MATHEMATICS OF COMPUTATION, VOLUME 32, NUMBER 144
OCTOBER 1978, PAGES 973–990

Numerical Solution of an Exterior Neumann Problem Using a Double Layer Potential

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Abstract. We give here a variational formulation in $H^{1/2}(\Gamma)/\mathbf{R}$ of the exterior Neumann problem for the Laplace operator using a double layer potential. This formulation is then applied to the construction of a finite element method. Optimal error estimates are given.

Introduction. Solving boundary value problems for partial differential operators by integral equation methods is not a new idea. However, the classical way to do it consists in representing the unknown solution as a potential of the type that will lead to an integral equation of the second kind. Then, Fredholm's theorems can be used. Thus, the Dirichlet problem is usually solved with the help of a double layer potential, and the Neumann problem with the use of a single layer potential.

We shall have a different point of view. Our aim will be to obtain a variational formulation of the problem in order to obtain the existence and unicity of a solution and error estimates. This philosophy leads to opposite choices for the representation of the solution. Thus, J. C. Nedelec and J. Planchard, for the three-dimensional case, and M. N. Leroux for the two-dimensional case, have solved the Dirichlet problem by using a single layer potential. We propose here the solution of a Neumann problem by using a double layer potential.

Let Ω be a bounded open set of $\mathbf{R}^3$. Let Γ be the boundary of Ω and Ω^c denote the complementary set of $\overline{\Omega}$.

We assume that Γ is sufficiently smooth, and we put the coordinates' origin in Ω. We shall write

$\vec{n}$, for the exterior normal to Γ,

r, for the distance to the origin,

$[v] = v|_\Gamma^{\text{int}} - v|_\Gamma^{\text{ext}}$, for the jump through Γ, of the function v defined in $\mathbf{R}^3$.

I. The Exterior Neumann Problem for the Laplace Operator. Let us consider the following problem.

$$(P_1)\quad \begin{cases} \text{Find } u_1 \in W_0^1(\Omega^c) = \{v \in \mathcal{D}'(\Omega^c) | v/r \in L^2(\Omega^c), Dv \in L^2(\Omega^c)\}, \text{ such that} \\ \Delta u_1 = 0 \quad \text{in } \Omega^c, \\ \partial u_1/\partial n = g_1 \in H^{-1/2}(\Gamma). \end{cases}$$

We have the

Received September 12, 1977.
AMS (MOS) subject classifications (1970). Primary 65R05.

PROPOSITION I-1. *Problem* (P_1) *has one and only one solution.*

Proof. It is a straightforward consequence of the fact that $\|\operatorname{grad} v\|_{L^2(\Omega^c)}$ is a norm on $W_0^1(\Omega^c)$ equivalent to the definition norm [4, Theorem II-2, p. 20]. □

For g_1 arbitrary in $H^{-1/2}(\Gamma)$, it is impossible to find a harmonic extension of u_1 in Ω. Let then λ and u_0 be defined by

$$\lambda = \langle g_1, 1\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \quad \text{and} \quad u_0(x) = \frac{\lambda}{4\pi}\frac{1}{|x-x_0|},$$

where x_0 is an arbitrary point of Ω. Let us take

$$u = u_1 - u_0 \quad \text{and} \quad g = g_1 - \frac{\partial u_0}{\partial n}.$$

Then, u is harmonic in Ω^c and can be harmonically extended in Ω. We are thus led to problem (P).

$$(P) \qquad \begin{cases} \text{Find } u \in (H^1(\Omega)/\mathbf{R}) \times W_0^1(\Omega^c) \text{ such that} \\ \Delta u = 0 \quad \text{in } \Omega \text{ and } \Omega^c, \\ \partial u/\partial n = g \in H_0^{-1/2}(\Gamma) = \{h \in H^{-1/2}(\Gamma) | \langle h, 1\rangle = 0\}. \end{cases}$$

We have then

THEOREM I-1. *Problem* (P) *has one and only one solution.*

Proof. We have only to split up Problem (P) into an interior problem and an exterior one. For an interior problem, the result is well known [9]. For an exterior problem, it is the result stated in Proposition I-1. □

In order to introduce the formulation on Γ which we are interested in, we shall need a problem (P') that we are going to define now. First, let us define

$$H^1(\Delta; \Omega) = \{v \in H^1(\Omega), \Delta v \in L^2(\Omega)\},$$

$$W^1(\Delta; \Omega^c) = \{v \in W_0^1(\Omega^c), r\Delta v \in L^2(\Omega^c)\},$$

$$K = \{v \in (H^1(\Delta; \Omega)/\mathbf{R}) \times W^1(\Delta; \Omega^c) | \operatorname{supp}(\Delta v) \subset \Gamma, [\partial v/\partial n] = 0\},$$

$$\|v\|_K = \left(\int_\Omega |\operatorname{grad} v|^2\, dx + \int_{\Omega^c} |\operatorname{grad} v|^2\, dx\right)^{1/2}$$

Problem (P') is the following:

$$(P') \qquad \text{Find } u \in K, \quad \text{such that } [u] = q \in H^{1/2}(\Gamma)/\mathbf{R}.$$

We have then

PROPOSITION I-2. *Problem* (P') *has one and only one solution.*

Proof. For $u \in H^1(\Delta; \Omega)$ we have the following Green's formula [9].

$$\forall v \in H^1(\Omega), \quad \int_\Omega \operatorname{grad} u \operatorname{grad} v\, dx = -\int_\Omega \Delta u v\, dx + \left\langle \frac{\partial u}{\partial n}, v\right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.$$

In the same way, one can prove for $u \in W^1(\Delta; \Omega^c)$ the following Green's formula [4].

$$\forall v \in W_0^1(\Omega^c), \quad \int_{\Omega^c} \operatorname{grad} u \operatorname{grad} v\,dx = -\int_{\Omega^c} \Delta u v\,dx - \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.$$

Let us now consider $u \in K$. We have

$$\forall v \in (H^1(\Omega)/\mathbf{R}) \times W_0^1(\Omega^c),$$

$$\int_{\Omega\cup\Omega^c} \operatorname{grad} u \operatorname{grad} v\,dx = \left\langle \frac{\partial u}{\partial n}, [v] \right\rangle_{H_0^{-1/2}(\Gamma)\times(H^{1/2}(\Gamma)/\mathbf{R})}.$$

If v belongs to K too, we have

$$\int_{\Omega\cup\Omega^c} \operatorname{grad} u \operatorname{grad} v\,dx = \left\langle \frac{\partial v}{\partial n}, [u] \right\rangle_{H_0^{-1/2}(\Gamma)\times(H^{1/2}(\Gamma)/\mathbf{R})},$$

which gives a variational formulation of Problem ($\mathcal{P}'$) and ends the proof. □

Thus, Problem ($\mathcal{P}$) defines an isomorphism J_0 of $H_0^{-1/2}(\Gamma)$ onto K, and Problem ($\mathcal{P}'$) defines an isomorphism J_1 of $H^{1/2}(\Gamma)/\mathbf{R}$ onto K. Therefore, $J = J_1^{-1} \circ J_0$ is an isomorphism of $H_0^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)/\mathbf{R}$. We shall see in a moment that a coercive bilinear form corresponds to this isomorphism.

