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Improved Error Estimates for Numerical Solutions of Symmetric Integral Equations

By E. Rakotch

Abstract. The most widely employed method for a numerical solution of a symmetric integral equation with kernel $K(x, t)$ in interval $I \equiv [a, b]$ is the replacement of the original problem by the sequence of eigenproblems

$$K^{(n)} y_i^{(n)} = \mu_{in} y_i^{(n)}, \qquad K^{(n)} \equiv \{w_{jn} K(x_{in}, x_{jn})\}, \qquad i = 1, \ldots, n,$$

with $w_{jn} > 0$ and $x_{jn} \in I$, $j = 1, \ldots, n$. The eigenvectors $y_i^{(n)}$ are further used to obtain an approximation, with improved error estimates, of the numerical eigensolution for some $N > n$, with no necessity of computing μ_{iN} and $y_i^{(N)}$, $i = 1, \ldots, N$, and of constructing another matrix.

1. Introduction. Let $K(x, t)$ be a Hermitian kernel defined in $I \times I$, where $I \equiv [a, b]$, i.e. $K(t, x) = \overline{K(x, t)}$, such that

$$F(x) \equiv \int_a^b |K(x, t)|^2 \, dt \quad \text{is bounded in } I.$$

It is known that all the *characteristic values* μ_i of $K(x, t)$ are real and there exists an orthonormal set $\{y_i(x)\}$ of *characteristic functions* [2], i.e.

$$(1) \qquad \int_a^b K(x, t) y_i(t) \, dt = \mu_i y_i(x), \qquad \int_a^b y_i(x) \overline{y_j(x)} \, dx = \delta_{ij}.$$

The first attempt to obtain a numerical solution for (1) with an error estimate for the characteristic values was made by Wielandt [4], which replaced the original problem by the sequence of eigenproblems

$$(2) \qquad K^{(n)} y_i^{(n)} = \mu_{in} y_i^{(n)}, \qquad K^{(n)} \equiv \{w_{jn} K(x_{in}, x_{jn})\}, \qquad i = 1, \ldots, n;$$

$w_{jn} > 0$ and $x_{jn} \in I$, $j = 1, \ldots, n$, are, respectively, the weights and the nodes of an integration rule S by which the approximation

$$\int_a^b f(x) \, dx \simeq \sum_{i=1}^n w_{in} f(x_{in})$$

is made. The eigenvalues μ_{kn}, $k = 1, \ldots, n$, which are all real, are then taken by Wielandt as approximations to the corresponding characteristic values of $K(x, t)$, where the correspondence is specified by the following assumptions:

Let $V \equiv \{\alpha_1, \ldots, \alpha_m\}$ be a subset of the set R of all eigenvalues of a square matrix A or of all characteristic values of a kernel $F(x, t)$ defined in $I \times I$, and let $W \equiv \{z^2 \mid z \in V\}$; then,

(a) if $\alpha_1, \ldots, \alpha_m$ are the m largest (smallest) real elements of R such that

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$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_m$ ($\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_m$), then every $\alpha_i \neq \alpha_m$ with multiplicity $r_i \geqslant 1$ occurs r_i times in V,

(b) if $\alpha_1, \ldots, \alpha_m$ are the m real elements of R of the largest modulus such that $|\alpha_1| \geqslant |\alpha_2| \geqslant \cdots \geqslant |\alpha_m|$ and there are r_i real elements of R of modulus $|\alpha_i|$, then every $\alpha_i^2 \neq \alpha_m^2$ occurs r_i times in W.

Wielandt obtained the error estimates and the convergence properties for the eigenvalues with integration rules S such that

$$\eta_n(x, t) \equiv \sum_{i=1}^{n} w_{in} K(x, x_{in}) K(x_{in}, t) - \int_a^b K(x, z) K(z, t)\, dz \tag{3}$$

converges to 0, as $n \longrightarrow \infty$, uniformly in $I \times I$, and they were further improved and extended to every integration rule satisfying (3) [3], together with an error estimate for the numerical solutions generated by the eigenvectors of (2), as defined in [3, Section 3]. The purpose of this paper is to obtain new numerical solutions for (1) with improved error estimates for those ones corresponding to simple characteristic values in terms of the error estimates, obtainable either by [4] or [3], for the eigensolution of (2) with n replaced by some $N > n$. This approach is due originally to Linz [1].

