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Roots of Two Transcendental Equations as Functions of a Continuous Real Parameter

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Abstract. The roots, λ and η, of the transcendental equations $j_l(\alpha\lambda)y_l(\lambda) = j_l(\lambda)y_l(\alpha\lambda)$ and

$$[xj_l(x)]'_{x=\alpha\eta}[xy_l(x)]'_{x=\eta} = [xj_l(x)]'_{x=\eta}[xy_l(x)]'_{x=\alpha\eta},$$

where $l = 1, 2, \ldots$ are considered as functions of the continuous real parameter α. The symbols j_l and y_l denote the spherical Bessel functions of the first and second kind. The two transcendental equations are invariant under the transformations $\lambda \to -\lambda$ and $\eta \to -\eta$, respectively. Therefore, only positive roots are discussed. All the λ-roots increase monotonically as α increases in the open interval (0, 1). For each order l, the smallest η-root decreases monotonically as α increases in (0, 1), tending towards $\sqrt{l(l+1)}$ as α approaches unity. For $\alpha \in (0, 1)$, all the other η-roots have a minimum value equal to $\sqrt{l(l+1)}/\alpha$.

In [1] roots of the transcendental equations,

$$j_l(\alpha\lambda)y_l(\lambda) = j_l(\lambda)y_l(\alpha\lambda) \tag{1}$$

and

$$[xj_l(x)]'_{x=\alpha\eta}[xy_l(x)]'_{x=\eta} = [xj_l(x)]'_{x=\eta}[xy_l(x)]'_{x=\alpha\eta}, \tag{2}$$

where j_l and y_l denote spherical Bessel functions of the first and second kind, are presented. Here we discuss the dependence of the roots λ_{ln} of Eq. (1) and η_{ln} of Eq. (2) on the continuous real parameter α whose domain is the open interval $(0, 1) = \{\alpha: 0 < \alpha < 1\}$. The subscript $n = 1, 2, \ldots$ orders the roots such that $\lambda_{ln+1} > \lambda_{ln}$ and $\eta_{ln+1} > \eta_{ln}$. Since

$$j_l(ze^{m\pi i}) = e^{ml\pi i}j_l(z), \qquad y_l(ze^{m\pi i}) = (-1)^m e^{ml\pi i}y_l(z)$$

($l, m = 0, 1, 2, \ldots$) [2, p. 439, 10.1.34, 10.1.35], it follows that Eqs. (1) and (2) are invariant under the transformations $\lambda \longrightarrow -\lambda$ and $\eta \longrightarrow -\eta$, respectively.

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Therefore, only positive roots need be considered.

(3a) If Eq. (1) is written as $F(\alpha, \lambda) = 0$,

where

(3b) $$F(\alpha, \lambda) = j_l(\alpha\lambda)y_l(\lambda) - j_l(\lambda)y_l(\alpha\lambda),$$

then

(4) $$\frac{d\lambda}{d\alpha} = -\frac{\partial F/\partial\alpha}{\partial F/\partial\lambda},$$

where

(5a) $$\frac{\partial F}{\partial\alpha} = \lambda[j_{l-1}(\alpha\lambda)y_l(\lambda) - j_l(\lambda)y_{l-1}(\alpha\lambda)]$$

and

(5b) $$\frac{\partial F}{\partial\lambda} = \alpha[j_{l-1}(\alpha\lambda)y_l(\lambda) - j_l(\lambda)y_{l-1}(\alpha\lambda)] - [j_{l-1}(\lambda)y_l(\alpha\lambda) - j_l(\alpha\lambda)y_{l-1}(\lambda)].$$

The expressions (5a) and (5b) for the partial derivatives $\partial F/\partial\alpha$ and $\partial F/\partial\lambda$ have been obtained by means of the formula [2, p. 439, 10.1.21]

(6) $$\frac{l+1}{z}f_l(z) + \frac{d}{dz}f_l(z) = f_{l-1}(z), \qquad f_l(z) = \begin{cases} j_l(z), \\ y_l(z), \end{cases}$$

and by utilizing Eqs. (3). By virtue of the relation [2, p. 439, 10.1.31]

(7) $$j_l(z)y_{l-1}(z) - j_{l-1}(z)y_l(z) = z^{-2}$$

and Eqs. (3) one obtains from Eqs. (4) and (5)

(8a) $$\frac{d\lambda}{d\alpha} = -\frac{\lambda/\alpha}{1 - \alpha\tau_l^2(\alpha, \lambda)},$$

where

(8b) $$\tau_l(\alpha, \lambda) = j_l(\alpha\lambda)/j_l(\lambda) = y_l(\alpha\lambda)/y_l(\lambda).$$

For $0 < \alpha < 1$ expression (5b) is finite and, if Eqs. (3) hold, expression (5a) cannot vanish. Therefore,

(9) $$\frac{d\lambda}{d\alpha} \neq 0 \quad \text{for } 0 < \alpha < 1,$$

which means that λ is a monotonic function of α. This implies that if, for given values of l and n,

(10) $$\lambda_{ln}(\alpha_2) > \lambda_{ln}(\alpha_1) \quad \text{and} \quad \alpha_2 > \alpha_1,$$

where $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1)$, then $\lambda_{ln}(\alpha)$ is a monotonically increasing function for $0 < \alpha < 1$. In particular, the roots λ_{ln} given in [1] for $l = 1(1)15$ and $n = 1(1)30$ satisfy condition (10).

From Eq. (5a) it follows that $\lim_{\alpha\to 1} \partial F/\partial\alpha \neq 0$, and from Eq. (5b) that $\lim_{\alpha\to 1} \partial F/\partial\lambda = 0$. Therefore, Eq. (4) entails that

$$\lim_{\alpha\to 1} \frac{d\lambda}{d\alpha} = \pm\infty. \tag{11}$$

Condition (10) excludes the minus sign in Eq. (11).