Let q and q' belong to $H^{1/2}(\Gamma)/\mathbf{R}$ and define $a(q, q') = \langle q, J^{-1}(q')\rangle$. Then, we have

THEOREM I-2. *The bilinear form a is symmetrical and positive definite on $H^{1/2}(\Gamma)/\mathbf{R}$.*

Proof. Define

$$u = J_1(q), \qquad v = J_0(J^{-1}(q')).$$

Then,

$$a(q, q') = \left\langle [u], \frac{\partial v}{\partial n} \right\rangle = \int_{\Omega\cup\Omega^c} \operatorname{grad} u \operatorname{grad} v\,dx = \left\langle [v], \frac{\partial u}{\partial n} \right\rangle = a(q', q),$$

and

$$a(q, q) = \int_{\Omega\cup\Omega^c} |\operatorname{grad} u|^2\,dx \geq C\|u\|^2_{(H^1(\Omega)/\mathbf{R})\times W_0^1(\Omega^c)},$$

hence,

$$a(q, q) \geq C\|q\|^2_{H^{1/2}(\Gamma)/\mathbf{R}},$$

which ends the proof. □

Thus, the jump through Γ of the solution u of Problem ($\mathcal{P}$) is the solution of the coercive variational problem.

$$(\mathcal{Q}) \qquad \begin{cases} \text{Find } q \in H^{1/2}(\Gamma)/\mathbf{R} \quad \text{such that} \\ a(q, q') = \langle g, q' \rangle_{H_0^{-1/2}(\Gamma)\times(H^{1/2}(\Gamma)/\mathbf{R})}, \quad \forall q' \in H^{1/2}(\Gamma)/\mathbf{R}. \end{cases}$$

In order to use these results, we have to find an explicit expression of a. This is what we shall do now.

PROPOSITION I-3. *Let q belong to $\mathcal{D}(\Gamma)$. The solution u of Problem $(\mathcal{P}')$ can be expressed by*

$$u(y) = -\frac{1}{4\pi}\int_\Gamma q(x)\,\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) d\gamma_x .$$

Proof. It is an immediate consequence of well-known facts about double layer potentials [13]. □

THEOREM I-3. *Let q and q' belong to $\mathcal{D}(\Gamma)$. Then, the bilinear form a has the following expression*

$$a(q, q') = \frac{1}{8\pi}\int_\Gamma\int_\Gamma (q(x)-q(y))\,(q'(x)-q'(y))\,\frac{\partial^2}{\partial n_x \partial n_y}\left(\frac{1}{|x-y|}\right) d\gamma_x d\gamma_y .$$

Proof.

$$a(q, q') = \int_\Gamma q(y)\,\frac{\partial}{\partial n_y}\left\{-\frac{1}{4\pi}\int_\Gamma q'(x)\,\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) d\gamma_x\right\} d\gamma_y = 2A.$$

Since a is symmetric, we have

$$a(q, q') = A + \int_\Gamma q'(y)\,\frac{\partial}{\partial n_y}\left\{-\frac{1}{8\pi}\int_\Gamma q(x)\,\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) d\gamma_x\right\} d\gamma_y = A + B.$$

On the other hand,

$$\int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) d\gamma_x = \Omega_y,$$

where Ω_y is the solid angle sustained by the surface Γ at the point y. Therefore,

$$\frac{\partial}{\partial n_y}\left\{\int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) d\gamma_x\right\} = 0,$$

and

$$C = \frac{1}{8\pi}\int_\Gamma q(y)q'(y)\,\frac{\partial}{\partial n_y}\left\{\int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) d\gamma_x\right\} d\gamma_y = 0.$$

Moreover, since

$$\forall w \in K, \quad \left\langle \frac{\partial w}{\partial n}, 1\right\rangle = 0,$$

we have

$$D = \frac{1}{8\pi}\int_\Gamma \frac{\partial}{\partial n_y}\left\{\int_\Gamma q(x)q'(x)\,\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) d\gamma_x\right\} d\gamma_y = 0.$$

Finally, we arrive at

$$a(q, q') = A + B + C + D.$$

Then, we can write

$$a(q, q') = \frac{1}{8\pi} \int_\Gamma \frac{\partial}{\partial n_y} \left\{ \int_\Gamma (q(y) - q(x))(q'(y) - q'(x)) \frac{\partial}{\partial n_x} \left(\frac{1}{|x - y|} \right) d\gamma_x \right\} d\gamma_y,$$

provided that we do not forget that the differentiation with respect to n_y does not concern $q(y)$ and $q'(y)$. This can be seen in the expressions of A, B and C.

It becomes possible then to interchange the order of integration and differentiation. This leads us to

$$a(q, q') = \frac{1}{8\pi} \int_\Gamma \int_\Gamma (q(y) - q(x))(q'(y) - q'(x)) \frac{\partial^2}{\partial n_x \partial n_y} \left(\frac{1}{|x - y|} \right) d\gamma_x d\gamma_y. \quad \square$$

Remark. The same arguments can be used for the exterior Neumann problem in $\mathbf{R}^2$. This time, the space $W_0^1(\Omega^c)$ becomes

$$W_0^1(\Omega^c) = \left\{ v \in \mathcal{D}'(\Omega^c) \;\middle|\; \frac{v}{r \operatorname{Log} r} \in L^2(\Omega^c),\, Dv \in L^2(\Omega^c) \right\}.$$

Problem $(\mathcal{P}_1)$ becomes

$$(\mathcal{P}_1) \qquad \begin{cases} \text{Find } u \in W_0^1(\Omega^c)/\mathbf{R} \quad \text{such that} \\ \Delta u = 0 \quad \text{in } \Omega^c, \\ \partial u/\partial n = g \in H_0^{-1/2}(\Gamma); \end{cases}$$

for now, $\|\operatorname{grad} u\|_{L^2(\Omega^c)}$ is a norm only on $W_0^1(\Omega^c)/\mathbf{R}$ equivalent to the quotient norm [4].

