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An Iterative Process for Nonlinear Monotonic Nonexpansive Operators in Hilbert Space

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Abstract. The following theorem is proved: Suppose H is a complex Hilbert space, and $T: H \longrightarrow H$ is a monotonic, nonexpansive operator on H, and $f \in H$. Define $S: H \longrightarrow H$ by $Su = -Tu + f$ for all $u \in H$. Suppose $0 \leqslant t_n \leqslant 1$ for all $n = 1, 2, 3, \ldots$, and $\Sigma_{n=1}^{\infty} t_n(1 - t_n)$ diverges. Then the iterative process $V_{n+1} = (1 - t_n)V_n + t_nSV_n$ converges to the unique solution $u = p$ of the equation $u + Tu = f$.

It is well known that the equation $u + Tu = f$ has a unique solution u for each f in a Hilbert space H provided that $T: H \longrightarrow H$ is monotonic and Lipschitzian (e.g., see [3]). The purpose of this paper is to show that if T is nonexpansive (Lipschitz constant 1), then the Mann iterative process [1] will, under a certain condition, converge to this unique solution.

The normal Mann iterative process is defined by $V_{n+1} = (1 - t_n)V_n + t_nTV_n$. We will use the condition that $\Sigma_{n=1}^{\infty} t_n(1 - t_n)$ diverges, which has been extensively used by Groetsch [2].

THEOREM. *Suppose H is a complex Hilbert space, and $T: H \longrightarrow H$ is a monotonic, nonexpansive operator on H, and $f \in H$. Define $S: H \longrightarrow H$ by $Su = -Tu + f$ for all $u \in H$. Suppose $0 \leqslant t_n \leqslant 1$ for all $n = 1, 2, 3, \ldots$, and $\Sigma_{n=1}^{\infty} t_n(1 - t_n)$ diverges. Then the iterative process $V_{n+1} = (1 - t_n)V_n + t_nSV_n$ converges to the unique solution $u = p$ of the equation $u + Tu = f$.*

Proof. We first observe that S is nonexpansive and satisfies $\operatorname{Re}(Sx - Sy, x - y) \leqslant 0$ for all $x, y \in H$. Since $Sp = p$, we get

$$\begin{aligned}\|V_{n+1} - p\|^2 &= \|(1 - t_n)(V_n - p) + t_n(SV_n - Sp)\|^2 \\ &= (1 - t_n)^2\|V_n - p\|^2 + 2t_n(1 - t_n)\operatorname{Re}(SV_n - Sp, V_n - p) \\ &\quad + t_n^2\|SV_n - Sp\|^2.\end{aligned}$$

Using $\operatorname{Re}(SV_n - Sp, V_n - p) \leqslant 0$, $t_n(1 - t_n) \geqslant 0$, and $\|SV_n - Sp\| \leqslant \|V_n - p\|$, we get

$$\|V_{n+1} - p\|^2 \leqslant \{(1 - t_n)^2 + t_n^2\}\|V_n - p\|^2,$$

which can also be written

$$\|V_{n+1} - p\|^2 \leqslant \{1 - 2t_n(1 - t_n)\}\|V_n - p\|^2.$$

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Upon iteration this yields

$$\|V_{n+1} - p\|^2 \leqslant \left\{ \prod_{k=1}^{n} [1 - 2t_k(1 - t_k)] \right\} \|V_1 - p\|^2.$$

We note that $0 \leqslant 2t(1 - t) \leqslant 1/2$ for $0 \leqslant t \leqslant 1$. From the divergence of $\Sigma_{n=1}^{\infty} t_n(1 - t_n)$ it now follows that $\lim_n \|V_{n+1} - p\| = 0$, whence $\{V_n\}$ converges to p.

A particular case is of some interest, viz. $t_n = 1/n$. $(1/n)(1 - 1/n) = (n - 1)/n^2 > 1/2n$ for $n > 2$ establishes the divergence of $\Sigma_{n=1}^{\infty} t_n(1 - t_n)$. There is however an alternate method in this particular case which gives the additional information of an error estimate. As before, we let p denote the unique solution of $u + Tu = f$, and we observe that

$$\|SV_n - Sp\| \leqslant \|V_n - p\| \leqslant \|V_1 - p\|.$$

We have

$$V_{n+1} = \frac{n}{n+1} V_n + \frac{1}{n+1} SV_n$$

and so

$$V_{n+1} - p = \frac{n}{n+1}(V_n - p) + \frac{1}{n+1}(SV_n - Sp),$$

whence

$$\|V_{n+1} - p\|^2 = \frac{n^2}{(n+1)^2} \|V_n - p\|^2 + \frac{2n}{(n+1)^2} \operatorname{Re}(SV_n - Sp,\ V_n - p) + \frac{1}{(n+1)^2} \|SV_n - Sp\|^2.$$

Thus, we get

$$(n + 1)^2 \|V_{n+1} - p\|^2 - n^2 \|V_n - p\|^2 \leqslant \|V_1 - p\|^2.$$

The left-hand side collapses upon summation from $n = 1$ to $n = N$ to yield

$$(N + 1)^2 \|V_{N+1} - p\|^2 - \|V_1 - p\|^2 \leqslant N \cdot \|V_1 - p\|^2.$$

Hence for each $N = 1, 2, 3, \ldots$, we have

$$\|V_{N+1} - p\|^2 \leqslant (1/(N + 1)) \|V_1 - p\|^2.$$

Thus $\{V_n\}$ converges to p and for each n we have

$$\|V_{n+1} - p\| \leqslant \frac{1}{\sqrt{n+1}} \|V_1 - p\|.$$

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