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Vector Groups and the Equality Problem for Vector Addition Systems

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Abstract. Our purpose is to demonstrate that results concerning the equality problem for vector addition systems, may be used to establish the decidability and undecidability of decision problems associated with the class of *HNN* extensions of the infinite cyclic group. We call these groups 'vector groups.'

By *vector groups* we understand the *HNN* groups $G(p_1, q_1, \ldots, p_k, q_k)$ given by

$$\text{(I)} \qquad \langle a_1, \ldots, a_k, b; a_1^{-1}b^{p_1}a_1 = b^{q_1}, \ldots, a_k^{-1}b^{p_k}a_k = b^{q_k}\rangle,$$

where the exponent pairs p_i, q_i occurring in (I) are positive and relatively prime. In [2] the conjugacy problem for vector groups was reduced to the reachability problem for self-dual vector addition systems, and subsequently an algebraic solution of both problems was presented in [3]. Here we demonstrate that recent results concerning the equality problem for vector addition systems, having surprising algebraic consequences for vector groups.

Let G be a vector group and call m a *conjugate power* of l in G when $b^m = xb^lx^{-1}$, x in G and a *positive* conjugate power if in addition x is given by a positive word in the generators $a_1, \ldots, a_k, b$ of G (i.e., one which involves no negative exponents). The set of (positive) conjugate powers of l in G is called a *(positive) conjugate power set.* Here we consider the question as to whether the (positive) conjugate powers of l in G_1 and G_2 coincide where G_1 is given in (I) and G_2 arises from G_1 by removing a particular defining relation, $a_i^{-1}b^{p_i}a_i = b^{q_i}$. We will prove:

THEOREM A. *It is decidable whether the removal of a particular defining relation from a vector group changes a conjugate power set.*

THEOREM B. *It is undecidable whether the removal of a particular defining relation from a vector group changes a positive conjugate power set.*

For $G = G(p_1, q_1, \ldots, p_k, q_k)$ let CP(l, G) and PCP(l, G) denote respectively the conjugate and positive conjugate powers of l on G. From [1, pp. 22–23], it follows that m lies in CP(l, G) if and only if there is a sequence $l = l_1, \ldots, l_n = m$ such that $l_{i+1} = l_i(p_j/q_j)$ or $l_{i+1} = l_i(q_j/p_j)$. Similarly, m lies in PCP(l, G) if and only if $l = l_1, \ldots, l_n = m$ where $l_{i+1} = l_i(p_j/q_j)$. These observations allow us to restrict our attention to those positive integers l whose prime divisors are among the prime divisors of the exponents of G.

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By a *vector addition system* (*of dimension* n) we mean a pair (d, W), d in N^n, W a finite subject of Z^n (where N and Z denote the nonnegative integers and integers respectively). Let us call d' *reachable* in (d, W) if $d = d'$ or $d' = d + w_1 + \cdots + w_t$, w_i in W and $d + w_1 + \cdots + w_s$ in N^n, $s = 1, \ldots, t$.

Let $R(d, W)$ denote the reachable vectors in (d, W) or *reachability set* of (d, W). Let K be a class of vector addition systems. By the *equality problem* for K we mean the problem of deciding for arbitrary (d_1, W_1), (d_2, W_2) in K whether $R(d_1, W_1) = R(d_2, W_2)$. The *special equality problem* is to decide the equality problem in those cases where $d_1 = d_2$ and W_2 arises from W_1 by removal of a vector.

Let VAS denote the class of vector addition systems and SDVAS denote the *self-dual* vector addition systems (d, W) defined by the property: $W = -W$ (i.e. w in W if and only if $-w$ in W).

LEMMA 1. *The problem of deciding whether the removal of a particular defining relation from a vector group changes a* (*positive*) *conjugate power set is reducible to the special equality problem for* (*VAS*) *SDVAS.*

Proof. Let G_1 be as in (I) and G_2 arise from G_1 by removal of one of the defining relations $a_i^{-1}b^{p_i}a_i = b^{q_i}$. It follows from the remarks above that we need consider only those positive integers l whose prime divisors are among $c_1, \ldots, c_n$, the prime divisors of the exponents of G_1. To obtain the indicated vector addition systems we take $d = (s_1, \ldots, s_n)$ and $w_i = (u_{1i}, \ldots, u_{ni})$

$$l = c_1^{s_1} \cdots c_n^{s_n},$$

$$p_i = c_1^{e_{1i}} \cdots c_n^{e_{ni}}, \qquad q_i = c_1^{e'_{1i}} \cdots c_n^{e'_{ni}}, \qquad i = 1, \ldots, k,$$

$$u_{ji} = \begin{cases} e_{ji} & \text{if } e_{ji} \neq 0, \\ -e'_{ji} & \text{otherwise,} \end{cases} \qquad j = 1, \ldots, n,$$

(and note $e_{ji} \cdot e'_j = 0$).

Let W'_1 consist of $w_1, \ldots, w_k$, $W'_2 = W'_1 - \langle w_i \rangle$, $W_1 = W'_1 \cup -W'_1$ and $W_2 = W'_2 \cup -W'_2$.

Our lemma follows from the observation that for $i = 1, 2$ we have m in $\mathrm{CP}(l, G_i)$ (respectively $\mathrm{PCP}(l, G_i)$) if and only if d' in $R(d, W_i)$ (respectively $R(d, W'_i)$) where $d' = (t_i, \ldots, t_n)$ and $m = c_1^{t_1} \cdots c_n^{t_n}$. □

A set $L \subseteq N^n$ is said to be *linear* if and only if $L = R(d, W)$ where W consists of nonnegative integral vectors. L is said to be *semilinear* if and only if it is the finite union of a finite number of linear sets.

It has been pointed out to the author by Professor John Hopcroft that:

LEMMA 2. *The reachability sets of self-dual vector addition systems are effectively computable semilinear sets.*

Remark. Lemma 2 seems to be known to several workers in this area (cf. [4], [7]).

LEMMA 3. *The equality problem for SDVAS is decidable.*

Lemma 3 is a consequence of the fact that semilinear sets in N^n form an effectively computable Boolean algebra [5]. Extending a previous result of Michael Rabin, M. Hack proved [6]:

LEMMA 4. *The special equality problem for VAS is undecidable.*

Theorems A and B are now seen to be consequences of Lemmas 1, 3 and 4.

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