Then, we can immediately find a harmonic extension of u inside Ω. Thus, Problem $(\mathcal{P})$ can be written

$$(\mathcal{P}) \qquad \begin{cases} \text{Find } u \in (H^1(\Omega)/\mathbf{R}) \times (W_0^1(\Omega^c)/\mathbf{R}) \quad \text{such that} \\ \Delta u = 0 \quad \text{in } \Omega \text{ and } \Omega^c, \\ \partial u/\partial n = g \in H_0^{-1/2}(\Gamma). \end{cases}$$

Concerning Problem $(\mathcal{P}')$, the only change will be the definition of

$$W^1(\Delta; \Omega^c) = \{ v \,|\, v \in W_0^1(\Omega^c),\, r \operatorname{Log} r \Delta v \in L^2(\Omega^c) \}.$$

Finally, for regular q and q', the solution of Problem $(\mathcal{P}')$ can be written as

$$u(y) = \frac{1}{2\pi} \int_\Gamma q(x) \frac{\partial}{\partial n_x} \operatorname{Log}|x - y| \, d\gamma_x + C,$$

where C is an arbitrary constant; the bilinear form a becomes

$$a(q, q') = -\frac{1}{4\pi} \int_\Gamma \int_\Gamma (q(x) - q(y))(q'(x) - q'(y)) \frac{\partial^2}{\partial n_x \partial n_y} \operatorname{Log}|x - y| \, d\gamma_x d\gamma_y.$$

II. Approximation. The problem is to approximate the solution q of

$$(\mathcal{Q})\qquad \begin{cases} \text{Find } q \in H^{1/2}(\Gamma)/\mathbf{R} \quad \text{such that} \\ a(q, q') = \langle g, q' \rangle, \forall q' \in H^{1/2}(\Gamma)/\mathbf{R}. \end{cases}$$

1. *Construction of an Approximate Surface* Γ_h [14]. Let $\{\mathcal{O}_i\}$ be a family of p bounded open sets of $\mathbf{R}^3$, covering Γ and such that, for each i, there exists a C^∞ mapping

$$\mathcal{O}_i \xrightarrow{\theta_i} Q = \{y \mid y = \{y', y_3\}, |y'| < 1, -1 < y_3 < 1\}.$$

Let us further assume that θ_i has a C^∞ inverse mapping $Q \xrightarrow{\theta_i^{-1}} \mathcal{O}_i$ and that θ_i is a mapping of $\mathcal{O}_i \cap \Omega$, $\mathcal{O}_i \cap \Omega^c$, $\mathcal{O}_i \cap \Gamma$, onto, respectively,

$$Q_- = \{y \in Q \mid y_3 < 0\}, \quad Q_+ = \{y \in Q \mid y_3 > 0\}, \quad Q_0 = \{y \in Q \mid y_3 = 0\}.$$

We shall assume that the usual compatibility relations [10] hold between the θ_i.

In what follows, we shall write φ_i for θ_i^{-1} considered as a mapping of Q_0 onto $\mathcal{O}_i \cap \Gamma$.

To define Γ_h, let us assume that we know a partition of Γ into p closed parts Γ_i such that

$$\Gamma_i \subset \mathcal{O}_i, \quad \bigcup_{i=1}^{P} \Gamma_i = \Gamma, \quad \Gamma_i \cap \Gamma_j \text{ is a curve of } \Gamma \text{ (or empty), when } i \neq j.$$

Let us denote by D_i the image of Γ_i by θ_i.

Let Σ_h be a set of nodes on Γ, and let σ_{ih} be the image of $\Sigma_h \cap \Gamma_i$ by θ_i. Now, we build on σ_{ih} a triangulation $\mathcal{T}_{ih}$ of D_i. Then, to each element T of each triangulation $\mathcal{T}_{ih}$, we append a C^0 Lagrange finite element with an interpolation space G such that $P_k \subset G$, where P_k is the space of polynomials of degree k or less.

Let φ_{ih} be the mapping the restriction of which, on each element T of $\mathcal{T}_{ih}$, is the G interpolate F_T of φ_i. Then, Γ_h is the surface defined by the mappings φ_{ih}.

2. *Approximation* n_h *of the Normal* n *to* Γ. We will use, as an approximation to n, the G interpolate n_h of n. As we shall see later, this is consistent with the approximation chosen for Γ.

3. *Construction of an Approximation* V_h *of* $H^{1/2}(\Gamma)$. To each element T of the triangulation

$$\mathcal{T}_h = \bigcup_{i=1}^{P} \mathcal{T}_{ih},$$

we associate a functional vector space P, such that $P_m \subset P$.

Then, we define V_h as the space of the images of the elements of P on every curved element of Γ_h by the mapping F_T, i.e.,

$$V_h = \{q_h \in C^0(\Gamma_h) \mid q_h|_T = p \circ F_T^{-1}, \forall T \in \mathcal{T}_h, \forall p \in P\}.$$

We want the elements of V_h to be continuous on Γ_h, in order to have the inclusion of V_h in $H^{1/2}(\Gamma_h)$.

4. *The Approximate Problem.* The kernel of Problem $(\mathcal{Q})$ is

$$\frac{\partial^2}{\partial n_x \partial n_y}\left(\frac{1}{|x-y|}\right) = \frac{(n_x, n_y)}{|x-y|^3} - 3\,\frac{(x-y, n_x)(x-y, n_y)}{|x-y|^5}.$$

Let us approximate this kernel by

$$\frac{(n_{hx}, n_{hy})}{|x-y|^3} - 3\,\frac{(x-y, n_{hx})(x-y, n_{hy})}{|x-y|^5},$$

where x, y belong to Γ_h, and n_{hx}, n_{hy} are the approximations of n_x and n_y, defined in II-2.

We shall write

$$\frac{\partial^2}{\partial n_{hx} \partial n_{hy}}\left(\frac{1}{|x-y|}\right)$$

for this kernel. Then, we approximate Problem $(\mathcal{Q})$ by

$$(\mathcal{Q}_h)\begin{cases} \text{Find } q_h \in V_h/\mathbf{R} \quad \text{such that} \\ -\dfrac{1}{8\pi}\displaystyle\int_{\Gamma_h}\int_{\Gamma_h} (q_h(x) - q_h(y))(q'_h(x) - q'_h(y))\,\frac{\partial^2}{\partial n_{hx}\partial n_{hy}}\left(\frac{1}{|x-y|}\right) d\gamma_{hx}\,d\gamma_{hy} \\ = \displaystyle\int_{\Gamma_h} g_h(y) q'_h(y)\,d\gamma_{hy}, \ \forall q'_h \in V_h/\mathbf{R}, \end{cases}$$

where g_h is an approximation of g defined on Γ_h and satisfying $\langle g_h, 1\rangle = 0$.

We shall write Problem $(\mathcal{Q}_h)$ more concisely as

$$\begin{cases} \text{Find } q_h \in V_h/\mathbf{R} \quad \text{such that} \\ a_h(q_h, q'_h) = \langle g_h, q'_h\rangle_{H_0^{-1/2}(\Gamma_h)\times(H^{1/2}(\Gamma_h)/\mathbf{R})}, \quad \forall q'_h \in V_h/\mathbf{R}. \end{cases}$$

According to Theorem I-2, that we apply now to Problem $(\mathcal{Q}_h)$, this problem would have one and only one solution if n_h was normal to Γ_h.

Unfortunately, this is not the case because, in order to get an optimal order of convergence, we have chosen another n_h. Thus, the existence of a unique solution to Problem $(\mathcal{Q}_h)$ will be a consequence of the uniform coercivity of the bilinear forms a_h. This uniform coercivity will appear during the error study of the following section.

5. *Error Estimates.* To compare q and q_h, we have to define a mapping of Γ_h onto Γ. For the same reasons as those of [14], we have to use ψ defined by

–for x belonging to Γ_h, $\psi(x)$ is the orthogonal projection of x onto Γ, i.e., $\psi(x)$ is the foot of the normal to Γ passing through x.

