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An Approximation for $\int_x^\infty e^{-t^2/2}\, t^p\, dt$, $x > 0$, p real

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Abstract. A new approximation is given for $\int_x^\infty e^{-t^2/2} t^p\, dt$, $x > 0$, p real, which extends an earlier approximation of Boyd's for $p = 0$.

Introduction. In [2] Boyd gives

$$g(x) = 4/[3x + \sqrt{x^2 + 8}] \tag{1}$$

as an approximation for

$$F(x) \equiv [1/Z(x)] \int_x^\infty Z(t)\, dt, \qquad x > 0, \tag{2}$$

where

$$Z(x) = \exp(-x^2/2). \tag{3}$$

It can be shown $g(x) > F(x)$, and in fact

$$g(x) - F(x) = 2x^{-7} + O(x^{-9}), \qquad (x \longrightarrow \infty). \tag{4}$$

For $x > c \equiv (4 - \pi)/\sqrt{\pi(\pi - 2)} \cong .453$, $g(x)$ serves as a much better approximation to $F(x)$ than the well-known estimate

$$h(x) = 2/[x + \sqrt{x^2 + 8/\pi}], \qquad [1, \text{p. } 298]. \tag{5}$$

More specifically, it is easy to conclude that

$$F(x) < g(x) < h(x), \qquad x > c, \tag{6}$$

as Table I below indicates.

TABLE I

x	$F(x)$	$g(x)$	$h(x)$
0.5	.8763645	.9148542	.9206969
1.0	.6556795	.6666667	.6936713
2.0	.4213692	.4226497	.4387303
5.0	.1928081	.1928216	.1951510
9.5	.1041337	.1041338	.1045309

Our objective is to generalize Boyd's result to the function

$$F(p, x) \equiv Y(p, x)/Z(p, x), \tag{7}$$

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where

$$(8) \qquad Y(p, x) \equiv \int_x^\infty Z(p, t)\, dt,$$

$$(9) \qquad Z(p, x) \equiv x^p \exp(-x^2/2), \qquad x > 0, \qquad p \text{ real}.$$

The corresponding approximation for $F(p, x)$ is given by

$$(10) \qquad g(p, x) = 4x/[3(x^2 - p) + \sqrt{(x^2 - p)^2 + 8(x^2 + p)}],$$

which reduces to (1) for $p = 0$, and also is exact for $p = 1$, i.e., $g(1, x) = F(1, x)$. For fixed p, it improves as x increases and, depending on the value of p, it bounds F either from above or below for all $x > x_m$. In particular,

$$(11) \qquad \begin{cases} \text{(a)} \quad p \leqslant 0, & x^2 \geqslant x_m^2 \equiv -p \qquad (\Rightarrow g(p, x) > F(p, x)), \\ \text{(b)} \quad 0 < p \leqslant 1, & x^2 \geqslant x_m^2 \equiv p + 2p^{2/3} \qquad (\Rightarrow g(p, x) \geqslant F(p, x))^*, \\ \text{(c)} \quad p \geqslant 1, & x^2 \geqslant x_m^2 \equiv p + 2p^{2/3} \qquad (\Rightarrow g(p, x) \leqslant F(p, x)). \end{cases}$$

By expanding (10) in powers of $1/x$ and subsequently taking the difference of the leading terms with those of the asymptotic series for $F(p, x)$,

$$(12) \qquad F(p, x) \cong \frac{1}{x}\left[1 + \frac{p-1}{x^2} + \frac{(p-1)(p-3)}{x^4} + \cdots\right], \qquad (x \longrightarrow \infty),$$

we find

$$(13) \quad g(p, x) - F(p, x) \cong \frac{2(1-p)}{x^7} + \frac{2(1-p)(4p-19)}{x^9} + O(x^{-11}), \qquad (x \longrightarrow \infty).^{**}$$

Table II is given to show the comparison between $F(p, x)$ and $g(p, x)$ for some selected values of p and x. The asterisked x values are close approximates of x_m given in (11).

Before deriving (10), we note that an approximation, $g_1(p, x)$, for F can also be obtained from the first two terms of the continued fraction expansion for the incomplete gamma function, namely

$$(14) \qquad g_1(p, x) = \frac{x^2 + 2}{x(x^2 + 3 - p)}, \qquad [4, \text{p. } 356].$$

The relationship between F and the incomplete gamma function is given below in (29).

From (12) and (14) one obtains

$$(15) \qquad g_1(p, x) - F(p, x) \cong 2(p-1)(p-3)/x^7 + O(x^{-9}), \qquad (x \longrightarrow \infty).$$

A comparison with (13) shows that at large x, g_1 is better than g for $2 < p < 4$, but it is not as good otherwise, especially at large $|p|$.

