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On the Fluctuations of Littlewood for Primes of the Form $4n \pm 1$

By Carter Bays and Richard H. Hudson

Abstract. Let $\pi_{b,c}(x)$ denote the number of primes $\leqslant x$ which are $\equiv c \pmod b$. Among the first 950,000,000 integers there are only a few thousand integers n with $\pi_{4,3}(n) < \pi_{4,1}(n)$. The authors find three new widely spaced regions containing hundreds of millions of such integers; the density of these integers and the spacing of the regions is of some importance because of their intimate connection with the truth or falsity of the analogue of the Riemann hypothesis for $L(s)$. The discovery that the majority of all integers n less than 2×10^{10} with $\pi_{4,3}(n) < \pi_{4,1}(n)$ are the 410,000,000 (consecutive) integers lying between 18,540,000,000 and 18,950,000,000 is a major surprise; results are carefully corroborated and some of the implications are discussed.

1. **Introduction and Summary.** In a letter written in 1853 [2] Chebyshev made several very interesting remarks on differences between $\pi_{4,3}(x)$ and $\pi_{4,1}(x)$ where $\pi_{b,c}(x)$ denotes the number of positive primes $\leqslant x$ which are $\equiv c \pmod b$. These remarks were "popularly" interpreted [9] as asserting that there are many more primes of the form $4n + 3$ than of the form $4n + 1$.

A very important interpretation of Chebyshev's assertion is the conjecture that

$$\lim_{x \to \infty} \sum_{p>2} (-1)^{(p-1)/2} e^{-p/x} = -\infty, \tag{1.1}$$

for it follows from the work of Hardy-Littlewood-Landau [3], [11], [12], that (1.1) is true if and only if no zero of

$$L(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \qquad (s = \sigma + it,\ \sigma > 0) \tag{1.2}$$

has real part $> \frac{1}{2}$ [10].

Due to a famous result of Littlewood [15], it is known that there are infinitely many integers x with $\pi_{4,3}(x) < \pi_{4,1}(x)$; indeed, using Ingham's [6] Ω-notation, we have

$$\pi_{4,1}(x) - \pi_{4,3}(x) = \Omega_{\pm}\left(\frac{x^{1/2}}{\log x} \log\log\log x\right). \tag{1.3}$$

On the other hand, let $N(x)$ be the number of integers $n \leqslant x$ with $\pi_{4,3}(n) - \pi_{4,1}(n) < 0$. Because of the results of Hardy-Littlewood-Landau, Knapowski and Turán conclude on p. 26 of [9] that we probably have

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(1.4) $$\lim_{x\to\infty} \frac{N(x)}{x} = 0.$$

The results of Leech [13] and Shanks [17] give $N(3 \cdot 10^6) = 3406$ so that (1.4) appears to be easily true. The results in the next section suggest, however, that $N(x)/x$ probably does not tend to zero nearly as fast as could be anticipated on the basis of the earlier numerical work of Leech, of Shanks, and of Lehmer [14]. The unexpected and rather astonishing fact that by far the majority of all integers less than 2×10^{10} with $\pi_{4,3}(x) < \pi_{4,1}(x)$ are the 410 million consecutive integers lying between 18,540,000,000 and 18,950,000,000 clearly calls for considerable corroboration which we provide in Section 4. A brief discussion of the implications of this unexpected turn of events appears in Section 5.

2. Notation. We define an integer n with

(2.1) $$\pi_{4,3}(n) - \pi_{4,1}(n) = \Delta(n) = -1$$

to be an axis crossing. Moreover, we define the lth axis crossing region for each $l > 1$ to be the lth set of consecutive positive integers $n_0(l), n_1(l), \ldots, n_f(l)$ with the properties that (1),

(2.2) $$\Delta(n_0(l)) = \Delta(n_f(l)) = -1,$$

and that (2) $\Delta(n) \geqslant 0$ for all n with $n_0(l) > n > n_f(l-1)$ and $n_0(l) > 2n_f(l-1)$. The choice of the factor 2 is somewhat arbitrary and is motivated only by the wide spacing of the regions discovered to date. We call $n_0 = n_0(l)$ an initial regional axis crossing and $n_f = n_f(l)$ a final regional axis crossing.

All axis crossings less than 2×10^{10} occur in the interior (including the boundary) of six rather widely spaced regions (this is, of course, what should be expected assuming the truth of the analogue of the Riemann hypothesis for $L(s)$ [6, p. 106]).

