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Positivity of the Weights of Extended Gauss-Legendre Quadrature Rules

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Abstract. We show that the weights of extended Gauss-Legendre quadrature rules are all positive.

1. Introduction. We consider extended Gauss-Legendre quadrature formulas, i.e., integration rules of the type

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} A_i^{(n)} f(\xi_i^{(n)}) + \sum_{j=1}^{n+1} B_j^{(n)} f(x_j^{(n)}) + R_n(f), \tag{1}$$

where $\xi_i^{(n)}$, $i = 1, \ldots, n$, are the zeros of the nth degree Legendre polynomial $P_n(x)$, while the nodes $x_j^{(n)}$, $j = 1, 2, \ldots, n + 1$, and the weights $A_i^{(n)}$, $B_j^{(n)}$ are chosen so that (1) has degree of exactness $p = 3n + 1$ ($3n + 2$ if n is odd), i.e., $R_n(f) = 0$ whenever f is a polynomial of degree up to p. If we denote by $E_{n+1}(x)$ the polynomial of degree $n + 1$, whose zeros are the abscissas $x_j^{(n)}$, $j = 1, 2, \ldots, n + 1$, then $E_{n+1}(x)$ has to satisfy the following orthogonality relation

$$\int_{-1}^{1} P_n(x)E_{n+1}(x)x^k\,dx = 0, \qquad k = 0, 1, \ldots, n.$$

Szegö [4] has studied $E_{n+1}(x)$ in a different context and gives some very interesting results. For instance, he proves that the nodes $x_j^{(n)}$ are in $(-1, 1)$ and interlace with the zeros of $P_n(x)$.

Formulas for the computation of the weights $A_i^{(n)}$ and $B_j^{(n)}$ are given in [2], [3]. In [2] it is shown that the $B_j^{(n)}$'s are positive; however, nothing has been said about the sign of $A_i^{(n)}$. In this note we show that the weights $A_i^{(n)}$ are also positive.

2. Positivity of $A_i^{(n)}$. We consider the Legendre function of second kind

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{x - t}\,dt, \qquad n \geqslant 1, \tag{2}$$

defined for any x in the complex plane cut along the segment $[-1, 1]$; we introduce the function

$$\overline{Q}_n(x) = \frac{1}{2} \lim_{\epsilon \to +0} [Q_n(x + i\epsilon) + Q_n(x - i\epsilon)], \tag{3}$$

which is analytic on $(-1, 1)$. It is known [5, p. 78] that

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(4) $$\lim_{\epsilon\to+0}\,[Q_n(x+i\epsilon)-Q_n(x-i\epsilon)] = -i\pi P_n(x), \qquad -1<x<1.$$

From (2), (3) and (4), and recalling Lebesgue's convergence theorem, it then follows that at the zeros $\xi_i^{(n)}$, $i = 1, \ldots, n$, of $P_n(x)$ we have

(5) $$\bar{Q}_n(\xi_i^{(n)}) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{\xi_i^{(n)} - t}\,dt.$$

Let now

$$E_{n+1}(\cos\phi) = \lambda_0 \cos(n+1)\phi + \lambda_1 \cos(n-1)\phi + \cdots + \begin{cases} \lambda_{n/2}\cos\phi, & n \text{ even}, \\ \tfrac{1}{2}\lambda_{(n+1)/2}, & n \text{ odd}, \end{cases}$$

and

$$e_{n+1}(\phi) = \lambda_0 \sin(n+1)\phi + \lambda_1 \sin(n-1)\phi + \cdots + \begin{cases} \lambda_{n/2}\sin\phi, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

where $x = \cos\phi$, $0 < \phi < \pi$, and, as known [4], $\lambda_0 = (2n+1)!/(2^{2n}(n!)^2)$. Then, Szegö in his paper [4, p. 507] gives the following inequality

(6) $$\bar{Q}_n(\cos\phi)E_{n+1}(\cos\phi) + \frac{\pi}{2}P_n(\cos\phi)e_{n+1}(\phi) > 1, \qquad 0<\phi<\pi,$$

which implies that at the nodes $\xi_i^{(n)}$

(7) $$|E_{n+1}(\xi_i^{(n)})| > |\bar{Q}_n(\xi_i^{(n)})|^{-1}, \qquad i = 1, \ldots, n.$$

We are now ready to prove the following

THEOREM. *The weights $A_i^{(n)}$ and $B_j^{(n)}$ of the extended Gauss-Legendre rules are always positive.*

Proof. The positivity of $B_j^{(n)}$ has already been proved in [2]. In that paper, the following expression for the weights $A_i^{(n)}$ has also been given

(8) $$A_i^{(n)} = H_i^{(n)} - \frac{h_n}{k_n|P_n'(\xi_i^{(n)})|\,|q_{n+1}(\xi_i^{(n)})|}, \qquad i = 1, \ldots, n,$$

where $H_i^{(n)} = 2|\bar{Q}_n(\xi_i^{(n)})|/|P_n'(\xi_i^{(n)})|$ are the weights of the n-point Gauss-Legendre rule, $h_n = 2/(2n+1)$, $k_n = (2n)!/(2^n(n!)^2)$ and $q_{n+1}(x) = 1/(2^n\lambda_0)E_{n+1}(x)$. Recalling (7), from (8) we have

$$A_i^{(n)} > H_i^{(n)}\left(1 - \frac{h_n}{k_n}\,2^{n-1}\lambda_0\right) = 0,$$

which proves the theorem.

What follows is an immediate consequence (see for example [5, Theorem 15.2.2]) of the theorem we have just proved.

COROLLARY. *The quadrature process defined by* (1) *is convergent for every function $f(x)$ which is Riemann-integrable in* $[-1, 1]$, *i.e.*, $\lim_{n\to\infty} R_n(f) = 0$.

Remark. In his paper, Szegö derives, although not explicitly stated, the analogue of (6) for rules of type (1) with a weight function of the form $(1-x^2)^{\mu-1/2}$, when

$0 < \mu < 1$. In a way very similar to the Legendre case, it may then be shown that the weights of that type of rules are positive, too. For $\mu = 0, 1$ see [2].

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