If Eq. (2) is written as

$$G(\alpha, \eta) = 0, \tag{12a}$$

where

$$G(\alpha, \eta) = s_l(\alpha\eta)t_l(\eta) - s_l(\eta)t_l(\alpha\eta), \tag{12b}$$

$$s_l(x) = xj_{l-1}(x) - lj_l(x), \qquad t_l(x) = xy_{l-1}(x) - ly_l(x), \tag{12c}$$

then

$$\frac{d\eta}{d\alpha} = -\frac{\partial G/\partial\alpha}{\partial G/\partial\eta}, \tag{13}$$

where

$$\frac{\partial G}{\partial\alpha} = \frac{1}{\alpha}[l(l+1) - (\alpha\eta)^2][j_l(\alpha\eta)t_l(\eta) - y_l(\alpha\eta)s_l(\eta)] \tag{14a}$$

and

$$\begin{aligned}\frac{\partial G}{\partial\eta} &= \frac{1}{\eta}[l(l+1) - (\alpha\eta)^2][j_l(\alpha\eta)t_l(\eta) - y_l(\alpha\eta)s_l(\eta)] \\ &\quad -\frac{1}{\eta}[l(l+1) - \eta^2][j_l(\eta)t_l(\alpha\eta) - y_l(\eta)s_l(\alpha\eta)].\end{aligned} \tag{14b}$$

The expression (12b) has been derived from Eq. (2) by means of Eq. (6), and the expressions (14a) and (14b) for the partial derivatives $\partial G/\partial\alpha$ and $\partial G/\partial\eta$ have been obtained by means of the formula [2, p. 439, 10.1.22]

$$\frac{l}{z}f_l(z) - \frac{d}{dz}f_l(z) = f_{l+1}(z), \qquad f_l(z) = \begin{cases} j_l(z), \\ y_l(z). \end{cases}$$

By virtue of Eqs. (7) and (12) the expressions (14a) and (14b) can be rewritten as

$$\frac{\partial G}{\partial\alpha} = \frac{1}{\alpha^2\eta}[l(l+1) - (\alpha\eta)^2]\rho_l^{-1}(\alpha, \eta), \tag{15a}$$

$$\frac{\partial G}{\partial\eta} = \frac{1}{\alpha\eta^2}[l(l+1) - (\alpha\eta)^2]\rho_l^{-1}(\alpha, \eta) - \frac{1}{\eta^2}[l(l+1) - \eta^2]\rho_l(\alpha, \eta), \tag{15b}$$

where

$$\rho_l(\alpha, \eta) = \frac{s_l(\alpha\eta)}{s_l(\eta)} = \frac{t_l(\alpha\eta)}{t_l(\eta)}. \tag{15c}$$

By substituting (15a) and (15b) in Eq. (13), one obtains

$$\frac{d\eta}{d\alpha} = -\frac{\eta/\alpha}{1 - \alpha\sigma_l(\alpha, \eta)\rho_l^2(\alpha, \eta)}, \tag{16a}$$

where

$$\sigma_l(\alpha, \eta) = [l(l+1) - \eta^2]/[l(l+1) - (\alpha\eta)^2]. \tag{16b}$$

The expression (16a) for the total derivative is analogous to the expression (8a). The definitions (12c) and the relations (15c) imply that $\rho_l(\alpha, \eta)$ and $\rho_l^{-1}(\alpha, \eta)$ are both nonzero. Since $\eta \neq 0$, it follows from Eq. (13) and Eqs. (15a) and (15b) that

$$\frac{d\eta}{d\alpha} = 0, \quad 0 < \alpha < 1, \tag{17}$$

if and only if

$$\alpha\eta = \sqrt{l(l+1)}. \tag{18}$$

From Eqs. (17) and (18) it is obvious that η is not a monotonic function of α, unless

$$\lim_{\alpha \to 1} \eta = \sqrt{l(l+1)}, \tag{19}$$

in which case one finds by means of l'Hospital's rule that

$$\lim_{\alpha \to 1} \frac{d\eta}{d\alpha} = -\sqrt{l(l+1)}. \tag{20}$$

The roots which satisfy Eq. (19) are, therefore, monotonically decreasing functions of α. All other roots have an extremum in the open interval (0, 1), in accordance with Eq. (17), and satisfy

$$\lim_{\alpha \to 1} \frac{d\eta}{d\alpha} = \pm\infty, \tag{21}$$

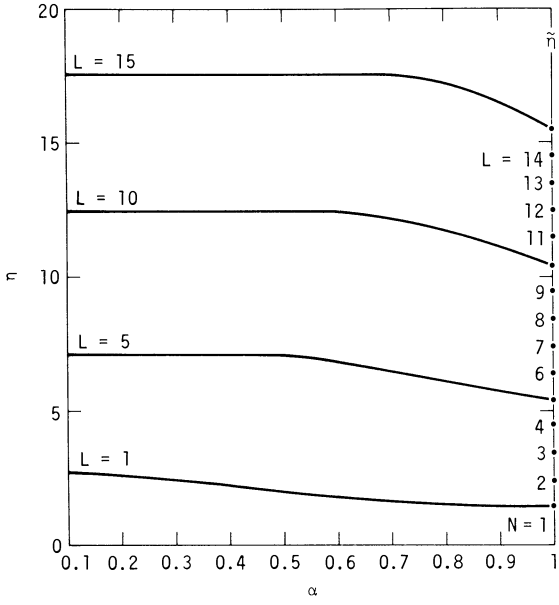
since

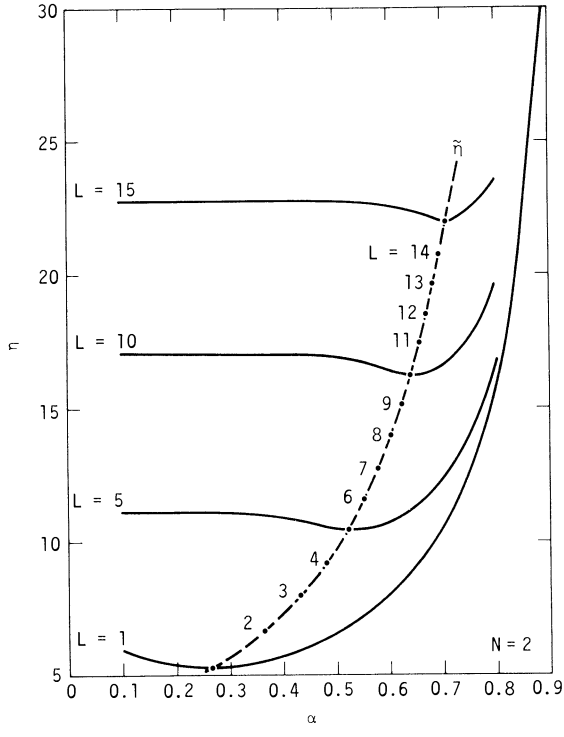
$$\lim_{\alpha \to 1} \frac{\partial G}{\partial \alpha} \neq 0, \quad \text{if } \lim_{\alpha \to 1} \eta \neq \sqrt{l(l+1)},$$

