
Q(\sqrt{-2},\sqrt{-n}), & n = 15,35,91,115,235,403,427,\\
Q(\sqrt{-3},\sqrt{-n}), & n = 5,10,22,35,58,115,187,235,\\
Q(\sqrt{-7},\sqrt{-n}), & n = 5,10,13,15,51,115,123,187,235,267,403,\\
Q(\sqrt{-11},\sqrt{-n}), & n = 6,13,51,58,91,123,403,427,\\
Q(\sqrt{-19},\sqrt{-n}), & n = 6,13,22,37,58,91,123,403,\\
Q(\sqrt{-43},\sqrt{-n}), & n = 5,6,10,15,22,35,37,58,115,235,267,427,\\
Q(\sqrt{-67},\sqrt{-n}), & n = 5,6,10,13,15,22,35,123,235,403,\\
Q(\sqrt{-163},\sqrt{-n}), & n = 5,6,10,13,15,22,35,37,51,58,91,115,187,\\
 & \qquad 235,267,403.
\end{array}
$$

Case II yields 94 fields with $H = 2$.

4. Consideration of Case III. In this case we consider those fields K with $h_1 = 1, h_2 = 4$. In order for $H = 2$ to hold we must have $h = 1, \lambda_0 = 2$. The class-number of $Q(\sqrt{-n})$ is 4 for the 54 values of n shown in Table 1; it is conjectured that this list is complete. We thus have $9 \times 54 = 486$ fields to consider.

We begin by considering the 54 fields K containing $Q(\sqrt{-1})$ as a subfield. Direct examination of tables and genus considerations show that $h \geqslant 2$ (so that $H > 2$) except in 13 cases. Of these, four have $N(\epsilon) = -1$, so that $\lambda_0 = 2, H = 2$:

$$
(4.1)\quad Q(\sqrt{-1},\sqrt{-17}),\quad Q(\sqrt{-1},\sqrt{-73}),\quad Q(\sqrt{-1},\sqrt{-97}),\quad Q(\sqrt{-1},\sqrt{-193}).
$$

The condition that 2ϵ be representable is satisfied for $n = 14$ and 46, so $\lambda_0 = 1$, $H = 4$. The discriminant of the real quadratic subfield in the remaining 7 cases is odd; therefore, 2ϵ cannot be represented.

Thus, the seven fields

$$
(4.2)\quad
\begin{array}{c}
Q(\sqrt{-1},\sqrt{-21}),\quad Q(\sqrt{-1},\sqrt{-33}),\quad Q(\sqrt{-1},\sqrt{-57}),\quad Q(\sqrt{-1},\sqrt{-93}),\\
Q(\sqrt{-1},\sqrt{-133}),\quad Q(\sqrt{-1},\sqrt{-177}),\quad Q(\sqrt{-1},\sqrt{-253}),
\end{array}
$$

all have $H = 2$.

Next we consider the fields K having $Q(\sqrt{-2})$ as a subfield. Again, direct examination of tables and appeal to genus considerations show that $h \geqslant 2$ (so that $H > 2$) except in six cases. Of these six fields, two have $N(\epsilon) = -1$, so that $\lambda_0 = 2$, $H = 2$, namely the fields

$$
(4.3)\quad Q(\sqrt{-2},\sqrt{-34}),\quad Q(\sqrt{-2},\sqrt{-82}).
$$

Of the remaining four fields $Q(\sqrt{-2},\sqrt{-n})$ $(n = 14,42,46,142)$ the condition that 2ϵ be representable is satisfied for the three values $n = 14,46,142$. Thus, just the field

TABLE 1

Imaginary quadratic fields of discriminant d with class-number 4					
Odd discriminants	O_1	-d = pq	39	323	1027
			55	355	1227
			155	667	1243
			203	723	1387
			219	763	1411
			259	955	1507
			291	1003	1555
	O_2	-d = pqr	195	627	
			435	715	
			483	795	
			555	1435	
			595		
Even discriminants	E_1	-d = 4p	4.17	4.97	
			4.73	4.193	
	E_2	-d = 8p	8.7	8.41	
			8.17	8.71	
			8.23		
	E_3	-d = 4pq	4.21	4.93	
			4.33	4.133	
			4.57	4.177	
			4.85	4.253	
	E_4	-d = 8pq	8.15	8.51	
			8.21	8.65	
			8.35	8.95	
			8.39		

$$Q(\sqrt{-2}, \sqrt{-42}) \tag{4.4}$$

has $H = 2$.

Finally, it remains to consider the $7 \times 54 = 378$ fields K which do not possess either $Q(\sqrt{-1})$ or $Q(\sqrt{-2})$ as a subfield. These are all of the form $Q(\sqrt{-p}, \sqrt{-n})$, where

p is a prime $\equiv 3 \pmod 4$ (indeed $p = 3,7,11,19,43,67,163$) and n is one of the integers listed in Table 1.