2. Error Estimation. The following theorem is first applied for the new error estimation [5, pp. 139–140]:

THEOREM 1. *Let A be a Hermitian matrix of order m with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $y \equiv (y_1, y_2, \ldots, y_m) \in C_m$—the m-dimensional complex Euclidean space—with*

$$|y| \equiv \left[\sum_{i=1}^{m} |y_i|^2 \right]^{1/2} = 1;$$

then for every number μ

$$\min_i |\mu - \lambda_i| \leqslant |Ay - \mu y|.$$

The smallest value of $|Ay - \mu y|$ with $|y| = 1$ is attained for $\mu = (Ay, y)$, which by Theorem 1 yields:

THEOREM 2. *Let A be a Hermitian matrix of order m with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$; then for every $y \in C_m$ such that $|y| = 1$ and for every μ*

$$\min_i |(Ay, y) - \lambda_i| \leqslant |Ay - \mu y|.$$

Define now a new scalar product $(u, v)_m$ of $u, v \in C_m$ and a new norm $|u|_m$ by

$$(u, v)_m \equiv \sum_{i=1}^{m} w_{im} u_i \bar{v}_i, \qquad |u|_m \equiv \sqrt{(u, u)_m}. \tag{4}$$

To obtain a numerical solution for a characteristic function, the eigenvectors $y_k^{(n)}$ of (2) were assumed to satisfy $|y_k^{(n)}|_n = 1$ [3, Section 3]. Now, given an approximate eigensolution $\tilde{\mu}_{kn}, \tilde{y}_k^{(n)}$ of (2), the numerical solution $\tilde{y}_{kn}(x)$ for a characteristic function generated by $\tilde{y}_k^{(n)}$ can be defined, as in [3, Section 3], by

$$\widetilde{y}_{kn}(x) \equiv \widetilde{\mu}_{kn}^{-1} \sum_{j=1}^{n} w_{jn}\, \widetilde{y}_{kj}^{(n)} K(x, x_{jn}).$$

Further, take $N > n$ and, using (4), obtain approximations $\widetilde{y}_k^{(N)}$ for $y_k^{(N)}$ and $\widetilde{\mu}_{kN}$ for μ_{kN} and a numerical solution $\widetilde{y}_{kN}(x)$ for the characteristic function as

$$\widetilde{y}_k^{(N)} \equiv |Y_k^{(N)}|_N^{-1}\, Y_k^{(N)}, \quad \text{where } Y_{ki}^{(N)} \equiv \widetilde{y}_{kn}(x_{iN}),\ i = 1, \dots, N,$$

$$\widetilde{\mu}_{kN} \equiv (K^{(N)} \widetilde{y}_k^{(N)}, \widetilde{y}_k^{(N)})_N,$$

$$\widetilde{y}_{kN}(x) \equiv \widetilde{\mu}_{kN}^{-1} \sum_{j=1}^{N} w_{jN} \widetilde{y}_{kj}^{(N)} K(x, x_{jN});$$

then for a simple characteristic value μ_k it can be shown that the error estimate for $\widetilde{\mu}_{kN}$ is of better order of magnitude than that obtained for μ_{kn} by [3, Theorem 1], provided that $|\widetilde{y}_k^{(n)} - y_k^{(n)}|_n$ and $|\widetilde{\mu}_{kn} - \mu_{kn}|$ are sufficiently small. Indeed,

$$(5) \qquad |\widetilde{\mu}_{kN} - \mu_k| \leqslant |\widetilde{\mu}_{kN} - \mu_{kN}| + |\mu_{kN} - \mu_k|,$$

and by [3, Theorem 1],

$$|\mu_{kN} - \mu_k| = O(\sigma_N), \quad \text{where } \sigma_m \equiv \max_{I \times I} |\eta_m(x, t)|;$$

therefore, it remains to prove that

$$(6) \qquad \widetilde{\mu}_{kN} - \mu_{kN} = O(\sigma_n^2 + |\widetilde{y}_k^{(n)} - y_k^{(n)}|_n^2 + (\widetilde{\mu}_{kn} - \mu_{kn})^2),$$

for which it will be shown below that

$$(7) \qquad |\widetilde{y}_k^{(N)} - y_k^{(N)}|_N = O(\sigma_n + |\widetilde{y}_k^{(n)} - y_k^{(n)}|_n + |\widetilde{\mu}_{kn} - \mu_{kn}|).$$