and

$$\lim_{\alpha \to 1} \frac{\partial G}{\partial \eta} = 0.$$

The minima of the η-roots, together with the corresponding values of the parameter α, are presented for $l = 1(1)15$ and $n = 1(1)30$ in the microfiche supplement. In the graphs, the solid curves represent η-roots as functions of α for $l = 1, 5, 10, 15$ and for $n = 1, 2, 3, 4, 5, 10$. The dashed curves connect the minima, $\tilde{\eta}$, of these roots.


20
15
10
5
0
η
L = 15
L = 14
13
12
11
L = 10
9
8
7
L = 5
6
4
3
L = 1
2
N = 1
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
α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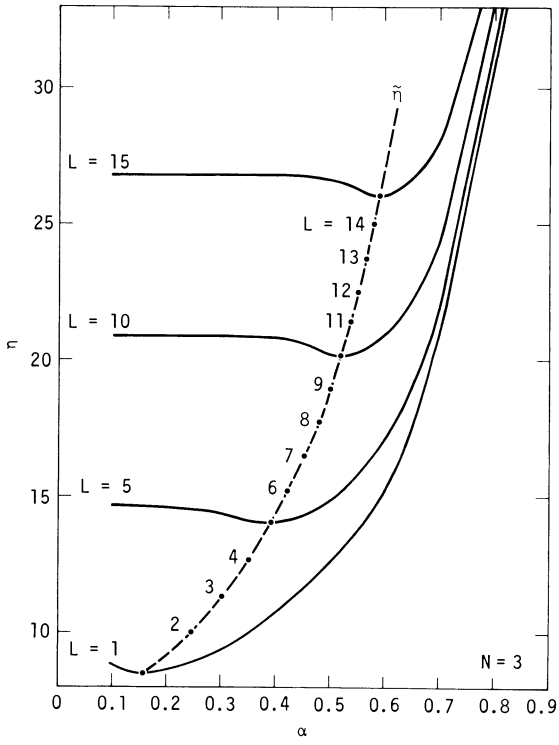
25
20
15
10
5
η
L = 15
L = 14
13
12
L = 10
11
9
8
7
L = 5
6
4
3
2
L = 1
N = 2
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
α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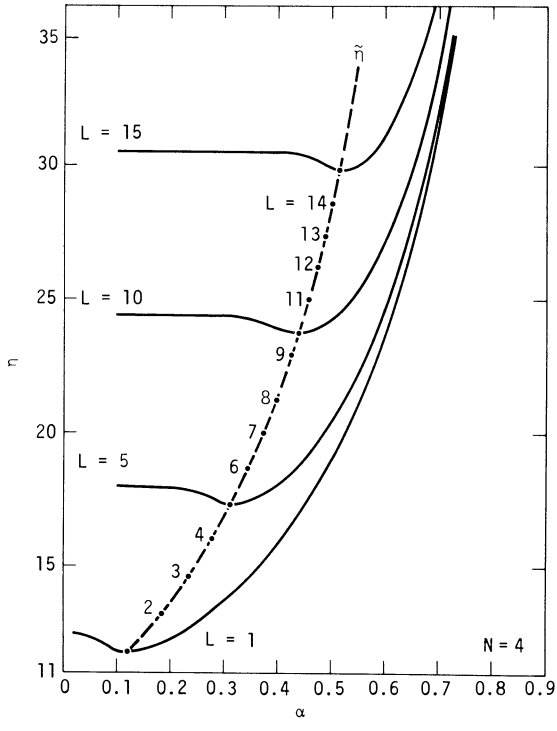
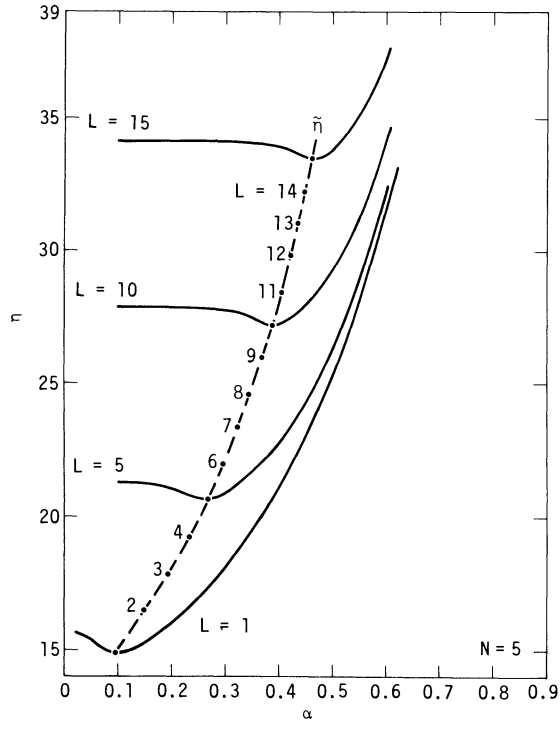
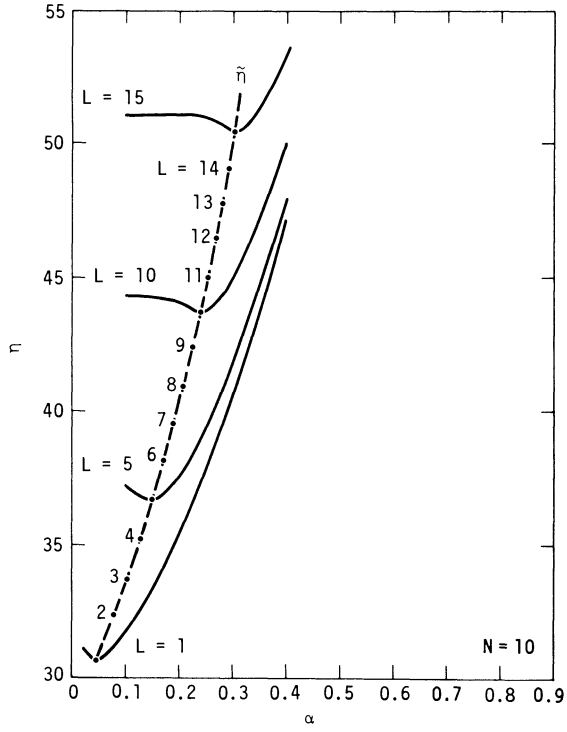
25
20
15
10
η
L = 15
L = 14
13
12
L = 10
11
9
8
7
L = 5
6
4
3
2
L = 1
N = 3
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
α