Any set of consecutive integers n with $\Delta(n) < 0$ for every n in the set will be called a negative block. In the exterior of axis crossing regions there exist only nonnegative blocks, i.e., (long) sets of consective integers with $\Delta(n) \geqslant 0$.

Finally, we define a deepest regional axis crossing to be an integer n_d lying within an axis crossing region with the property that

(2.3) $$\Delta(n) \geqslant \Delta(n_d)$$

for every integer n lying within the region.

3. Description of Axis Crossing Regions. Outside the ranges considered below, all integers less than 2×10^{10} lie inside nonnegative blocks.

Description of Region 1 (Leech and Shanks)

$n_0 = n_d = 26861 \qquad n_f = 26862 \qquad \Delta(n_d) = -1$

Description of Region 2 (Leech and Shanks)

$n_0 = 616841 \quad n_d = 623681 \quad n_f = 633798 \quad \Delta(n_d) = -8$

Range	max $\Delta(n)$	min $\Delta(n)$
616769–633882	12	−8

Description of Region 3 (Lehmer)

$n_0 = 12{,}306{,}137 \quad n_d = 12{,}366{,}589 \quad \Delta(n_d) = -24 \quad n_f = 12{,}382{,}326$

Range-steps of 40000	max $\Delta(n)$	min $\Delta(n)$
12,284,449–12,324,449	68	−12
12,324,449–12,364,449	35	−18
12,364,449–12,404,449	34	−24

Description of Region 4

$n_0 = 951{,}784{,}481 \quad n_d = 951{,}867{,}557 \quad \Delta(n_d) = -48 \quad n_f = 952{,}223{,}490$

Range-steps of 40000	max $\Delta(n)$	min $\Delta(n)$
951,780,000–951,820,000	35	−19
951,820,000–951,860,000	9	− 37
951,860,000–951,900,000	19	−48
951,900,000–951,940,000	23	−22
951,940,000–951,980,000	53	0

negative block	max $\Delta(n)$	min $\Delta(n)$
951,850,000–951,880,000	−12	−48

Description of Region 5

$n_0 = 6{,}309{,}280{,}709 \quad n_d = 6{,}345{,}026{,}777 \quad n_f = 6{,}403{,}150{,}198$

$\Delta(n_d) = -1374$

630–631 denotes interval from 6,300,000,000 to 6,310,000,000

Range-steps of 10,000,000	max $\Delta(n)$	min $\Delta(n)$
630–631	811	−29
631–632	207	−347
632–633	229	−471
633–634	15	−717
634–635	−559	−1374
635–636	−86	−930
636–637	−19	−389
637–638	187	−589
638–639	64	−626
639–640	361	−226
640–641	674	−50

negative block	max $\Delta(n)$	min $\Delta(n)$
6,340,000,000–6,370,000,000	−19	−1374

Description of Region 6

$n_0 = 18{,}465{,}126{,}293$ $\quad n_d = 18{,}699{,}356{,}297$ $\quad n_f = 19{,}033{,}524{,}538$

$\Delta(n_d) = -2719$

1846–47 denotes interval from 18,460,000,000 to 18,470,000,000

Range	max $\Delta(n)$	min $\Delta(n)$	Range	max $\Delta(n)$	min $\Delta(n)$	Range	max $\Delta(n)$	min $\Delta(n)$
1846–47	1010	−295	1866–67	−1435	−1906	1886–87	−898	−1984
1847–48	171	−372	1867–68	−1720	−2374	1887–88	−1453	−1922
1848–49	74	−587	1868–69	−2041	−2548	1888–89	−1195	−1944
1849–50	364	−454	1869–70	−2221	−2719	1889–90	−867	−1592
1850–51	628	−478	1870–71	−1862	−2635	1890–91	−1032	−1846
1851–52	−56	−717	1871–72	−1723	−2229	1891–92	−938	−1983
1852–53	24	−713	1872–73	−1829	−2450	1892–93	−538	−1294
1853–54	28	−467	1873–74	−1944	−2612	1893–94	−324	−1130
1854–55	−126	−1115	1874–75	−2088	−2672	1894–95	−54	−772
1855–56	−739	−1431	1875–76	−1538	−2260	1895–96	17	−685
1856–57	−789	−1471	1876–77	−1318	−1994	1896–97	−498	−939
1857–58	−820	−1505	1877–78	−1300	−1755	1897–98	−140	−649
1858–59	−694	−1270	1878–79	−1003	−1618	1898–99	0	−540
1859–60	−767	−1217	1879–80	−1087	−1566	1899–1900	244	−607
1860–61	−848	−1335	1880–81	−882	−1489	1900–01	256	−125
1861–62	−616	−1805	1881–82	−529	−1474	1901–02	735	−81
1862–63	−1354	−1839	1882–83	−469	−851	1902–03	948	169
1863–64	−1294	−1831	1883–84	−281	−949	1903–04	762	−60
1864–65	−878	−1731	1884–85	−418	−1201			
1865–66	−665	−1636	1885–86	−690	−1392			

