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Evaluation of the Integral $\int_0^\infty \frac{t^{2\alpha-1} J_\nu(x\sqrt{1+t^2})}{(1+t^2)^{\alpha+\beta-1}}\,dt$

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Abstract. Methods are developed for evaluating the integral

$$I_\nu^{\alpha\beta}(x) = \int_0^\infty \frac{t^{2\alpha-1} J_\nu(x\sqrt{1+t^2})}{(1+t^2)^{\alpha+\beta-1}}\,dt,$$

where $J_\nu(t)$ is the Bessel function of the first kind and order ν, $\alpha > 0$, $\beta > 1/4$, and ν is real. Only $I_\nu^{1/2,1}(x)$ and $I_\nu^{\alpha,\nu/2+1-\alpha}(x)$ are included in previously published tables of integrals of Bessel functions. The integrals $I_1^{1/2,1/2}(x)$ and $I_2^{1/2,1}(x)$ are used in a technique developed by I. S. Fedorova for calculating the diameter distribution of long circular cylinders from small-angle x-ray, light, or neutron scattering data.

The $I_\nu^{\alpha\beta}(x)$ are shown to be proportional to a G function. From this result, power series expansions and recurrence relations are developed for use in evaluating the $I_\nu^{\alpha\beta}(x)$. A convenient expression is obtained for the quantity required in Fedorova's method for computing diameter distributions.

I. Introduction. In the method which Fedorova has recently developed [1], [2] for using light, small-angle x-ray, or small-angle neutron scattering data to calculate the diameter distribution of assemblies of independently-scattering long circular cylinders, the quantity

$$\Phi(a) = aI_1^{1/2,1/2}(a) - I_2^{1/2,1}(a)$$

must be evaluated, where

$$I_\nu^{\alpha\beta}(x) = \int_0^\infty \frac{t^{2\alpha-1} J_\nu(x\sqrt{1+t^2})}{(1+t^2)^{\alpha+\beta-1}}\,dt, \tag{1}$$

$x > 0$, and $J_\nu(x)$ is the Bessel function of the first kind and order ν. The conditions

$$\alpha > 0, \beta > 1/4, \tag{2}$$

ensure that the integral $I_\nu^{\alpha\beta}(x)$ exists for all real ν.

As Fedorova has pointed out [2], $I_\nu^{1/2,1}(x)$ can be expressed [3] in terms of Bessel functions. Only $I_\nu^{1/2,1}(x)$ and $I_\nu^{\alpha,\nu/2+1-\alpha}(x)$ have been listed in tables of integrals of Bessel functions.

In Section II, $I_\nu^{\alpha\beta}(x)$ is shown to be expressible in terms of a G function [4]. A series expansion and some recurrence relations are developed which are useful for evaluating $I_\nu^{\alpha\beta}(x)$. By use of some of the recurrence relations, in Section III the quantity $\Phi(a)$ used in Fedorova's diameter distribution method is expressed in terms of Bessel functions.

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II. Some Techniques for Calculating the $I_\nu^{\alpha\beta}(x)$.

(a) *Power Series.* The Bessel functions $J_\nu(x)$ are special cases of the G function [4] and can be expressed [5]

$$J_\nu(x) = G_{0,2}^{1,0}\left(\frac{x^2}{4}\,\middle|\,\frac{\nu}{2}, -\frac{\nu}{2}\right).$$

With this result, the integral $I_\nu^{\alpha\beta}(x)$ defined in (1) can be written [6], [7]

$$I_\nu^{\alpha\beta}(x) = \tfrac{1}{2}\Gamma(\alpha) G_{1,3}^{2,0}\left(\frac{x^2}{4}\,\middle|\,\begin{matrix}\alpha+\beta-1\\ \beta-1, \frac{\nu}{2}, \frac{-\nu}{2}\end{matrix}\right)$$

$$= \tfrac{1}{2}\Gamma(\alpha)\left\{\frac{\Gamma\left(\frac{\nu}{2}-\beta+1\right)}{\Gamma\left(\frac{\nu}{2}+\beta\right)\Gamma(\alpha)}\left(\frac{x^2}{4}\right)^{\beta-1} {}_1F_2\left(\begin{matrix}1-\alpha\\ \beta-\frac{\nu}{2}, \beta+\frac{\nu}{2}\end{matrix}\,\middle|\,-\frac{x^2}{4}\right)\right. \tag{3}$$

$$\left. + \frac{\Gamma\left(\beta-1-\frac{\nu}{2}\right)}{\Gamma(1+\nu)\Gamma\left(\alpha+\beta-1-\frac{\nu}{2}\right)} {}_1F_2\left(\begin{matrix}2+\frac{\nu}{2}-\alpha-\beta\\ 2+\frac{\nu}{2}-\beta, 1+\nu\end{matrix}\,\middle|\,-\frac{x^2}{4}\right)\right\},$$

where the ${}_1F_2$ functions are hypergeometric functions.