35
30
25
20
15
11
η
L = 15
L = 14
13
12
L = 10
11
9
8
7
L = 5
6
4
3
2
L = 1
N = 4
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
α


39
35
30
25
20
15
η
L = 15
L = 14
13
12
L = 10
11
9
8
7
L = 5
6
4
3
2
L = 1
$\tilde{\eta}$
N = 5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
α


55
50
45
40
35
30
η
L = 15
L = 14
13
12
L = 10
11
9
8
7
6
L = 5
4
3
2
L = 1
$\tilde{\eta}$
N = 10
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
α

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1. ROBERT L. PEXTON & ARNO D. STEIGER, "Roots of two transcendental equations involving spherical Bessel functions," *Math. Comp.*, v. 31, 1977, pp. 752–753.

2. M. ABRAMOWITZ & I. A. STEGUN, *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series No. 55, U. S. Government Printing Office, Washington, D. C., 1965.

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n	eta min	alpha(eta min)
1	1.414213562373 1e+00	1.0000000000000e+00
2	5.318627581256 9e+00	2.658982118658 9e-01
3	8.534486721139 3e+00	1.657061429357 0e-01
4	1.170848373874 0e+01	1.207853718662 0e-01
5	1.486844115350 6e+01	9.511511985502 6e-02
6	1.802189137604 2e+01	7.947983481887 e-02
7	2.117178312941 7e+01	6.879709279697 6e-02
8	2.431951313852 7e+01	5.815139284728 0e-02
9	2.746583034495 4e+01	5.148992564965 0e-02
10	3.061117300308 3e+01	4.619926071538 2e-02
11	3.375581485370 3e+01	4.189540582886 2e-02
12	3.689993578008 8e+01	3.832563749707 8e-02
13	4.004365889606 1e+01	3.531679175581 5e-02
14	4.318707132143 6e+01	3.274622518038 5e-02
15	4.633023644962 8e+01	3.052463511408 3e-02
16	4.947320151497 8e+01	2.858544664721 0e-02
17	5.261600243592 4e+01	2.687801233275 6e-02
18	5.575886701388 1e+01	2.536311641060 3e-02
19	5.890121710457 2e+01	2.400992087926 3e-02
20	6.204367012746 4e+01	2.279364116812 7e-02
21	6.518604013741 0e+01	2.169503714693 5e-02
22	6.832833859985 9e+01	2.069732107281 2e-02
23	7.147057496111 0e+01	1.978735393051 8e-02
24	7.461275707417 8e+01	1.895404509670 0e-02
25	7.775489152121 2e+01	1.818809768369 7e-02
26	8.089898386064 2e+01	1.748185994432 2e-02
27	8.403903881880 0e+01	1.682805494030 4e-02
28	8.718106044004 2e+01	1.622156871268 7e-02
29	9.032305220549 2e+01	1.565728269628 9e-02
30	9.346501712781 3e+01	1.513093995841 4e-02

l	n	eta min	alpha(eta min)
	1	2.4494697427632e+00	1.0000000000000e+00
	2	6.7034385724047e+00	3.6540794941669e-01
	3	1.0000355229542e+01	2.4494027326887e-01
	4	1.3216748457379e+01	1.8533225060024e-01
	5	1.6403073741932e+01	1.4933114252370e-01
	6	1.9574585003654e+01	1.2513822854972e-01
	7	2.2737644928046e+01	1.0772638395314e-01
	8	2.5895410151089e+01	9.4591856532821e-02
	9	2.9049832782507e+01	8.4320850494820e-02
	10	3.2201365992068e+01	7.6067883063922e-02
	11	3.5351281919472e+01	6.9289983553155e-02
	12	3.8499830117426e+01	6.3623390942561e-02
	13	4.1647322840569e+01	5.8815058825272e-02
	14	4.4793983838692e+01	5.4683453733521e-02
2	15	4.7939977689399e+01	5.1094928718017e-02
	16	5.1085428173618e+01	4.7948893262838e-02
	17	5.4230430195222e+01	4.5168178344988e-02
	18	5.7375057747555e+01	4.2692588712689e-02
	19	6.0519369377814e+01	4.0474758590456-02
	20	6.3663412021395e+01	3.8475627758688e-02
	21	6.6807223747094e+01	3.6665043170421e-02
	22	6.9950835758059e+01	3.5017304666181e-02
	23	7.3094273873883e+01	3.3511376650509e-02
	24	7.6237559844405e+01	3.2129697674064e-02
	25	7.9380711197815e+01	3.0857493033529e-02
	26	8.2523743894211e+01	2.9682242069909e-02
	27	8.5666670834796e+01	2.8593284088748e-02
	28	8.8809503262599e+01	2.7581392224888e-02
	29	9.1952250680788e+01	2.6638714325317e-02
	30	9.5094922107668e+01	2.5758365310083e-02