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*For $0 < p < 1$, computer results indicate $3p$ as a sharper estimate for x_m^2.

**Since the algebra is somewhat lengthy, the first two terms on the right-hand side of (12) and (13) were also computed by A. Morris, as well as three additional ones, using his algebraic computer program, "FLAP," [3].

TABLE II (see Table I for $p = 0$)

p	x	F(p,x)	g(p,x)	p	x	F(p,x)	g(p,x)
-100	1	.0099980	.0100000	-40	1	.0249672	.0250000
	5	.0401906	.0401948		6.3*	.0790482	.0790602
	10*	.0499988	.0500000		8	.0767483	.0767535
	13	.0482807	.0482812		12	.0650181	.0650189
-10	.75	.0772836	.0795903	-4	1	.2185598	.2500000
	3*	.1579338	.1583358		2*	.2473026	.2500000
	8	.1070674	.1070723		5	.1683551	.1684001
	12	.0774887	.0774891		10	.0953216	.0953223
-1	1*	.4614553	.5000000	-.8	.90*	.5032524	.5568742
	2	.3613286	.3636364		2	.3721367	.3742823
	5	.1860899	.1861126		5	.1873965	.1874189
	10	.0980755	.0980759		10	.0982647	.0982650
-.4	.64*	.6399281	.7794144	.2	.5	1.018194	.9765504
	.80	.6074871	.6594522		.77*	.8242077	.8262394
	1	.5652979	.5882353		1	.7091266	.7142857
	5	.1900660	.1900842		3	.3099818	.3101590
	10	.0986452	.0986454		7	.1406465	.1406479
.6	.8	.9957427	.9786360	1.4	1.4	.8090514	.8122124
	1.35*	.6549141	.6556292		1.98*	.5444214	.5441366
	3	.3212850	.3213781		4	.2557314	.2557085
	7	.1417435	.1417442		8	.1257627	.1257624
3	2	.7500000	.7583057	9	3	.9344615	1.000000
	2.68*	.4770367	.4765579		4.2*	.3920753	.3917566
	4	.2812500	.2811137		7	.1692794	.1692546
	8	.1289062	.1289046		10	.1084996	.1084977
19	5	.5265147	.5305361	39	7	.4765338	.4798227
	5.77*	.3404467	.3402561		7.87*	.2989724	.2988727
	8	.1712925	.1712553		11	.1311763	.1311612
	10	.1213368	.1213307		20	.0552170	.0552169
99	11	.3867447	.3881067	199	15	.4202082	.4231882
	11.9*	.2519044	.2518658		16.3*	.2245087	.2244941
	15	.1167631	.1167549		20	.0980926	.0980889
	20	.0660856	.0660851		30	.0427010	.0427009

Derivation of (10) and (11). We now derive (10) treating 11(a), (b) and (c) separately. The final results depend on the function $H(p, x)$ given below by (19) or (20).

Let

(16) $$f(p, x) \equiv 1/F(p, x) = Z(p, x)/Y(p, x).$$

For brevity, denote $f(p, x)$ by f, $\partial f/\partial x$ by f', $\partial^2 f/\partial x^2$ by f''. Then

(17) $$f' = f(f + p/x - x) = fu, \qquad u \equiv f + p/x - x,$$

(18) $$f'' = fH, \qquad H = H(p, x),$$

where

(19) $$H = 2f^2 + 3(p/x - x)f + (p/x - x)^2 - 1 - p/x^2,$$

or

(20) $$H = 2u^2 + (x - p/x)u - 1 - p/x^2 = u' + u^2.$$

We shall use the following properties of H, which are easily found:

(21) $$\lim_{x\to 0_+} H(p, x) = \begin{cases} \infty, & p < 0, \\ (4-\pi)/\pi, & p = 0, \\ -\infty, & 0 < p < 1, \\ \infty, & p > 1. \end{cases}$$

Also

(22) $$H(p, x) \cong 2(1-p)/x^4, \qquad (x \longrightarrow \infty),$$

which follows after some tedious algebra from (12) and (19).

For $p < 0$, we have

(23) $$u = f + p/x - x > 0, \qquad x \geqslant s \equiv \sqrt{-p}.$$

Indeed, let $S \equiv uY = Z - (x - p/x)Y$, so that $\partial S/\partial x \leqslant 0$ for $x^2 \geqslant -p$, and $S \cong Z/x^2$ $(x \longrightarrow \infty)$. Hence $S > 0$ for $x^2 \geqslant -p$, and since Y is always positive (23) follows.