When this G function is expanded in a power series [7], $I_\nu^{\alpha\beta}(x)$ can be expressed

$$I_\nu^{\alpha\beta}(x) = \frac{\frac{\pi}{2}\Gamma(\alpha)}{\sin\left(\frac{2\beta-\nu}{2}\pi\right)}\left[S_\nu^{\alpha\beta}(x) - T_\nu^{\alpha\beta}(x)\right], \tag{4}$$

where

$$S_\nu^{\alpha\beta}(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+2\beta-2}}{k!\,\Gamma\left(\beta+\frac{\nu}{2}+k\right)\Gamma\left(\beta-\frac{\nu}{2}+k\right)\Gamma(\alpha-k)},$$

$$T_\nu^{\alpha\beta}(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\,\Gamma\left(\alpha+\beta-\frac{\nu}{2}-1-k\right)\Gamma\left(2-\beta+\frac{\nu}{2}+k\right)\Gamma(1+\nu+k)}.$$

In $S_\nu^{\alpha\beta}(x)$ and $T_\nu^{\alpha\beta}(x)$ and in all other series introduced below, terms are defined to be zero if their numerator is finite and in their denominator they contain a gamma function with argument equal to zero or a negative integer.

The expression for $I_\nu^{\alpha\beta}(x)$ can also be obtained by the residue theorem from the Mellin-Barnes representation for the function $G_{1,3}^{2,0}$ (see [4, p. 146]).

Equation (4) does not apply when $2\beta - \nu$ is an even integer. The series expansion for $\beta = j + 1 + \nu/2$, where j is an integer, can be obtained from (4) by finding the limit of this equation as β approaches $j + 1 + \nu/2$. This series can be expressed

$$I_\nu^{\alpha, j+1+\nu/2}(x) = \tfrac{1}{2}\Gamma(\alpha)\left[U_\nu^{\alpha j}(x) + V_\nu^{\alpha j}(x)\right], \tag{5}$$

where

$$U_\nu^{\alpha j}(x) = (-1)^j \sum_{k=0}^{\infty} \frac{[-2\gamma - 2\log_e(x/2) + P_k^{\alpha j}](x/2)^{2k+2j_+ +\nu}}{k!(k+|j|)!\Gamma(\alpha + j_- - k)\Gamma(1+\nu+j_+ + k)},$$

$$V_\nu^{\alpha 0}(x) = 0,$$

$$V_\nu^{\alpha j}(x) = \sum_{k=0}^{|j|-1} \frac{(-1)^k(|j|-1-k)!(x/2)^{2k+2j_- +\nu}}{k!\Gamma(\alpha + j_+ - k)\Gamma(1+\nu+k+j_-)}, \qquad j \neq 0,$$

$$j_+ = \frac{j+|j|}{2}, \qquad j_- = \frac{j-|j|}{2}.$$

$$P_k^{\alpha j} = Q(1+k) + Q(1+|j|+k) + Q(1+j_+ +\nu+k) - Q(\alpha + j_- - k)$$

(6)
$$Q(z) = \psi(z) + \gamma = \sum_{i=0}^{\infty}\left[\frac{1}{1+i} - \frac{1}{z+i}\right],$$

$$\psi(z) = \frac{d}{dz}[\log_e \Gamma(z)]$$

and γ is Euler's constant.

The relation [8]

(7)
$$\frac{Q(z)}{\Gamma(z)} = \frac{Q(1-z)}{\Gamma(z)} - \Gamma(1-z)\cos(z\pi)$$

was employed in obtaining $V_\nu^{\alpha j}(x)$ and also is useful in evaluating $U_\nu^{\alpha j}(x)$ when any of the $Q(z)$ has an argument equal to zero or a negative integer.

The expression for $Q(z)$ can often be simplified [8].

The power series (4) and (5) can often be used for calculating the $I_\nu^{\alpha\beta}(x)$. Other techniques, such as rational approximations and Chebyshev expansions, may also be convenient.

(b) *Recurrence Relations.* When the G function in (3) is expressed as a contour integral [9], the $I_\nu^{\alpha\beta}(x)$ can be written

(8)
$$I_\nu^{\alpha\beta}(x) = \frac{\Gamma(\alpha)}{4\pi i}\int_L \frac{\Gamma(\beta-1-s)\Gamma\left(\frac{\nu}{2}-s\right)}{\Gamma(\alpha+\beta-1-s)\Gamma\left(1+\frac{\nu}{2}+s\right)}\left(\frac{x}{2}\right)^{2s} ds.$$

The integration contour L is the first path described in [9], which also lists the conditions under which (8) is valid.

From (8),

(9)
$$\frac{x}{2\nu}[I_{\nu-1}^{\alpha\beta}(x) + I_{\nu+1}^{\alpha\beta}(x)] = \frac{\Gamma(\alpha)}{4\pi i}\int_L \frac{\Gamma(\beta-1-s)\Gamma\left(\frac{\nu-1}{2}-s\right)}{\Gamma(\alpha+\beta-1-s)\Gamma\left(\frac{\nu+3}{2}+s\right)}\left(\frac{x}{2}\right)^{2s+1} ds.$$

The recurrence relation

$$\frac{x}{2\nu}\left[I^{\alpha\beta}_{\nu-1}(x) + I^{\alpha\beta}_{\nu+1}(x)\right] = I^{\alpha,\beta+1/2}_{\nu}(x) \tag{10}$$

then is obtained from (9) by the change of variable $r = s + ½$.