l	n	eta min	alpha(eta min)
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	2	7.999210300566 2e+00	4.330554496476 6e-01
	3	1.137697145499 7e+01	3.044636342290 6e-01
	4	1.463962311347 8e+01	2.366250543669 1e-01
	5	1.785640464812 5e+01	1.939977102775 5e-01
	6	2.104960832288 6e+01	1.645684595187 6e-01
	7	2.422894517727 3e+01	1.429736668296 8e-01
	8	2.739939169600 7e+01	1.264298730998 0e-01
	9	3.058378077577 9e+01	1.133400884056 5e-01
	10	3.372385019533 4e+01	1.027196359571 4e-01
	11	3.688072771474 7e+01	9.392714921274 8e-02
	12	4.003517826856 0e+01	8.652844418227 e-02
	13	4.318773954531 5e+01	8.021030161619 4e-02
	14	4.633880081012 7e+01	7.475598162559 1e-02
3	15	4.948685095649 3e+01	6.999789948170 5e-02
	16	5.263750697927 4e+01	6.581051577691 0e-02
	17	5.578554395884 1e+01	6.209678144488 9e-02
	18	5.893288853434 2e+01	5.878044842684 3e-02
	19	6.207964825500 6e+01	5.580092208171 3e-02
	20	6.522590815563 0e+01	5.310928913204 7e-02
	21	6.837173752445 5e+01	5.066569522090 5e-02
	22	7.151719339052 9e+01	4.843732605978 5e-02
	23	7.466232312382 2e+01	4.639691708228 9e-02
	24	7.780718639861 0e+01	4.452162667627 6e-02
	25	8.095175669868 7e+01	4.279217346736 0e-02
	26	8.409612247909 2e+01	4.119216811689 5e-02
	27	8.724028807781 6e+01	3.970759028278 1e-02
	28	9.038427443402 2e+01	3.832637521106 9e-02
	29	9.352809986347 8e+01	3.703808403679 6e-02
	30	9.667177951783 2e+01	3.583638652516 e-02

l	n	eta min	alpha(eta min)
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	2	9.2488036878346e+00	4.8353869464671e-01
	3	1.3704835415667e+01	3.5200286738480e-01
	4	1.8014969353310e+01	2.7924723777727e-01
	5	1.9264276727992e+01	2.3214657961430e-01
	6	2.2481327202837e+01	1.9892668787883e-01
	7	2.5678958372451e+01	1.7415586044912e-01
	8	2.8863910055485e+01	1.5493887415755e-01
	9	3.2040094528742e+01	1.3957936207048e-01
	10	3.5209950656866e+01	1.2701341159506e-01
	11	3.8375082746969e+01	1.1653749346299e-01
	12	4.1536591606198e+01	1.0766737916032e-01
	13	4.4695258752587e+01	1.0005839723975e-01
	14	4.7851654919077e+01	9.3458334148788e-02
4	15	5.1006206926120e+01	8.7676269459975e-02
	16	5.4159240525936e+01	8.2573630644061e-02
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	18	6.0461711185309e+01	7.3984413906032e-02
	19	6.3611507426639e+01	7.0303882874097e-02
	20	6.6760526640290e+01	6.6987727330189e-02
	21	6.9908874489882e+01	6.3970933413422e-02
	22	7.3056638276603e+01	6.1214641961315e-02
	23	7.6203890762261e+01	5.8686451705618e-02
	24	7.9350693082054e+01	5.6359129092610e-02
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	26	8.5643146555804e+01	5.2216258376175e-02
	27	8.8788796614483e+01	5.0368198972860e-02
	28	9.1934326765228e+01	4.8644897015783e-02
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	30	9.8224484791104e+01	4.5529747134949e-02

i	n	eta min	alpha(eta min)
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	3	1.3599383314456e+01	3.9124763227218e-01
	4	1.7356994042392e+01	3.1556302673575e-01
	5	2.0839600531543e+01	2.6537202075578e-01
	6	2.3881969421748e+01	2.2934564058783e-01
	7	2.7099211467424le+01	2.0211750196080e-01
	8	3.0299942560431e+01	1.8076688330767e-01
	9	3.3489109121023e+01	1.6355244193741e-01
	10	3.8689844839058e+01	1.4936593347924e-01
	11	3.9944231449585e+01	1.3746596121404e-01
	12	4.3013712272470e+01	1.2733673253674e-01
	13	4.6179320097691e+01	1.1880775696707e-01
	14	4.9341814953205e+01	1.1100575810286e-01
5	15	5.2501769009360e+01	1.0432459093093e-01
	16	5.5659821549420e+01	9.8405727933102e-02
	17	5.8815715615844e+01	9.3125205019096e-02
	18	6.1970323111200e+01	8.8384654138777e-02
	19	6.5123682404764e+01	8.4104999209181e-02
	20	6.8275910935337e+01	8.0221933329297e-02
	21	7.1427214404267e+01	7.6682814693133e-02
	22	7.4577693569113e+01	7.3443214873720e-02
	23	7.7727449469444e+01	7.0467069387178e-02
	24	8.0876567134364e+01	6 7723269781818e-02
	25	8.4025118797493e+01	6.5185573712216e-02
	26	8.7173186149056e+01	6.2831554904020e-02
	27	9.0320762209456e+01	6.0641932606472e-02
	28	9.3467952803635e+01	5.8600037882061e-02
	29	9.6614777743903e+01	5.6691385137480e-02
	30	9.9761271786473e+01	5.4903325478599e-02