The error estimate for $\widetilde{y}_{kN}(x)$ will be deduced from

$$(8) \qquad \begin{aligned} |\widetilde{y}_{kN}(x) - y_k(x)| \leqslant{}& |\widetilde{y}_{kN}(x) - \widetilde{\mu}_{kN}^{-1} \mu_{kN} y_{kN}(x)| + |\widetilde{\mu}_{kN}^{-1} (\mu_{kN} - \widetilde{\mu}_{kN}) y_{kN}(x)| \\ &+ |y_{kN}(x) - y_k(x)|. \end{aligned}$$

To bound $|\widetilde{\mu}_{kN} - \mu_{kN}|$, observe that for sufficiently large n and N it follows by (6) and [3, Theorem 2] that μ_{kN} is a well-separated eigenvalue of $K^{(N)}$ such that

$$|\widetilde{\mu}_{kN} - \mu_{kN}| = \min_i |\widetilde{\mu}_{kN} - \mu_{iN}|,$$

which by Theorem 2 implies

$$(9) \qquad \begin{aligned} &|\widetilde{\mu}_{kN} - \mu_{kN}| \leqslant \epsilon_{kN} \equiv |H\widetilde{z}_k - \widetilde{\mu}_{kN} \widetilde{z}_k| = |K^{(N)} \widetilde{y}_k^{(N)} - \widetilde{\mu}_{kN} \widetilde{y}_k^{(N)}|_N, \\ &\quad \text{where } H_{ij} \equiv K(x_{iN}, x_{jN}) \sqrt{w_{iN} w_{jN}} \text{ and } \widetilde{z}_{ki} \equiv y_{ki}^{(N)} \sqrt{w_{iN}},\ i, j = 1, \dots, N. \end{aligned}$$

Further, to obtain a bound for $|\widetilde{y}_k^{(N)} - y_k^{(N)}|_N$, let $\widetilde{y}_k^{(N)}$ have the expansion

$$\widetilde{y}_k^{(N)} = \sum_{j=1}^{N} c_j y_j^{(N)}, \qquad c_k > 0,$$

with $\{y_j^{(N)}\}$ forming a set satisfying (see Definition (4) and [3, Section 3])

$$(y_p^{(N)}, y_q^{(N)})_N = \delta_{pq}, \qquad p, q = 1, \dots, N;$$

then

$$|\widetilde{y}_k^{(N)}| = \sum_{j=1}^{N} |c_j|^2 = 1,$$

and by a procedure similar to that applied in [6, pp. 172–173]

$$|\widetilde{y}_k^{(N)} - y_k^{(N)}|_N^2 \leqslant \delta_k^{(N)}(1 + \delta_k^{(N)}), \qquad |\widetilde{\mu}_{kN} - \mu_{kN}| \leqslant d_{kN}\delta_k^{(N)}(1 - \delta_k^{(N)})^{-1},$$

(10)

where $d_{kN} \equiv \min_{j \neq k} |\widetilde{\mu}_{kN} - \mu_{jN}|$ and $\delta_k^{(N)} \equiv \epsilon_{kN}^2 d_{kN}^{-2}$ with ϵ_{kN} defined in (9),

provided that $\delta_k^{(N)} < 1$. To establish (7), observe that

$$\widetilde{y}_{ki}^{(N)} - y_{ki}^{(N)} = d_i^{(1)} + d_i^{(2)} + d_i^{(3)},$$

where

$$d_i^{(1)} \equiv \widetilde{y}_{ki}^{(N)} - y_{kn}(x_{iN}), \quad d_i^{(2)} \equiv y_{kn}(x_{iN}) - y_k(x_{iN}), \quad d_i^{(3)} \equiv y_k(x_{iN}) - y_{kN}(x_{iN}).$$