From this result, with (21) and (22), we have $H > 0$ for $x \in [s, \infty)$. In fact, if this were not the case, there would exist a point $\zeta \in (s, \infty)$ for which $H(p, \zeta) = 0$, $H'(p, \zeta) \geqslant 0$. But this is impossible since

(24) $$H'(p, x) = (f + 2u)H(p, x) - 2u^3 + 2p/x^3$$

is negative if $H = 0$, $u > 0$, and $p \leqslant 0$.

Thus factoring H, as given in (19),

(25) $$H = (f - \eta_+)(f - \eta_-) > 0, \qquad x^2 \geqslant -p, p < 0,$$

we get (10) with 11(a) from $f - \eta_+ > 0$, where

(26) $$\eta_\pm(p, x) = [3(x - p/x) \pm \sqrt{(x - p/x)^2 + 8(1 + p/x^2)}]/4.$$

Now consider the case $p > 1$. From (21) and (22) we know H has at least one positive zero such that $H'(p, x_0) \leqslant 0$, where x_0 denotes the largest such zero. Moreover, if z denotes the largest zero of H with $H'(p, z) > 0$, then $z < x_0$. Thus $H \leqslant 0$ for all $x \geqslant x_0$. In order to get an estimate, x_m, of x_0 we have from (24) and $H'(p, x_0) \leqslant 0$, that

(27) $$u(p, x_0) \geqslant p^{1/3}/x_0, \qquad p \geqslant 1.$$

Inequality 11(c) now follows directly by using (27) and $f(p, x_0) = \eta_+(p, x_0)$ in the expression for u given in (17).

When $0 < p < 1$, the analysis used to obtain 11(b) is similar to that for $p > 1$. First, it is shown (27) holds with the inequality reversed. Then proceeding as above, one obtains 11(b) with $H(p, x) > 0$ for all $x > x_m > x_0$. The details are omitted.

Relation of $F(p, x)$ to the Incomplete Gamma Function. The quantity $F(p, x)$ can be related to the normalized incomplete gamma function. Let

(28) $$r = t^2/2, \qquad y = x^2/2,$$

so that

(29) $$F(p, x) = \frac{1}{\sqrt{2}} e^{y} y^{-p/2} \Gamma\left(\frac{p+1}{2}, y\right),$$

where

(30) $$\Gamma(a, z) \equiv \int_z^\infty e^{-r} r^{a-1}\, dr, \qquad z > 0,\ a \text{ real}.$$

Thus, we have from (10) and (11)

(31) $$\Gamma(a, y) = \left(\frac{4e^{-y} y^a}{3[y - a + ½] + \sqrt{(y - a + ½)^2 + 4(y + a - ½)}}\right),$$

with

(32) $$\begin{cases} \text{(a) } a \leqslant ½, & y \geqslant y_m \equiv ½ - a, & (\Rightarrow g(2a-1, \sqrt{2y}) > \Gamma(a, y)), \\ \text{(b) } ½ < a \leqslant 1, & y \geqslant y_m \equiv a - ½ + (2a-1)^{2/3}, & (\Rightarrow g(2a-1, \sqrt{2y}) \geqslant \Gamma(a, y)), \\ \text{(c) } a \geqslant 1, & y \geqslant y_m \equiv a - ½ + (2a-1)^{2/3}, & (\Rightarrow g(2a-1, \sqrt{2y}) \leqslant \Gamma(a, y)). \end{cases}$$

An Application. The function $g(p, x)$ is particularly good for giving quick estimates when $p < 0$ and $F(p, x)$ cannot be evaluated by the usual recurrence relationship on p. In addition, g can often be used to establish properties of F. For example, a result not obtained easily by direct means, is to find, for $p < 0$, a good estimate to $x = X$, where $F(p, X) \geqslant F(p, x)$ for all $x > 0$. In fact, by using (10) and noting that $F'(p, X) = 0$ requires $F(p, X) = X/(X^2 - p)$, we find

(33) $$g(p, X) - F(p, X) \cong -X(X^2 + p)/(X^2 - p)^3.$$

Therefore, with $X^2 \cong -p$,

(34) $$F(p, X) \cong F(p, \sqrt{-p}) \cong g(p, \sqrt{-p}) = 1/(2\sqrt{-p}).$$

Clearly (34) also implies that $F(p, X) \leqslant 1/(2\sqrt{-p})$. Thus, if $p = -100$, then $X = 9.950374$, $\sqrt{-p} = 10$, $F(-100, X) = .04999938$, $1/(2\sqrt{-p}) = .05$, $g(p, X) = .05000063$. Even for the case of $p = -4$, where one does not expect the right side of (33) to hold, we find $X = 1.791507$, $\sqrt{-p} = 2$, $F(-4, X) = .2484926$, $1/(2\sqrt{-p}) = .250$ $g(-4, X) = .2524567$.

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