i	n	eta min	alpha(eta min)
	1	6.4807406964079e+00	1.0000000000000e+00
	2	1.1665896600581e+01	5.5552672790635e-01
	3	1.5288862215004e+01	4.2444106956562e-01
	4	1.8673386677181e+01	3.4705798448135e-01
	5	2.1990018414825e+01	2.9471283634936e-01
	6	2.5258026234685e+01	2.5658143624035e-01
	7	2.8495743227345e+01	2.2742837927416e-01
	8	3.1713135526174e+01	2.0435509106500e-01
	9	3.4916153701342e+01	1.8560886565777e-01
	10	3.8108597250752e+01	1.7005980712869e-01
	11	4.1293017630852e+01	1.5694519485943e-01
	12	4.4471197250480e+01	1.4572894590415e-01
	13	4.7644421665436e+01	1.3602307409493e-01
	14	5.0813642892917e+01	1.2753938370577e-01
6	15	5.3979581818798e+01	1.2005911272456e-01
	16	5.7142794809749e+01	1.1341308593847e-01
	17	6.0303718405926e+01	1.0746634307602e-01
	18	6.3462700115425e+01	1.0211889324943e-01
	19	6.6620020128568e+01	9.7279176528330e-02
	20	6.9775906943222e+01	9.2879347360993e-02
	21	7.2930548612421e+01	8.8861817384484e-02
	22	7.6084102288813e+01	8.5176844491060e-02
	23	7.9236698551465e+01	8.1789635571433e-02
	24	8.2388448551136e+01	7.8660793016214e-02
	25	8.5539448603809e+01	7.5763182434702e-02
	26	8.8689773485492e+01	7.3072017705268e-02
	27	9.1839498789619e+01	7.0585941509164e-02
	28	9.4988662709170e+01	6.8226450915739e-02
	29	9.8137377750737e+01	6.6037434940118e-02
	30	1.0128562976207e+02	6.3984799360305e-02

l	n	eta min	alpha(eta min)
	1	7.4833147735479e+00	1.0000000000000e+00
	2	1.2846726290389e+01	5.8250752794089e-01
	3	1.6518517158763e+01	4.5302581954887e-01
	4	1.9969981032539e+01	3.7474695185267e-01
	5	2.3319452000847e+01	3.2090440089546e-01
	6	2.6613663848553e+01	2.8118318530407e-01
	7	2.9872395405081e+01	2.5050936398207e-01
	8	3.3107068773752e+01	2.2603374598802e-01
	9	3.6324575691060e+01	2.0601244722005e-01
	10	3.9529357766830e+01	1.8931030495586e-01
	12	4.5911679913639e+01	1.6299299413625e-01
	13	4.9093270631506e+01	1.5243056079351e-01
	14	5.2269736312211e+01	1.4318725703090e-01
7	15	5.5442147252570e+01	1.3497519748949e-01
	16	5.8611176266879e+01	1.2767726216395e-01
	17	6.1777361090828e+01	1.2113361013504e-01
	18	6.4941120649097e+01	1.1523230117914e-01
	19	6.8102800780622e+01	1.0986282872650e-01
	20	7.1262662809518e+01	1.0501028699060e-01
	21	7.4420999214828e+01	1.0055380875482e-01
	22	7.7577943644516e+01	9.6481685977759e-02
	23	8.0733879659154e+01	9.2891362578063e-02
	24	8.3888344672802e+01	8.9205855478550e-02
	25	8.7042056572743e+01	8.5973551960987e-02
	26	9.0194916342062e+01	8.2968254498597e-02
	27	9.3347011132672e+01	8.0186624327286e-02
	28	9.6498416562910e+01	7.7548575822167e-02
	29	9.9649196570435e+01	7.5096587636469e-02
	30	1.0279941491797e+02	7.2795305104790e-02

l	n	eta min	alpha(eta min)
	1	6.4852613742386e+00	1.0000000000000e+00
	2	1.4014091803578e+01	6.0546206291092e-01
	3	1.7751846239743e+01	4.7799430322022e-01
	4	2.1247259775215e+01	3.9935665681299e-01
	5	2.4831300257219e+01	3.4449161873841e-01
	6	2.7951846535746e+01	3.0356762917317e-01
	7	3.1231916213762e+01	2.7166822367460e-01
	8	3.4484290891050e+01	2.4808222567111e-01
	9	3.7718751338840e+01	2.2497361330666e-01
	10	4.0934350569466e+01	2.0728999820567e-01
	11	4.4140532970826e+01	1.9223332395752e-01
	12	4.7337738987663e+01	1.7924963140723e-01
	13	5.0527712424969e+01	1.6793311855720e-01
	14	5.3711861534342e+01	1.5797773475527e-01
8	15	5.6891172965905e+01	1.4914934834133e-01
	16	6.0066409866894e+01	1.4128499974062e-01
	17	6.3238219597693e+01	1.3417963738100e-01
	18	6.6407105273821e+01	1.2777767091228Be-01
	19	6.9573475314457e+01	1.2198144199908e-01
	20	7.2737864886456e+01	1.1665594967160e-01
	21	7.5899951732003e+01	1.1179560982304e-01
	22	7.9060586039739e+01	1.0732833959793e-01
	23	8.2219709470578e+01	1.0320252198501e-01
	24	8.5377542106087e+01	9.9385402352018e-02
	25	8.8534207872231e+01	9.5841839873738e-02
	26	9.1689826780497e+01	9.2543322275714e-02
	27	9.4844510334110e+01	8.9465181952518e-02
	28	9.7998344232805e+01	8.6585966719202e-02
	29	1.0115141056911e+02	8.3886930755585e-02
	30	1.0430377961536e+02	8.1351619332779e-02

l	n	eta min	alpha(eta min)
	1	9.4866329605051e+00	1.0000000000000e+00
	2	1.5170352064579e+01	6.2535351454736e-01
	3	1.8971457260793e+01	5.0005821113757e-01
	4	2.2510740110602e+01	4.2143585389948e-01
	5	2.5927964239716e+01	3.6569193400568e-01
	6	2.9274817836973e+01	3.2406121306495e-01
	7	3.2576393660336e+01	2.9121802189579e-01
	8	3.5846745736419e+01	2.6484978903991e-01
	9	3.9094493240399e+01	2.4268417580013e-01
	10	4.2325271998333e+01	2.2414109696405e-01
	11	4.5542946767469e+01	2.0830520756884e-01
	12	4.8750268497938e+01	1.9460063037204e-01
	13	5.1949254997770e+01	1.8261730569404e-01
	14	5.5141423451991e+01	1.7204548570939e-01
9	15	5.8327938667556e+01	1.6264646406548e-01
	16	6.1509711001152e+01	1.5423309305302e-01
	17	6.4687463013130e+01	1.4665643910907e-01
	18	6.7861776012409e+01	1.3979841466228e-01
	19	7.1033123269254e+01	1.3355505912164e-01
	20	7.4201894309817e+01	1.2785162769636e-01
	21	7.7368412636511e+01	1.2261894301646e-01
	22	8.0532949406611e+01	1.1780063999154e-01
	23	8.3695733610720e+01	1.1334906298366e-01
	24	8.6856960029067e+01	1.0922383593350e-01
	25	9.0018795429035e+01	1.0538956798644e-01
	26	9.3175363455057e+01	1.0181694594346e-01
	27	9.6332646523440e+01	9.8479730703863e-02
	28	9.9489298951730e+01	9.5355310374717e-02
	29	1.0264482949335e+02	9.242367576960e-02
	30	1.0579952340589e+02	8.9868012436216e-02