Now

$$\widetilde{y}_{ki}^{(N)} = [\sigma_N^*(\widetilde{y}_{kn})]^{-1}\widetilde{y}_{kn}(x_{iN}), \quad \text{where } \sigma_N^*(u) \equiv \left[\sum_{i=1}^{N} w_{iN}|u(x_{iN})|^2\right]^{1/2},$$

$$\widetilde{y}_{kn}(x_{iN}) - y_{kn}(x_{iN}) = \mu_{kn}^{-1}\widetilde{\mu}_{kn}^{-1}(\mu_{kn} - \widetilde{\mu}_{kn})\widetilde{y}_{kn}(x_{iN}) + \mu_{kn}^{-1}\sum_{j=1}^{n} w_{jn}[\widetilde{y}_{kj}^{(n)} - y_{kj}^{(n)}]K(x_{iN}, x_{jn}),$$

$$\sigma_N^*(y_{kn}) = \left[1 + \mu_{kn}^{-2}\sum_{p,q=1}^{n} w_{pn}w_{qn}[\eta_N(x_{pn}, x_{qn}) - \eta_n(x_{pn}, x_{qn})]\, y_{kq}^{(n)}\overline{y_{kp}^{(n)}}\right]^{1/2},$$

$$\|y_{km}\|^2 = 1 - \mu_{km}^{-2}\sum_{i,j=1}^{m} w_{im}w_{jm}\eta_m(x_{im}, x_{jm})y_{kj}^{(n)}\overline{y_{ki}^{(n)}}. \tag{11}$$

Hence

$$\begin{aligned}|\sigma_N^*(\widetilde{y}_{kn}) - 1| &\leqslant |\sigma_N^*(\widetilde{y}_{kn}) - \sigma_N^*(y_{kn})| + |\sigma_N^*(y_{kn}) - 1| \\ &\leqslant |\sigma_N^*(\widetilde{y}_{kn} - y_{kn})| + |\sigma_N^*(y_{kn}) - 1| \\ &= O(|\widetilde{\mu}_{kn} - \mu_{kn}|) + O(|\widetilde{y}_k^{(n)} - y_k^{(n)}|_n) + O(\sigma_N) + O(\sigma_n),\end{aligned}$$

$$d_i^{(1)} = O(|\widetilde{y}_k^{(n)} - y_k^{(n)}|_n + |\widetilde{\mu}_{kn} - \mu_{kn}| + \sigma_n),$$

and by [1, Theorem 3], [1, Eq. (10)] and (11),

$$d_i^{(2)} = O(\sigma_n), \qquad d_i^{(3)} = O(\sigma_N),$$

which establishes (7).

For the sake of error estimation of $\widetilde{y}_{kN}(x)$, apply the Cauchy-Schwarz inequality to the first two terms at the right-hand of (8) to obtain

$$|\widetilde{y}_{kN}(x) - \widetilde{\mu}_{kN}^{-1}\mu_{kN}y_{kN}(x)| + |\widetilde{\mu}_{kN}^{-1}(\mu_{kN} - \widetilde{\mu}_{kN})y_{kN}(x)|$$
$$\leqslant |\widetilde{\mu}_{kN}^{-1}|[|\widetilde{y}_k^{(N)} - y_k^{(N)}|_N + |\mu_{kN}^{-1}(\widetilde{\mu}_{kN} - \mu_{kN})|]\sqrt{G_N(x)},$$

where

$$G_N(x) \equiv \sum_{j=1}^{N} w_{jN}|K(x, x_{jN})|^2,$$

with bounds for $|\widetilde{y}_k^{(N)} - y_k^{(N)}|_N$, $|\widetilde{\mu}_{kN} - \mu_{kN}|$ and μ_{kN} obtained from (10); the bound for the last term of the right-hand of (8) is obtained by application of [3, Theorem 3] or of the remark in [3, Section 4] with

$$|y_{kj}^{(N)}| \leqslant |\widetilde{y}_{kj}^{(N)}| + |\widetilde{y}_{kj}^{(N)} - y_{kj}^{(N)}| \leqslant |\widetilde{y}_{kj}^{(N)}| + w_{jN}^{-1/2}|\widetilde{y}_k^{(N)} - y_k^{(N)}|_N$$

and $|\widetilde{y}_k^{(N)} - y_k^{(N)}|_N$ bounded by (10).