Similar calculations show that

$$(2\beta + 2\alpha - \nu - 4)I^{\alpha\beta}_{\nu}(x) = x\left[I^{\alpha,\beta-3/2}_{\nu+1}(x) - I^{\alpha,\beta-1/2}_{\nu+1}(x)\right] + (2\beta - \nu - 4)I^{\alpha,\beta-1}_{\nu}(x), \tag{11}$$

$$I^{\alpha+1,\beta-1}_{\nu}(x) + I^{\alpha,\beta}_{\nu}(x) = I^{\alpha,\beta-1}_{\nu}(x), \tag{12}$$

and that

$$x^{-\nu}\frac{d}{dx}\left[x^{\nu}I^{\alpha\beta}_{\nu}(x)\right] = I^{\alpha,\beta-1/2}_{\nu-1}(x), \tag{13}$$

$$x^{\nu}\frac{d}{dx}\left[x^{-\nu}I^{\alpha\beta}_{\nu}(x)\right] = -I^{\alpha,\beta-1/2}_{\nu+1}(x). \tag{14}$$

Integral relations corresponding to (13) and (14) are

$$I^{\alpha\beta}_{\nu}(x) = x^{-\nu}\int_0^x y^{\nu}I^{\alpha,\beta-1/2}_{\nu-1}(y)\,dy, \tag{15}$$

$$I^{\alpha\beta}_{\nu}(x) = -x^{\nu}\int_x^{\infty} y^{-\nu}I^{\alpha,\beta-1/2}_{\nu+1}(y)\,dy. \tag{16}$$

After the $I^{\alpha\beta}_{\nu}(x)$ have been expressed in terms of known functions or evaluated by the power series or by some other technique for $\alpha_0 \leqslant \alpha < \alpha_0 + 1$ and $\beta_0 \leqslant \beta < \beta_0 + ½$ for the necessary values of ν, the recurrence relations (10), (11), and (12) can be used to find the $I^{\alpha\beta}_{\nu}(x)$ for larger values of α and β. Relations (13)–(16) are convenient when the $I^{\alpha\beta}_{\nu}(x)$ are expressed in terms of functions which can be easily integrated or differentiated.

When $\nu = 0$, (10) cannot be used. Equation (11) then can be employed.

When the recurrence relations are used successively many times, the possibility of round-off errors must be considered. This effect can be especially noticeable if (10) is used when $x/(2\nu)$ is large with respect to 1.

The special cases $I^{1/2,1}_{\nu}(x)$ and $I^{\alpha,\nu/2+1-\alpha}_{\nu}(x)$, which, as was mentioned in Section I, are often found in tables of integrals of Bessel functions, also follow from (3).

Thus, with the G-function representation [10] of the product $J_\nu(x)Y_\nu(x)$, the expression

$$I^{1/2,1}_{\nu}(x) = -\frac{\pi}{2}J_{\nu/2}(x/2)Y_{\nu/2}(x/2) \tag{17}$$

can be obtained. Also, substitution of (17) in (13) gives

$$I^{1/2,1/2}_{\nu}(x) = -\frac{\pi}{4}\left[J_{(\nu-1)/2}\left(\frac{x}{2}\right)Y_{(\nu+1)/2}\left(\frac{x}{2}\right) + J_{(\nu+1)/2}\left(\frac{x}{2}\right)Y_{(\nu-1)/2}\left(\frac{x}{2}\right)\right]. \tag{18}$$

Another useful equation

$$I^{1/2,3/2}_{0}(x) = \int_x^{\infty}\frac{\sin y}{y}\,dy$$

can be obtained either from (16) and (17) or by letting $\alpha = 1/2$, $\beta = 3/2$, and $\nu = 0$ in the equation expressing $I_\nu^{\alpha\beta}(x)$ in terms of the hypergeometric functions ${}_1F_2$ [4, p. 223, Eq. 25].

From (8) and the G-function representation [9], [5] of $J_\nu(x)$,

$$I_\nu^{\alpha,\nu/2+1-\alpha}(x) = \tfrac{1}{2}\Gamma(\alpha)(x/2)^{-\alpha}J_{\nu-\alpha}(x).$$

This result is one of Sonine's formulas [11].

III. The Expression for $\Phi(a)$. With (17), (18), and the relation [12]

$$Y_{\nu-1}(x)J_\nu(x) - Y_\nu(x)J_{\nu-1}(x) = 2/(\pi/x),$$

the quantity $\Phi(a)$ employed in Fedorova's method of computing the diameter distribution for assemblies of cylinders [1], [2] can be written

$$\Phi(a) = aI_1^{1/2,1/2}(a) - I_2^{1/2,1}(a) \tag{19}$$
$$= \frac{\pi}{2}Y_1(a/2)[J_1(a/2) - aJ_0(a/2)] - 1.$$

Equation (19) is convenient for calculating diameter distributions by Fedorova's method.

Substitution of the asymptotic expansions for the Bessel functions in (19) gives the asymptotic approximation previously obtained [2] by Fedorova by a different method.

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