negative block	max $\Delta(n)$	min $\Delta(n)$
18,540,000,000–18,950,000,000	−54	−2719

length of negative block exceeds 410,000,000

4. Corroboration of Results in Section 3. It would be unfortunate if the large amount of interesting data in Section 3 were marred by a faulty prime count; since prime counts of earlier authors have occasionally been in error, we have performed a sextuple corroboration of our results as follows.

(a) We computed primes of the form $6n \pm 1$ with an independent sieve size and found our prime counts to agree at $x = 19{,}920{,}000{,}000$. Here, we have

$$\pi_{4,1}(x) = 439{,}415{,}099, \qquad \pi_{4,3}(x) = 439{,}418{,}549,$$
$$\pi_{6,1}(x) = 439{,}412{,}098, \qquad \pi_{6,5}(x) = 439{,}421{,}549,$$
$$\pi(x) = 878{,}833{,}649.$$

(b) We checked our prime counts in (a) against those of earlier authors. We have:

$$\pi_{4,1}(10^{10}) = 227523275, \qquad \pi_{4,3}(10^{10}) = 227529235,$$

$$\pi(10^{10}) = 455052511 \qquad \text{(corrected value of Lehmer [14])}.$$

(c) Our prime count for the modulus 6 agrees with that of Brent [1] at 10^{11}, namely,

$$\pi_{6,1}(10^{11}) = 2059018668, \qquad \pi_{6,5}(10^{11}) = 2059036143,$$
$$\pi(10^{11}) = 4118054813.$$

(d) Although all the above forms of corroboration work off the same sieve, we note that our values for n_d for regions 1, 2, and 3 agree with Leech [13], Shanks [17], and Lehmer [14].

(e) We have used the generalization of Hudson and Brauer [4] of the formula of Meissel to primes of the form $4n \pm 1$ which, of course, works on an entirely different principle than the sieve used above, to verify certain special values.

(f) The entire program was rerun with an independent sieve size to double check all values listed in the paper.

5. Discussion of Numerical Work. Unfortunately, we did not compute the exact length of the negative blocks listed in Section 3. However, it is quite clear that the negative block which includes the 410,000,000 (consecutive) integers between 18,540,000,000 and 18,950,000,000 easily contains the majority of all integers less than 20 billion with $\Delta(n) < 0$. Taken as an isolated fact (and, perhaps, in any case), this turn of events is rather astonishing. However, to put it into perspective, one should note that the negative block which contains the 30,000,000 integers between 6,340,000,000 and 6,370,000,000 certainly dwarfs anything that occurs before 6 billion. The extraordinary length of negative blocks remains a major surprise, and it is not likely that its cause will be fully understood for some time.

Since even the most powerful analytic tools have proven quite unsuccessful in such investigations, we would like to recommend for consideration a new tool which we feel has some promise for increasing our understanding of the way in which Littlewood fluctuations are propagated.

The true importance of the formula of Meissel lies in the generalization referred to in the 1966 lecture of Brun (see A. L. Whiteman's excellent review, MR 36 #2548). Since it is well known (see [6], [11], or [17]) that the so-called Chebyshev phenomenon is an effect arising primarily from products of two primes $\leqslant x$, a meticulous row-by-row analysis of the effect of products of two primes $\leqslant x$ (following the pattern in [4] and [5]; e.g., for $x = 18{,}699{,}356{,}297$) would satisfy the hopes of Viggo Brun that "younger mathematicians will continue to explore reasons for the subtle influence of the distribution of primes not exceeding $a^{1/2}$ on the distribution of primes between $a^{1/2}$ and a". We certainly recognize that until such a meticulous analysis has been completed, or until vastly more numerical information is available, all conjectures regarding the true order of magnitude of $N(x)$ are highly speculative. However, the data in this paper certainly suggests to us that $N(x) \neq o(x^\delta)$ for any $\delta < 1$ and that, in fact, we could well have $N(x) > x/\log x$ for infinitely many x. If anything like the latter is true, the 7th and 8th axis crossing regions should prove to be very interesting.

Department of Mathematics and Computer Science
University of South Carolina
Columbia, South Carolina 29208

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