n	eta min	alpha(eta min)
1	1.0488088461701e+01	1.0000000000000e+00
2	1.6317251755978e+01	6.4276071966962e-01
3	2.0179310985512e+01	5.1974482601191e-01
4	2.3761377957145e+01	4.4139226892229e-01
5	2.7211316627735e+01	3.8543110038819e-01
6	3.0584345712318e+01	3.4292342186619e-01
7	3.3907484839254e+01	3.0931484689656e-01
8	3.7195982360422e+01	2.8196831528938e-01
9	4.0459250724115e+01	2.5922596919103e-01
10	4.3703479910221e+01	2.3998291447836e-01
11	4.6932937333223e+01	2.2346967988039e-01
12	5.0150875178617e+01	2.0913155095813e-01
13	5.3358942032762e+01	1.9655727947638e-01
14	5.6559435391109e+01	1.8543481576616e-01
15	5.9753463297499e+01	1.7552288777265e-01
16	6.2942051526036e+01	1.6663088485241e-01
17	6.6126016808039e+01	1.5860759483137e-01
18	6.9306018114271e+01	1.5133012640271e-01
19	7.2482593422393e+01	1.4469803005781e-01
20	7.5656186541759e+01	1.3862829943077e-01
21	7.8827167046647e+01	1.3305189672753e-01
22	8.1995845319704e+01	1.2790999886407e-01
23	8.5162484025730e+01	1.2315385620044e-01
24	8.8327307028137e+01	1.1874117795033e-01
25	9.1490506342440e+01	1.1463581198551e-01
26	9.4652247644859e+01	1.1080654440532e-01
27	9.7812674885424e+01	1.0722627223363e-01
28	1.0097191273338e+02	1.0387134597911e-01
29	1.0413007163660e+02	1.0072103396129e-01
30	1.0728724797993e+02	9.7757083709175e-02

l	n	eta min	alpha(eta min)
	1	1.1489125293076e+01	1.0000000000000e+00
	2	1.7456124324063e+01	6.5617180096864e-01
	3	2.1376939271597e+01	5.3745417653600e-01
	4	2.5000725519282e+01	4.5955187517899e-01
	5	2.8482869090826e+01	4.0336966253062e-01
	6	3.1861863340240e+01	3.6038555238664e-01
	7	3.5226541612815e+01	3.2814988052659e-01
	8	3.8533272413865e+01	2.9816116237757e-01
	9	4.1812218274983e+01	2.7477913794280e-01
	10	4.5070096828627e+01	2.5491680982572e-01
	11	4.8311560942631e+01	2.3761316663676e-01
	12	5.1539952734766e+01	2.2291687678095e-01
	13	5.4757744052045e+01	2.0981735993645e-01
	14	5.7966607672265e+01	1.9820179434485e-01
11	15	6.1168591515936e+01	1.8762720033631e-01
	16	6.4364234552460e+01	1.7850170009731e-01
	17	6.7554646226936e+01	1.7007157812036e-01
	18	7.0740582277651e+01	1.6241212853217e-01
	19	7.3922564840288e+01	1.5542104375877e-01
	20	7.7101211803239e+01	1.4901355016827e-01
	21	8.0276856676887e+01	1.4311677024635e-01
	22	8.3449675124662e+01	1.3767696207918e-01
	23	8.6620557632969e+01	1.3263739702252e-01
	24	8.9789159337049e+01	1.2795470855931e-01
	25	9.2955697731806e+01	1.2359759384202e-01
	26	9.6120960778905e+01	1.1952776249360e-01
	27	9.9284511791734e+01	1.1571921023469e-01
	28	1.0244869337526e+02	1.1214735112037e-01
	29	1.0560763062854e+02	1.0879087378620e-01
	30	1.0876743378889e+02	1.0563019550032e-01

l	n	eta min	alpha(eta min)
	1	1.1469125293076e+01	1.0000000000000e+00
	2	1.7456124324083e+01	6.5817160998864e-01
	3	2.1376939271597e+01	5.3745417653600e-01
	4	2.5000725519282e+01	4.5955187517899e-01
	5	2.8482869090826e+01	4.0336968253082e-01
	6	3.1861863340240e+01	3.6036555236841e-01
	7	3.5228541612815e+01	3.2814968052859e-01
	8	3.8533272413885e+01	2.9818118237757e-01
	9	4.1812218274983e+01	2.7477913794260e-01
	10	4.5070099628827e+01	2.5491680962572e-01
	11	4.8311560942831e+01	2.3761316663878e-01
	12	5.1539952734766e+01	2.2291687678095e-01
	13	5.4757744052045e+01	2.0981735993645e-01
	14	5.7966807872285e+01	1.9820179434488e-01
11	15	6.1168591515938e+01	1.8782720033831e-01
	16	6.4364234552460e+01	1.7850170009731e-01
	17	6.7554846228936e+01	1.7007157812036e-01
	18	7.0740582277851e+01	1.6241212853217e-01
	19	7.3922584640288e+01	1.5542104375677e-01
	20	7.7101211803239e+01	1.4901355016827e-01
	21	8.0276858678867e+01	1.4311677024635e-01
	22	8.3449875124662e+01	1.3767696207916e-01
	23	8.6620557832989e+01	1.3263739702252e-01
	24	8.9789159337049e+01	1.2795670855931e-01
	25	9.2955697731806e+01	1.2359759384202e-01
	26	9.6120960778905e+01	1.1952778249380e-01
	27	9.9284511791734e+01	1.1571921023469e-01
	28	1.0244669337526e+02	1.1214735112037e-01
	29	1.0560783062884e+02	1.0879067378820e-01
	30	1.0876743378869e+02	1.0563019550032e-01