We begin by looking at those fields for which $p \nmid n$. Genus considerations show that in cases O_1, E_1, E_2 (see Table 1) we have $2 \mid h$, and in cases O_2, E_3, E_4 we have $4 \mid h$, so that certainly $h \neq 1$. In many cases this can also be directly verified from tables. Thus, we need only consider those fields for which $p \mid n$. However, when $p \mid n$ we know from Lemma 6 that $\lambda_0 = 2$, giving $H = 2h$. However, the only such fields with $h = 1$ are given by:

$$
\begin{aligned}
&p = 3, \quad n = 21,33,39,42,57,93,177,219,291,483,627,723,1227,\\
&p = 7, \quad n = 14,21,42,133,203,259,483,763,\\
&p = 11, \quad n = 33,55,253,627,1243,1507,\\
&p = 19, \quad n = 57,133,323,627,1387.
\end{aligned} \tag{4.5}
$$

(There are none corresponding to $p = 43,67,163$.) Those listed in (4.5) all have $H = 2$. Case III yields 46 fields with $H = 2$.

5. Consideration of Case IV. In this case we consider those fields $K = Q(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_1 = h_2 = 2$. There are $17 + 16 + \cdots + 2 + 1 = 153$ cases to consider. For $H = 2$ to hold, we must have $h = 1$, $\lambda_0 = 2$. If $(n_1, n_2) = 1$, genus considerations show that $2 \mid h$. If $(n_1, n_2) = 2$, genus considerations show that $2 \mid h$ except when $n_1 = 2p_1$, $n_2 = 2p_2$, p_1, p_2 distinct primes with $p_1 \equiv p_2 \pmod 4$. There are just 2 fields to be considered individually, namely, $Q(\sqrt{-6}, \sqrt{-22})$ and $Q(\sqrt{-10}, \sqrt{-58})$. The second of these is ruled out as $h = 4$. The first, on the other hand, has $h = 1$, $N(\epsilon) = 1$, and Lemma 6 shows that $\lambda_0 = 2$, so that $H = 2$ for

$$Q(\sqrt{-6}, \sqrt{-22}). \tag{5.1}$$

The only fields left to be considered are those for which $(n_1, n_2) > 2$. We need only consider those 19 fields for which $h = 1$, namely,

$$
\begin{aligned}
n_1 &= 5, & n_2 &= 10,15,35,115,235,\\
n_1 &= 10, & n_2 &= 15,35,115,235,\\
n_1 &= 13, & n_2 &= 91,403,\\
n_1 &= 15, & n_2 &= 35,115,235,\\
n_1 &= 35, & n_2 &= 115,235,\\
n_1 &= 51, & n_2 &= 187,\\
n_1 &= 91, & n_2 &= 403,\\
n_1 &= 115, & n_2 &= 235.
\end{aligned}
$$

The first of these, $Q(\sqrt{-5}, \sqrt{-10})$, has $N(\epsilon) = -1$, $\lambda_0 = 2$, $H = 2$. The remaining eighteen have $N(\epsilon) = +1$, and by Lemma 6, $\lambda_0 = 2$, $H = 2$. Thus all 19 fields listed in (5.2) have $H = 2$. Case IV yields 20 fields with $H = 2$.

6. Main Theorem and Concluding Remarks.

THEOREM. *If the list of imaginary quadratic fields with class-number* 4 *given in Table* 1 *is complete, then there are exactly* 160 *imaginary bicyclic biquadratic fields with class-number* 2. *These are listed in* (3.1), (3.2), (3.3), (4.1), (4.2), (4.3), (4.4), (4.5), (5.1), (5.2).

The following tables were used in the proof of the Theorem:

(i) that of E. L. Ince [7] giving the class-number, the cycles of binary quadratic forms, the fundamental unit ϵ and its norm, and the representation of a suitable multiple of ϵ as the square of an integer of $Q(\sqrt{m})$, for fields $Q(\sqrt{m})$ of radicand m, $2 \leqslant m \leqslant 2025$;

(ii) that of M. N. and G. Gras [5] giving the class-number and the norm of ϵ of $Q(\sqrt{m})$ for radicands m, $2 \leqslant m < 10^4$;

(iii) that of H. C. Williams and J. Broere [13] giving (among other things) the class-number of $Q(\sqrt{m})$ for radicands m, $2 \leqslant m < 1.5 \times 10^5$;

(iv) an unpublished table of A. O. L. Atkin giving the class-number, the cycle of forms, and the norm of ϵ, for fields $Q(\sqrt{d})$ of (odd and even) discriminant d, $4 \leqslant d < 4000$;

(v) an unpublished table of A. O. L. Atkin giving the class-number and the norm of ϵ, for fields $Q(\sqrt{d})$ of odd discriminant d, $5 \leqslant d < 900000$;

(vi) that of D. A. Buell [3] giving the class-numbers and class groups of the imaginary quadratic fields $Q(\sqrt{d})$ of discriminant d, $0 < -d < 4000000$.

Where necessary, further machine computations were carried out by the first and second authors independently on different computers. No numerical results in this paper are presented on the strength of a single source; needed information contained in only one of the tables (this applied particularly in determining λ_0) was verified by a separate computation. The determination of λ_0 was made by expanding $\sqrt{m}$ as a continued fraction to find the integer n such that $n\epsilon_m$ is representable in $Q(\sqrt{m})$. We remark that no discrepancies were found among the various tables.

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