For Γ_h sufficiently close to Γ, i.e., for h sufficiently small, ψ is regular and bijective [14].

Then, we have the following estimates,

THEOREM II-1. *Let q and q_h be the solutions, respectively, of Problems $(\mathcal{Q})$ and $(\mathcal{Q}_h)$. Then, if q belongs to $H^{m+1}(\Gamma)$, we have*

$$\|q - q_h \circ \psi^{-1}\|_{H^{1/2}(\Gamma)/\mathbf{R}}$$

$$\leqslant C\{\|g - \hat{g}_h\|_{H_0^{-1/2}(\Gamma)} + h^{m+1/2}\|q\|_{H^{m+1}(\Gamma)/\mathbf{R}} + h^{k+1}\|q\|_{H^{1/2}(\Gamma)/\mathbf{R}}\}$$

and

$$\|q - q_h \circ \psi^{-1}\|_{L^2(\Gamma)/\mathbf{R}} \leqslant C\{\sqrt{h}\|g - \hat{g}_h\|_{H_0^{-1/2}(\Gamma)} + \|g - \hat{g}_h\|_{H_0^{-1}(\Gamma)}$$
$$+ h^{m+1}\|q\|_{H^{m+1}(\Gamma)/\mathbf{R}} + h^{k+1}\|q\|_{H^{1/2}(\Gamma)/\mathbf{R}}\},$$

where

$$\hat{g}_h(x) = (g_h \circ \psi^{-1}(x)J(\psi^{-1}(x))).$$

Proof. Let $\widetilde{V}_h$ be the subspace of $H^{1/2}(\Gamma)$, the image of V_h by the mapping ψ^{-1} i.e.,

$$\widetilde{V}_h = \{\widetilde{v}_h | \widetilde{v}_h = v_h \circ \psi^{-1}, \forall v_h \in V_h\}.$$

Let us consider Problem (Q_h) on Γ. Defining J as Jacobian mapping, we obtain: find $\widetilde{q}_h \in \widetilde{V}_h/\mathbf{R}$ such that

$$-\frac{1}{8\pi}\int_\Gamma\int_\Gamma (\widetilde{q}_h(x) - \widetilde{q}_h(y))(\widetilde{q}_h'(x) - \widetilde{q}_h'(y))\frac{\partial^2}{\partial n_{hx}\partial n_{hy}}\left(\frac{1}{|\psi^{-1}(x) - \psi^{-1}(y)|}\right)$$
$$\cdot J(\psi^{-1}(x))J(\psi^{-1}(y))\,d\gamma_x\,d\gamma_y$$
$$= \int_\Gamma \widetilde{g}_h(y)\widetilde{q}_h'(y)J(\psi^{-1}(y))\,d\gamma_y, \quad \forall \widetilde{q}_h' \in \widetilde{V}_h/\mathbf{R}.$$

Define $\hat{g}_h = \widetilde{g}_h J(\psi^{-1})$, then, $\langle \hat{g}_h, 1\rangle = 0$; and we have the following estimate

$$\|q - \widetilde{q}_h\|_{H^{1/2}(\Gamma)/\mathbf{R}}$$

$$\leqslant C\left\{\|g - \hat{g}_h\|_{H_0^{-1/2}(\Gamma)} + \inf_{\widetilde{q}_h' \in \widetilde{V}_h/\mathbf{R}}\left[\|q - \widetilde{q}_h'\|_{H^{1/2}(\Gamma)/\mathbf{R}} + \sup_{w_h \in \widetilde{V}_h/\mathbf{R}} \frac{|a(\widetilde{q}_h', w_h) - a_h(q_h', w_h)|}{\|w_h\|_{H^{1/2}(\Gamma)/\mathbf{R}}}\right]\right\}.$$

Let us choose $\widetilde{q}_h' = \Pi_h q$, where Π_h is the $\widetilde{V}_h$ interpolation operator. We know then [1] that

$$\|q - \Pi_h q\|_{L^2(\Gamma)} \leqslant Ch^{m+1}\|q\|_{H^{m+1}(\Gamma)},$$

and

$$\|q - \Pi_h q\|_{H^1(\Gamma)} \leqslant Ch^m\|q\|_{H^{m+1}(\Gamma)};$$

hence, by interpolating between $L^2(\Gamma)$ and $H^1(\Gamma)$,

$$\|q - \Pi_h q\|_{H^{1/2}(\Gamma)} \leqslant Ch^{m+1/2} \|q\|_{H^{m+1}(\Gamma)},$$

and since

$$(q + C) - \Pi_h(q + C) = q - \Pi_h q, \quad \forall C \in \mathbf{R},$$

we have

$$\|q - \Pi_h q\|_{H^{1/2}(\Gamma)/\mathbf{R}} \leqslant Ch^{m+1/2} \|q\|_{H^{m+1}(\Gamma)/\mathbf{R}}.$$

It remains to estimate the error due to the change of bilinear form. For this, we shall need

LEMMA II-1. *The quantity*

$$b(q) = \left\{ \int_\Gamma \int_\Gamma \frac{(q(x) - q(y))^2}{|x - y|^3} d\gamma_x \, d\gamma_y \right\}^{1/2}$$

is a norm on $H^{1/2}(\Gamma)/\mathbf{R}$ equivalent to the usual norm.

Proof of Lemma II-1. According to Theorems I-2 and I-3, $a(q)$ defined by

$$a(q) = \left\{ \int_\Gamma \int_\Gamma (q(x) - q(y))^2 \frac{\partial^2}{\partial n_x \partial n_y} \left(\frac{1}{|x - y|} \right) d\gamma_x \, d\gamma_y \right\}^{1/2}$$

is a norm on $H^{1/2}(\Gamma)/\mathbf{R}$ equivalent to the usual one. But

$$\frac{\partial^2}{\partial n_x \partial n_y} \left(\frac{1}{|x - y|} \right) = \frac{(n_x, n_y)}{|x - y|^3} - 3 \frac{(x - y, n_x)(x - y, n_y)}{|x - y|^5},$$

so that

$$\left| \frac{\partial^2}{\partial n_x \partial n_y} \left(\frac{1}{|x - y|} \right) \right| \leqslant \frac{4}{|x - y|^3} \quad \text{and} \quad a(q) \leqslant 4b(q).$$

On the other hand,

$$\frac{\partial^2}{\partial n_x \partial n_y} \left(\frac{1}{|x - y|} \right) = \frac{1}{|x - y|^3} + \frac{f(x, y)}{|x - y|},$$

where $f(x, y)$ is bounded in a neighborhood of $x = y$, so that

$$b^2(q) \leqslant a^2(q) + \int_\Gamma \int_\Gamma (q(x) - q(y))^2 \frac{f(x, y)}{|x - y|} d\gamma_x \, d\gamma_y,$$

and since

$$\int_\Gamma \frac{|f(x, y)|}{|x - y|} d\gamma_x$$

is a bounded function of y, by developing $(q(x) - q(y))^2$, we obtain

$$b^2(q) \leqslant a^2(q) + C\|q\|^2_{L^2(\Gamma)}.$$

Moreover, $b(q)$ equals 0 when q is a constant, so that Lemma II-1 is proved. □

We shall use three other lemmas proved in [5]. Let us give them.