n	eta min	alpha(eta min)
1	1.3490737563232e+01	1.0000000000000e+00
2	1.9713770642102e+01	6.8433064740817e-01
3	2.3746196725668e+01	5.6612203313957e-01
4	2.7450369142494e+01	4.9145924024563e-01
5	3.0995295647161e+01	4.3525113348830e-01
6	3.4445425757667e+01	3.9165541625388e-01
7	3.7832891272619e+01	3.5656753623543e-01
8	4.1176121110549e+01	3.2783497870556e-01
9	4.4486664463742e+01	3.0325351936033e-01
10	4.7772221226443e+01	2.8239711733906e-01
11	5.1036166765256e+01	2.6432645250134e-01
12	5.4286390264996e+01	2.4650133866640e-01
13	5.7525767622463e+01	2.3451634685596e-01
14	6.0752566467507e+01	2.2206037266067e-01
15	6.3970443557288e+01	2.1069016759990e-01
16	6.7180778595963e+01	2.0061246819107e-01
17	7.0384655132810e+01	1.9167157298366e-01
18	7.3582964753788e+01	1.8334050016746e-01
19	7.6778433416659e+01	1.7571458451955e-01
20	7.9965865537420e+01	1.6870682516177e-01
21	8.3151167464647e+01	1.6224351348633e-01
22	8.6333386941000e+01	1.5628330862642e-01
23	8.9512627795179e+01	1.5071323337666e-01
24	9.2689281743625e+01	1.4554600965560e-01
25	9.5863537421732e+01	1.4072655984734e-01
26	9.9035687665635e+01	1.3622097126017e-01
27	1.0220591533920e+02	1.3199566305394e-01
28	1.0537439604913e+02	1.2802871059570e-01
29	1.0854129199000e+02	1.2429129334922e-01
30	1.1170673510227e+02	1.2076924055547e-01

l	n	eta min	alpha(eta min)
	1	1.44913767461 89e+01	1.00000000000000e+00
	2	2.08340721385 29e+01	6.95561417433 17e-01
	3	2.49198547049 23e+01	5.81523978473 38e-01
	4	2.88825760625 57e+01	5.05565251771 69e-01
	5	3.22380751307 82e+01	4.49511228179 76e-01
	6	3.57133203914 49e+01	4.05769518453 56e-01
	7	3.91219599490 98e+01	3.70415407741 43e-01
	8	4.24533731433 41e+01	3.41107018439 78e-01
	9	4.58097522444 36e+01	3.16338247560 57e-01
	10	4.91092552557 60e+01	2.95084433061 27e-01
	11	5.23675964242 58e+01	2.78818469548 25e-01
	12	5.56489230016 15e+01	2.64407137542 78e-01
	13	5.88963319006 50e+01	2.48048854175 58e-01
	14	6.21321904775 93e+01	2.33234602462 86e-01
14	15	6.53583442708 38e+01	2.21721907246 28e-01
	16	6.85762562933 07e+01	2.11317699678 63e-01
	17	7.17871032014 54e+01	2.01868601910 26e-01
	18	7.49918433695 37e+01	1.93239372377 88e-01
	19	7.81912661280 45e+01	1.85332422197 36e-01
	20	8.13860260563 35e+01	1.78057304085 59e-01
	21	8.45768801821 92e+01	1.71340098689 82e-01
	22	8.77638866706 63e+01	1.65116136733 81e-01
	23	9.09474507708 98e+01	1.59337910225 70e-01
	24	9.41283072554 47e+01	1.53953440455 09e-01
	25	9.73065522298 72e+01	1.48924964126 00e-01
	26	1.00462440944 92e+02	1.44218000776 21e-01
	27	1.03656196074 77e+02	1.39802320507 07e-01
	28	1.06828012604 73e+02	1.35651467866 19e-01
	29	1.09980629856 57e+02	1.31742108477 84e-01
	30	1.13116649655 093e+02	1.28053592995 71e-01

n	eta min	alpha(eta min)
1	1.54919333846630e+01	1.0000000000000e+00
2	2.19494907106590e+01	7.0579921825545e-01
3	2.80666362027330e+01	5.9386469289119e-01
4	2.98674095125020e+01	5.1869022582441e-01
5	3.34729362210920e+01	4.8281987286044e-01
6	3.69729750767720e+01	4.1900894636181e-01
7	4.04026909374990e+01	3.8343893190274e-01
8	4.37621225605290e+01	3.5384153345722e-01
9	4.71243203278970e+01	3.2874803340909e-01
10	5.04377989668830e+01	3.0714927499151e-01
11	5.37265979388950e+01	2.8633662580827e-01
12	5.70011128899720e+01	2.7178299855450e-01
13	6.02586349128910e+01	2.5709087932296e-01
14	6.35036859590550e+01	2.4395329422009e-01
15	6.67362366685260e+01	2.3212979782152e-01
16	6.99638528120670e+01	2.2142767736294e-01
17	7.31817955506225e+01	2.1169108078951e-01
18	7.63930949791300e+01	2.0279232552447e-01
19	7.95986005889870e+01	1.9482570032887e-01
20	8.27990202245390e+01	1.8710285873936e-01
21	8.59949489378450e+01	1.8014934105056e-01
22	8.91868909113040e+01	1.7370191097071e-01
23	9.23752784167880e+01	1.6770649015362e-01
24	9.55604756653290e+01	1.6211653797492e-01
25	9.87428062710190e+01	1.5689176730818e-01
26	1.01922547596480e+02	1.5199711693004e-01
27	1.05099941468910e+02	1.4740192209728e-01
28	1.08275206029790e+02	1.4307923970327e-01
29	1.11448512590760e+02	1.3900529513315e-01
30	1.14620043302840e+02	1.3515902575519e-01