3. Numerical Results. To illustrate the superiority of the new numerical solution and of the new error estimates, the first example presented in [3, Section 2] is taken for comparison. The error estimates for the $\widetilde{\mu}_{kN}$ presented in the following table are those obtained, using the triangle inequality in (5), with the estimate (10) for $|\widetilde{\mu}_{kN} - \mu_{kN}|$. The bounds for $|\mu_{kN} - \mu_k|$ and the μ_{jN} and the best error estimate for μ_{kn} are obtained in [3, Section 5], using the fact that μ_k is the nearest characteristic value to μ_{kn} and μ_{kN}, and

$$\eta_m(x, t) = \frac{1}{6(m-1)^2}\begin{cases} 3tA_m(x)B_m(x) + F_m(t), & x \leqslant t, \\ 3xA_m(t)B_m(t) + F_m(x), & x \geqslant t, \end{cases}$$

where (the formula for F_n in [3, Section 2, Example 1] is in error)

$$A_m(z) \equiv (m-1)z - [(m-1)z], \quad B_m(z) \equiv 1 - A_m(z),$$
$$F_m(z) \equiv 1 - z + A_m(z)B_m(z)\left(3z - \frac{2A_m(z) - 1}{m-1}\right).$$

The numerical solution for $\widetilde{y}_{kN}(x)$ is the function defined by (8) generated by the approximate eigenvector $\widetilde{y}_k^{(n)}$ corresponding to the kth negative eigenvalue μ_{kn}.

Case	k	Best error estimate for μ_{kn} by [3]	Error estimate for $\widetilde{\mu}_{kN}$	Actual error for $\widetilde{\mu}_{kN}$	Error estimate for $\widetilde{y}_{kN}(x)$	Actual maximal error for $\widetilde{y}_{kN}(l/m)$, $l = 0, 1, \ldots, m$
$n = 101$	1	$9.074 \cdot 10^{-5}$	$9.126 \cdot 10^{-7}$	$1.25 \cdot 10^{-7}$	0.001552	$1.75 \cdot 10^{-4}$
$N = 1000$	2	$4.342 \cdot 10^{-4}$	$5.046 \cdot 10^{-6}$	$9.11 \cdot 10^{-8}$	0.0745	$9.167 \cdot 10^{-4}$
$m = 3000$	3	$1.006 \cdot 10^{-3}$	$1.137 \cdot 10^{-5}$	$8.53 \cdot 10^{-8}$	0.721	$2.15 \cdot 10^{-3}$
$n = 201$	1	$2.27 \cdot 10^{-5}$	$2.274 \cdot 10^{-7}$	$3.14 \cdot 10^{-8}$	0.0004524	$5.177 \cdot 10^{-5}$
$N = 2001$	2	$1.09 \cdot 10^{-4}$	$1.26 \cdot 10^{-6}$	$2.3 \cdot 10^{-8}$	0.0163	$2.085 \cdot 10^{-4}$
$m = 6000$	3	$2.52 \cdot 10^{-4}$	$2.9 \cdot 10^{-6}$	$2.167 \cdot 10^{-8}$	0.165	$5.652 \cdot 10^{-4}$

Department of Mathematics
Technion, Israel Institute of Technology
Haifa, Israel

1. P. LINZ, "On the numerical computation of eigenvalues and eigenvectors of symmetric integral equations," *Math. Comp.*, v. 24, 1970, pp. 905–910. MR 43 #1461.

2. S. G. MIHLIN, *Lectures on Linear Integral Equations*, Fizmatgiz, Moscow, 1959; English transl., Russian Monographs and Texts on Advanced Math. and Phys., vol. 2, Gordon and Breach, New York; Hindustan, Delhi, 1960. MR 23 #A490; 24 #A3483.

3. E. RAKOTCH, "Numerical solution for eigenvalues and eigenfunctions of a Hermitian kernel and an error estimate," *Math. Comp.*, v. 29, 1975, pp. 794–805. MR 51 #9556.

4. H. WIELANDT, *Error Bounds for Eigenvalues of Symmetric Integral Equations*, Proc. Sympos. Appl. Math., Vol. 6, Amer. Math. Soc., Providence, R. I.,1956, pp. 261–282. MR 19, 179.

5. J. H. WILKINSON, *Rounding Errors in Algebraic Processes*, Prentice-Hall, Englewood Cliffs, N. J., 1963. MR 28 #4661.

6. J. H. WILKINSON, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965. MR 32 #1894.