LEMMA II-2. *Let h be the greatest diameter of the elements T of $\mathcal{T}_h$, then*

$$\max_{i=1,P} \sup_{x\in S_i} |\Phi_i(x) - \Phi_{ih}(x)| \leqslant Ch^{k+1} \max_{i=1,P} \sup_{x\in S_i} \|D^{k+1}\Phi_i(x)\|,$$

$$\max_{i=1,P} \sup_{x\in S_i} |D^l\Phi_i(x) - D^l\Phi_{ih}(x)| \leqslant Ch^{k+1-l} \max_{i=1,P} \sup_{x\in S_i} \|D^{k+1}\Phi_i(x)\|,$$

with $1 \leqslant |l| \leqslant k + 1$.

LEMMA II-3. *For each triangle T of $\mathcal{T}_h$, the mapping $\psi \circ F_T$ is bounded as also its derivatives up to the order $k + 1$. $D(\psi \circ F_T)$ is a linear mapping of rank 2, having a bounded inverse when considered as a mapping of $\mathbf{R}^2$ onto the tangent plane to Γ. Moreover,*

$$\sup_{x\in T} |\psi \circ F_T(x) - F_T(x)| \leqslant Ch^{k+1} \sup_{x\in T} |D^{k+1}\Phi_i(x)|,$$

$$\sup_{x\in T} |D(\psi \circ F_T)(x) - DF_T(x)| \leqslant Ch^k \sup_{x\in T} |D^{k+1}\Phi_i(x)|.$$

LEMMA II-4. *Let T and T' be two triangles of $\mathcal{T}_h$. Then,*

$$\sup_{x\in T} |J(F_T)(x) - J(\psi \circ F_T)(x)| \leqslant Ch^{k+1} \sup_{x\in T} |D^{k+1}\Phi_i(x)|,$$

$$C|F_T(x) - F_{T'}(y)| \leqslant |\psi \circ F_T(x) - \psi \circ F_{T'}(y)| \leqslant C|F_T(x) - F_{T'}(y)|,$$

$$\big||F_T(x) - F_{T'}(y)|^2 - |\psi \circ F_T(x) - \psi \circ F_{T'}(y)|^2\big| \leqslant Ch^{k+1}|F_T(x) - F_{T'}(y)|^2.$$

The first inequality of this last lemma shows why we are interested in the mapping ψ. It enables us to obtain an error on the Jacobian mapping of order $k + 1$, instead of k.

We are now in a position to resume the proof of Theorem II-1.

The error due to the change of bilinear form can be split up into three parts

—the error due to the change of Jacobian mapping;

—the error due to the change of normal;

—the error due to the substitution of $\psi^{-1}(x)$ to x.

Using Lemmas II-1 and II-4, we can see immediately that the first error is bounded by

$$Ch^{k+1}\|q\|_{H^{1/2}(\Gamma)/\mathbf{R}}.$$

As for the second and third errors, we have to estimate

$$\frac{\partial^2}{\partial n_x \partial n_y}\left(\frac{1}{|x-y|}\right) - \frac{\partial^2}{\partial n_{hx}\partial n_{hy}}\left(\frac{1}{|\psi^{-1}(x)-\psi^{-1}(y)|}\right)$$

$$= \frac{(n_x, n_y)}{|x-y|^3} - \frac{(n_{hx}, n_{hy})}{|\psi^{-1}(x)-\psi^{-1}(y)|^3} - 3\,\frac{(x-y, n_x)(x-y, n_y)}{|x-y|^5}$$

$$+ 3\,\frac{(\psi^{-1}(x)-\psi^{-1}(y), n_{hx})(\psi^{-1}(x)-\psi^{-1}(y), n_{hy})}{|\psi^{-1}(x)-\psi^{-1}(y)|^5}.$$

The error on the normal gives two kinds of terms

$$\frac{(n_x - n_{hx}, n_y)}{|x-y|^3} \quad \text{and} \quad \frac{(x-y, n_x - n_{hx})(x-y, n_y)}{|x-y|^5}.$$

We shall examine the first one. Let us recall that n_h is the G-interpolate of n, so that $|n_x - n_{hx}| \leqslant Ch^{k+1}$, both terms are bounded by $Ch^{k+1}/|x-y|^3$, and the error due to the change of normal can eventually be estimated by $Ch^{k+1}\|q\|_{H^{1/2}(\Gamma)/\mathbf{R}}$.

Let us now examine the error due to the substitution of $\psi^{-1}(x)$ to x. There appear three kinds of terms.

$$\frac{1}{|x-y|^3} - \frac{1}{|\psi^{-1}(x)-\psi^{-1}(y)|^3}, \quad \frac{1}{|x-y|^5} - \frac{1}{|\psi^{-1}(x)-\psi^{-1}(y)|^5},$$

$$(x-y, n_x) - (\psi^{-1}(x)-\psi^{-1}(y), n_x).$$

For the first term, we put

$$A = |x-y|; \qquad B = |\psi^{-1}(x)-\psi^{-1}(y)|.$$

Then,

$$\frac{1}{A^3} - \frac{1}{B^3} = \frac{B^3 - A^3}{A^3B^3} = \frac{(B-A)(B^2+AB+A^2)}{A^3B^3}$$

$$= \frac{(B^2-A^2)(B^2+AB+A^2)}{A^3B^3(A+B)}.$$

According to Lemma II-4, this last term is bounded above by $Ch^{k+1}/|x-y|^3$. An analogous argument gives the same bound for the second term.

The third term can be bounded in two ways.

On the one hand, we have

$$|x-y-\psi^{-1}(x)+\psi^{-1}(y)| \leqslant |x-\psi^{-1}(x)| + |y-\psi^{-1}(y)| \leqslant Ch^{k+1},$$

and, on the other hand, we also have

$$|x-y-\psi^{-1}(x)+\psi^{-1}(y)| = |(I-\psi^{-1})(x) - (I-\psi^{-1})(y)| \leqslant Ch^k|x-y|.$$

Thus, terms like

$$\frac{(x-y-\psi^{-1}(x)+\psi^{-1}(y), n_x)(x-y, n_{hx})}{|x-y|^5}$$
$$= \frac{(x-y-\psi^{-1}(x)+\psi^{-1}(y), n_x)(x-y, n_x)}{|x-y|^5}$$
$$+ \frac{(x-y-\psi^{-1}(x)+\psi^{-1}(y), n_x)(x-y, n_{hx}-n_x)}{|x-y|^5}$$

can be bounded above by

$$\frac{Ch^{k+1}|x-y|^2}{|x-y|^5} + \frac{Ch^k|x-y|Ch^{k+1}|x-y|}{|x-y|^5} \leqslant C\frac{h^{k+1}}{|x-y|^3};$$

and the error due to the substitution of $\psi^{-1}(x)$ to x is bounded by $Ch^{k+1}\|q\|_{H^{1/2}(\Gamma)/\mathbf{R}}$, which completes the estimation of the error in $H^{1/2}(\Gamma)/\mathbf{R}$. □

The error in $L^2(\Gamma)/\mathbf{R}$ will be given by the following theorem.

THEOREM II-2. *Let $s \leqslant m$. We have the following error estimate*

$$\|q-\tilde{q}_h\|_{H^{-s}(\Gamma)/\mathbf{R}} \leqslant C\{h^{s+1/2}\|g-\hat{g}_h\|_{H_0^{-1/2}(\Gamma)} + \|g-\hat{g}_h\|_{H_0^{-s-1}(\Gamma)}$$
$$+ h^{m+s+1}\|q\|_{H^{m+1}(\Gamma)/\mathbf{R}} + h^{k+1}\|q\|_{H^{1/2}(\Gamma)/\mathbf{R}}\}.$$

Proof. We shall need

LEMMA II-5. *Let q be the solution of*

$$\begin{cases} \text{Find } q \in H^{1/2}(\Gamma)/\mathbf{R} \text{ such that} \\ a(q, q') = \langle g, q'\rangle_{H_0^{-1/2}(\Gamma)\times(H^{1/2}(\Gamma)/\mathbf{R})}, \quad \forall q' \in H^{1/2}(\Gamma)/\mathbf{R}, \end{cases}$$

then,

$$\forall s \geqslant 0, \quad g \in H_0^s(\Gamma) \Rightarrow q \in H^{s+1}(\Gamma)/\mathbf{R} \quad \textit{and} \quad \|q\|_{H^{s+1}(\Gamma)/\mathbf{R}} \leqslant C\|g\|_{H_0^s(\Gamma)}.$$

Proof of Lemma II-5. Let u be the solution of

$$\begin{cases} \text{Find } u \in (H^1(\Omega)/\mathbf{R}) \times W_0^1(\Omega^c) \text{ such that} \\ \Delta u = 0 \quad \text{in } \Omega \text{ and } \Omega^c, \\ \partial u/\partial n = g \in H_0^{-1/2}(\Gamma). \end{cases}$$

Then, the classical regularity theorems tell us that

$$g \in H_0^s(\Gamma) \Rightarrow u|_{\text{int}} \in H^{s+3/2}(\Omega)/\mathbf{R} \quad \text{and} \quad u|_{\text{ext}} \in H_{\text{loc}}^{s+3/2}(\Omega^c),$$

hence $q = [u] \in H^{s+1}(\Gamma)/\mathbf{R}$. The closed graph theorem applied to the mapping $g \longrightarrow q$ ends the proof of this lemma. □

Proof of Theorem II-2. We shall use a classical duality argument. We have

$$\|q-\tilde{q}_h\|_{H^{-s}(\Gamma)/\mathbf{R}} = \sup_{g\in H_0^s(\Gamma)} \frac{|(g, q-\tilde{q}_h)|}{\|g\|_{H_0^s(\Gamma)}},$$

but

$$(g, q - \widetilde{q}_h) = a(q - \widetilde{q}_h, Jg),$$

where J is the mapping $\partial u/\partial n \longrightarrow [u]$, defined according to Proposition I-2.

Then, Lemma II-5 tells us that $\|Jg\|_{H^{s+1}(\Gamma)/\mathbf{R}} \leqslant C\|g\|_{H_0^s(\Gamma)}$, so that

$$\|q - \widetilde{q}_h\|_{H^{-s}(\Gamma)/\mathbf{R}} \leqslant C \sup_{g \in H_0^s(\Gamma)} \frac{|a(q - \widetilde{q}_h, Jg)|}{\|Jg\|_{H^{s+1}(\Gamma)/\mathbf{R}}}.$$

Now, we have

$$a(q - \widetilde{q}_h, Jg) = a(q - \widetilde{q}_h, Jg - \Pi_h Jg) + a(q - \widetilde{q}_h, \Pi_h Jg),$$

where Π_h is the $\widetilde{V}_h$ interpolation operator. Therefore,

$$\|Jg - \Pi_h Jg\|_{H^{1/2}(\Gamma)/\mathbf{R}} \leqslant Ch^{s+1/2} \|Jg\|_{H^{s+1}(\Gamma)/\mathbf{R}},$$

so that

$$\sup_{g \in H_0^s(\Gamma)} \frac{|a(q - \widetilde{q}_h, Jg - \Pi_h Jg)|}{\|Jg\|_{H^{s+1}(\Gamma)/\mathbf{R}}} \leqslant Ch^{s+1/2} \|q - \widetilde{q}_h\|_{H^{1/2}(\Gamma)/\mathbf{R}}.$$

It remains to estimate the second term

$$\begin{aligned} a(q - \widetilde{q}_h, \Pi_h Jg) &= (g, \Pi_h Jg) - a(\widetilde{q}_h, \Pi_h Jg) = (g, \Pi_h Jg) - (\hat{g}_h, \Pi_h Jg) \\ &\quad - \frac{1}{8\pi} \int_\Gamma \int_\Gamma (\widetilde{q}_h(x) - \widetilde{q}_h(y))(\Pi_h Jg(x) - \Pi_h Jg(y)) \\ &\quad \cdot \left\{ \frac{\partial^2}{\partial n_x \partial n_y} \left(\frac{1}{|x - y|} \right) - \frac{\partial^2}{\partial n_{hx} \partial n_{hy}} \left(\frac{1}{|\psi^{-1}(x) - \psi^{-1}(y)|} \right) \right. \\ &\quad \left. \cdot J(\psi^{-1}(x)) J(\psi^{-1}(y)) \right\} d\gamma_x \, d\gamma_y. \end{aligned}$$

The last term can be studied in the same way as for the proof of Theorem II-1. This study leads to the estimate $Ch^{k+1} \|q\|_{H^{1/2}(\Gamma)/\mathbf{R}} \|Jg\|_{H^{1/2}(\Gamma)/\mathbf{R}}$. As for the difference $(g, \Pi_h Jg) - (\hat{g}_h, \Pi_h Jg)$, it can be bounded above by $\|g - \hat{g}_h\|_{H_0^{-s-1}(\Gamma)} \cdot \|Jg\|_{H^{s+1}(\Gamma)/\mathbf{R}}$, which gives

$$\sup_{g \in H_0^s(\Gamma)} \frac{|a(q - \widetilde{q}_h, \Pi_h Jg)|}{\|Jg\|_{H^{s+1}(\Gamma)/\mathbf{R}}} \leqslant Ch^{k+1} \|q\|_{H^{1/2}(\Gamma)/\mathbf{R}} + \|g - \hat{g}_h\|_{H_0^{-s-1}(\Gamma)},$$

and ends the proof of Theorem II-2. □

The L^2 estimate announced in Theorem II-1 can be obtained by choosing $s = 0$.

Finally, we must not forget that our first problem was to find the solution of Problem $(\mathcal{P})$. In that respect, the following theorem is the more interesting.

THEOREM II-3. *Let u be the solution of Problem* (P) *satisfying* $\int_\Gamma q(x)\,d\gamma_x = 0$, *and let* u_h *be defined by*

$$u_h(y) = -\frac{1}{4\pi}\int_{\Gamma_h} q_h(x)\frac{\partial}{\partial n_{hx}}\left(\frac{1}{|x-y|}\right)d\gamma_{hx},$$

where q_h *is the solution of Problem* (Q_h) *satisfying* $\int_{\Gamma_h} q_h(x)\,d\gamma_{hx} = 0$. *Let us assume that* $\exists\delta > 0$: $d(y, \Gamma) \geqslant \delta$, *then, for sufficiently small h, e.g. for h such that*

$$\sup_{z\in\Gamma_h} d(z, \Gamma) \leqslant \frac{d(y, \Gamma)}{2},$$

we have the following error estimate

$$|u(y) - u_h(y)| \leqslant \frac{C}{e(y, \Gamma)}\{h^{m+1/2}\|g - \hat{g}_h\|_{H_0^{-1/2}(\Gamma)} + \|g - \hat{g}_h\|_{H_0^{-m-1}(\Gamma)} + h^{2m+1}\|q\|_{H^{m+1}(\Gamma)/\mathbf{R}} + h^{k+1}\|q\|_{H^{1/2}(\Gamma)/\mathbf{R}}\},$$

where

$$\frac{1}{e(y, \Gamma)} = \sum_{n=0}^{m}\frac{1}{d^{2+n}(y, \Gamma)} + \begin{cases} 1 & \textit{when } y \in \Omega, \\ 0 & \textit{when } y \in \Omega^c. \end{cases}$$

Proof. We have

$$u(y) - u_h(y) = -\frac{1}{4\pi}\int_\Gamma [q(x) - \tilde{q}_h(x)J(\psi^{-1}(x))]\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right)d\gamma_x$$
$$-\frac{1}{4\pi}\int_\Gamma \tilde{q}_h(x)J(\psi^{-1}(x))\left\{\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) - \frac{\partial}{\partial n_{hx}}\left(\frac{1}{|\psi^{-1}(x) - y|}\right)\right\}d\gamma_x.$$

Now, since

$$\int_\Gamma [q(x) - \tilde{q}_h(x)J(\psi^{-1}(x))]\,d\gamma_x = \int_\Gamma q\,d\gamma - \int_{\Gamma_h} q_h\,d\gamma_h = 0,$$

we have

$$\int_\Gamma [q(x) - \tilde{q}_h(x)J(\psi^{-1}(x))]\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right)d\gamma_x$$
$$= \int_\Gamma [q(x) - \tilde{q}_h(x)J(\psi^{-1}(x))]\left[\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) + \frac{4\pi}{\mathrm{mes}(\Gamma)}\right]d\gamma_x.$$

On the other hand, we have

$$\int_\Gamma \frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right)d\gamma_x = \begin{cases} -4\pi & \text{when } y \in \Omega, \\ 0 & \text{when } y \in \Omega^c, \end{cases}$$

so that

$$\left|\int_\Gamma [q(x) - \tilde{q}_h(x)J(\psi^{-1}(x))] \frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) d\gamma_x\right|$$

$$\leqslant \|q - \tilde{q}_h J(\psi^{-1})\|_{H^{-m}(\Gamma)/\mathbf{R}} \times \begin{cases} \left\| \dfrac{\partial}{\partial n_x}\left(\dfrac{1}{|x-y|}\right) + \dfrac{4\pi}{\operatorname{mes}(\Gamma)} \right\|_{H_0^m(\Gamma)} & \text{when } y \in \Omega, \\ \left\| \dfrac{\partial}{\partial n_x}\left(\dfrac{1}{|x-y|}\right)\right\|_{H_0^m(\Gamma)}, & \text{when } y \in \Omega^c. \end{cases}$$

Now,

$$\|q - \tilde{q}_h J(\psi^{-1})\|_{H^{-m}(\Gamma)/\mathbf{R}} \leqslant \|q - \tilde{q}_h\|_{H^{-m}(\Gamma)/\mathbf{R}} + \|\tilde{q}_h(1 - J(\psi^{-1}))\|_{H^{-m}(\Gamma)/\mathbf{R}},$$

and

$$\left\|\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right)\right\|_{H^m(\Gamma)} = \left\|\frac{(n_x, x-y)}{|x-y|^3}\right\|_{H^m(\Gamma)} \leqslant C \sum_{n=0}^{m} \frac{1}{d^{2+n}(y, \Gamma)},$$

so that we can proceed to the second term

$$\left|\int_\Gamma \tilde{q}_h(x)J(\psi^{-1}(x)) \left\{\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) - \frac{\partial}{\partial n_{hx}}\left(\frac{1}{|\psi^{-1}(x) - y|}\right)\right\} d\gamma_x\right|$$

$$\leqslant \|\tilde{q}_h J(\psi^{-1})\|_{L^2(\Gamma)} \left\|\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) - \frac{\partial}{\partial n_{hx}}\left(\frac{1}{|\psi^{-1}(x) - y|}\right)\right\|_{L^2(\Gamma)};$$

but since

$$\int_\Gamma \tilde{q}_h J(\psi^{-1}) d\gamma = \int_{\Gamma_h} q_h \, d\gamma_h = 0,$$

we have

$$\|\tilde{q}_h J(\psi^{-1})\|_{L^2(\Gamma)} = \|q_h\|_{L^2(\Gamma_h)/\mathbf{R}} \leqslant C\|q\|_{L^2(\Gamma)/\mathbf{R}}.$$

On the other hand,

$$\left\|\frac{\partial}{\partial n_x}\left(\frac{1}{|x-y|}\right) - \frac{\partial}{\partial n_{hx}}\left(\frac{1}{|\psi^{-1}(x) - y|}\right)\right\|_{L^2(\Gamma)}$$

$$= \left\|\frac{(n_x, x-y)}{|x-y|^3} - \frac{(n_{hx}, \psi^{-1}(x) - y)}{|\psi^{-1}(x) - y|^3}\right\|_{L^2(\Gamma)}$$

$$\leqslant \left\|\frac{(n_x, x-y) - (n_{hx}, \psi^{-1}(x) - y)}{|x-y|^3}\right\|_{L^2(\Gamma)}$$

$$+ \left\|(n_{hx}, \psi^{-1}(x) - y)\left\{\frac{1}{|x-y|^3} - \frac{1}{|\psi^{-1}(x) - y|^3}\right\}\right\|_{L^2(\Gamma)} \leqslant \frac{Ch^{k+1}}{d^3(y, \Gamma)}$$

and this ends the proof of Theorem II-3. □

The same type of estimate can be obtained for the derivatives. More specifically, we have

THEOREM II-4. *Under the hypotheses of Theorem* II-3, *we have the following estimate*

$$|\partial^\alpha u(y) - \partial^\alpha u_h(y)| \leqslant \frac{C}{e_\alpha(y, \Gamma)} \{h^{m+1/2}\|g - \hat{g}_h\|_{H_0^{-1/2}(\Gamma)} + \|g - \hat{g}_h\|_{H_0^{-m-1}(\Gamma)}$$

$$+ h^{2m+1}\|q\|_{H^{m+1}(\Gamma)/\mathbf{R}} + h^{k+1}\|q\|_{H^{1/2}(\Gamma)/\mathbf{R}}\},$$

where

$$\frac{1}{e_\alpha(y, \Gamma)} = \sum_{n=0}^{m} \frac{1}{d^{2+|\alpha|+n}(y, \Gamma)} + \begin{cases} 1, & \textit{when } y \in \Omega, \\ 0, & \textit{when } y \in \Omega^c. \end{cases}$$

Proof. The proof is the same as that of Theorem II-3. □

Remark. For the two-dimensional case, the same techniques give the same results. □

III. A Few Numerical Remarks.

1. *Condition Number for the Matrix* a_h. For h small enough, we have

$$\alpha\|q_h\|^2_{H^{1/2}(\Gamma_h)/\mathbf{R}} \leqslant a_h(q_h, q_h) \leqslant \beta\|q_h\|^2_{H^{1/2}(\Gamma_h)/\mathbf{R}}.$$

If we choose q_h such that $\int_{\Gamma_h} q_h\, d\gamma_h = 0$, we get, according to Lemma II-1,

$$\alpha\|q_h\|^2_{H^{1/2}(\Gamma_h)} \leqslant a_h(q_h, q_h) \leqslant \beta\|q_h\|^2_{H^{1/2}(\Gamma_h)}.$$

Now, since q_h belongs to V_h, we have [15]

$$\|q_h\|_{H^{1/2}(\Gamma_h)} \leqslant \frac{C}{\sqrt{h}}\|q_h\|_{L^2(\Gamma_h)},$$

so that

$$\alpha\|q_h\|^2_{L^2(\Gamma_h)} \leqslant a_h(q_h, q_h) \leqslant \frac{\beta}{h}\|q_h\|^2_{L^2(\Gamma_h)},$$

which shows that the eigenvalues λ of the matrix a_h are bounded above and below by $\alpha \leqslant \lambda \leqslant \beta/h$, so that the condition number of a_h is of $O(h^{-1})$ order.

2. *Computation of the Elements of* a_h *Near the Diagonal.* To compute these elements, we have to integrate a function with a singularity of order $1/|x - y|$. When k and m are not too high, we can use primitives. In that case, the following theorem is of some interest.

THEOREM III-1. *Let us assume that, when* $|x - y| \leqslant Ch$ *we define in the kernel of* a_h, *i.e. in*

$$\frac{(n_{hx}, n_{hy})}{|x - y|^3} - 3\,\frac{(x - y, n_{hx})(x - y, n_{hy})}{|x - y|^5}$$

n_{hx} *and* n_{hy} *as the normals to* Γ_h *in* x *and* y, *respectively* (*instead of using interpolates of order* k *of* n_x *and* n_y).

Then, the estimate of Theorem II-1 *remains valid.*

Proof. We only have to estimate

$$\frac{(n_x - n_{hx}, n_y)}{|x-y|^3} \quad \text{and} \quad \frac{(x-y, n_x - n_{hx})(x-y, n_y)}{|x-y|^5},$$

when $|x-y| \leqslant Ch$. Let us examine the first term. We have $n_y = n_x + O(x-y)$, so that

$$(n_x - n_{hx}, n_y) = (n_x - n_{hx}, n_x) + (n_x - n_{hx}, O(x-y)).$$

But

$$|(n_x - n_{hx}, O(x-y))| \leqslant Ch^k. \quad Ch = Ch^{k+1},$$

so that it remains to estimate $(n_x - n_{hx}, n_x)$.

Let (x_1, x_2) be the two components of x. We get

$$n_x = \frac{\partial(\psi \circ \varphi_{ih})/\partial x_1 \wedge \partial(\psi \circ \varphi_{ih})/\partial x_2}{|\partial(\psi \circ \varphi_{ih})/\partial x_1 \wedge \partial(\psi \circ \varphi_{ih})/\partial x_2|} = \frac{1}{J(\psi \circ \varphi_{ih})}\left(\frac{\partial(\psi \circ \varphi_{ih})}{\partial x_1} \wedge \frac{\partial(\psi \circ \varphi_{ih})}{\partial x_2}\right),$$

and

$$n_{hx} = \frac{1}{J(\varphi_{ih})}\left(\frac{\partial \varphi_{ih}}{\partial x_1} \wedge \frac{\partial \varphi_{ih}}{\partial x_2}\right).$$

According to Lemma II-4, the error on J is of order $k+1$. We have now to estimate terms like

$$R = \left(\frac{\partial(\psi \circ \varphi_{ih})}{\partial x_1} \wedge \left(\frac{\partial(\psi \circ \varphi_{ih})}{\partial x_2} - \frac{\partial \varphi_{ih}}{\partial x_2}\right), \frac{\partial(\psi \circ \varphi_{ih})}{\partial x_1} \wedge \frac{\partial(\psi \circ \varphi_{ih})}{\partial x_2}\right).$$

Such terms have been shown to be of order $k+1$ in [14, p. 67] by using the fact that ψ is the orthogonal projection onto Γ, so that we obtain

$$(n_x - n_{hx}, n_y) \leqslant Ch^{k+1}.$$

For the second term, it is much easier. We have

$$\frac{|(x-y, n_x - n_{hx})(x-y, n_y)|}{|x-y|^5} \leqslant \frac{Ch^k |x-y|\,|x-y|^2}{|x-y|^5} \leqslant \frac{Ch^{k+1}}{|x-y|^3},$$

since $(x-y, n_y) = O(|x-y|^2)$.

However, the involved primitives are difficult to compute, so that we are led to the use of numerical integration. The kernel being singular near the diagonal, we use extrapolation to the limit techniques [12] which give excellent results.

Finally, for more details on the numerical aspects and results of the method described in this paper, we refer to [5].

Conclusion. We have shown how to use a double layer potential to solve the Neumann problem without introducing Cauchy type integrals.

Thanks to the variational formulation thus obtained, we have been able to prove error estimates. We have seen that the error on the jump of the solution through Γ is optimal when $k = m$, whereas the error on the solution, far enough from Γ, is optimal when $k = 2m$. As for the condition number of the matrix, it is of order $O(h^{-1})$.

These results can be compared with those obtained by J. C. Nedelec [14] for the Dirichlet problem, by the use of a single layer potential. There, the error on the jump of $\partial u/\partial n$ through Γ was optimal for $k = m + 1$, and the condition number of the matrix was of the same order $O(h^{-1})$. However, the smallest eigenvalue of the matrix was only of order $O(h)$, so that the coercivity of a_h was very sensitive to numerical errors. This last fact appeared in the numerical experiments of M. Djaoua [3].

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