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Analysis of Some Difference Approximations for a Singular Perturbation Problem Without Turning Points

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Abstract. Some three point difference schemes are considered for a singular perturbation problem without turning points. Bounds for the discretization error are obtained which are uniformly valid for all h and $\epsilon > 0$. The degeneration of the order of the schemes at $\epsilon = 0$ is considered.

1. Introduction. We consider the two point boundary value problem

$$Ly \equiv -\epsilon y'' + py' + qy = f, \quad y(0) = \alpha, \quad y(1) = \beta, \tag{1.1}$$
$$p(x) > a > 0, \qquad q(x) \geqslant 0,$$

where $\epsilon > 0$ is a small parameter. It is well known that the solution $y(x, \epsilon)$ of this problem converges, as $\epsilon \longrightarrow 0$, and for $0 \leqslant x < 1$, to the solution $v(x)$ of the reduced problem

$$pv' + qv = f, \qquad v(0) = \alpha. \tag{1.2}$$

The loss of a boundary condition at $x = 1$ in the reduced problem results in a "boundary layer" in the solution y, for small ϵ. It is also well known [1, p. 300] that a reasonable difference approximation to (1) may give inaccurate results for small ϵ. In particular, the usual centered three point $O(h^2)$ difference approximation has this property. In this paper we analyse three difference operators, L_h^k, $k = 1, 2, 3$, on a uniform mesh of size h, for use in the approximate solution of (1.1). Each of the operators results in a tridiagonal, diagonally dominant matrix with negative off-diagonal entries. The operator L_h^1, which has been frequently proposed for such problems, uses a one sided difference approximation to the first derivative, and gives an $O(h)$ approximation to (1.1). The operators L_h^2 and L_h^3 give $O(h^2)$ approximations to (1.1). L_h^3 was proposed independently by Il'in [2] and by K. E. Barrett and others [3]. The operator L_h^2 was considered by Samarskii (see [9]). Each of the three approximate schemes behaves reasonably for ϵ small, and upon setting $\epsilon = 0$ in L_h^2 and L_h^3, an $O(h)$ approximation is obtained for the reduced problem (1.2).

Our purpose is to give bounds for the discretization error for the three schemes that are uniform in ϵ and h. Our error bounds contain a term that gives the effect of the boundary layer, and the bounds demonstrate that the boundary layer does not

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"pollute" the error away from $x = 1$. In the case of the second order schemes, our bounds contain a term of the form $h^2/(h + \epsilon)$, reflecting the loss of one order of accuracy in the error as $\epsilon \longrightarrow 0$. We also show that, among a certain class of difference schemes, this loss of an order of accuracy as $\epsilon \longrightarrow 0$ is unavoidable. Finally, we give some numerical results illustrating our bounds. To obtain our error bounds we utilize the positivity of the difference schemes and a comparison function that is designed to handle the effect of the boundary layer in the truncation error. This technique may be of use in other problems.

The literature on the numerical solution of singular perturbation problems is large. A useful discussion of a variety of problems is contained in Dorr [1]. Il'in [2] gives an $O(h)$ error bound for his scheme that is uniform in ϵ. We give a different proof of this result. Abrahamsson, Keller, and Kreiss [4] give an asymptotic expansion of the difference solution in h and ϵ. Some other methods for the numerical solution of singular perturbation problems are given, e.g., in [5].

In Section 2 we given some properties of solutions of (1.1), and in Section 3 we state the difference approximations that are being studied. The main results are contained in Section 4, and some numerical examples are presented in Section 5. Throughout the paper we let $c, c_1, \ldots$ denote positive constants that may take different values in different formulas, but that are always independent of h and ϵ. We assume that the parameter ϵ satisfies $0 < \epsilon \leqslant 1$. We assume that the functions $p(x)$, $q(x)$, and $f(x)$ are sufficiently differentiable for our purposes, but we shall not write out the assumptions in each instance.

2. Differentiability Properties of the Solution. To estimate the error in our difference approximations we shall require a bound for the derivatives of the solution of (1.1) that is valid for all $\epsilon \in (0, 1]$. To analyse the Il'in scheme we require more precise information on the behavior of the solution. These results are contained in Lemmas 2.3 and 2.4. To obtain the results, we require some information about the solutions of

$$Ly = g(x, \epsilon), \quad y(0) = \alpha, \quad y(1) = \beta, \tag{2.1}$$

where g satisfies

$$|g^{(i)}(x, \epsilon)| \leqslant K(1 + \epsilon^{-i-1}\exp(-a\epsilon^{-1}(1 - x))). \tag{2.2}$$

We will say that g is of class (K, j) if (2.2) holds for $0 \leqslant i \leqslant j$. Our first result is

LEMMA 2.1. *The problem* (2.1) *has a unique solution* y. *If* g *is of class* $(K, 0)$, *then* $|y(x)| \leqslant c$ *where* c *depends only on* α, β, *and* K.

Proof. The existence and uniqueness of a solution follows easily from the maximum principle [6]. A computation shows that

$$L(1 + x) \geqslant c_1, \qquad L(\exp(-a\epsilon^{-1}(1 - x))) \geqslant c_2\epsilon^{-1} \exp(-a\epsilon^{-1}(1 - x)).$$

Hence we may choose c_3 and c_4 so that

$$z(x) = c_3 e^{-x} + c_4 \exp(-a\epsilon^{-1}(1 - x))$$

satisfies $L(z \pm y) \geq 0$, $z(0) \geq |\alpha|$, $z(1) \geq |\beta|$. From the maximum principle, $|y(x)| \leq z(x) \leq c$.

LEMMA 2.2. *Let g be of class (K, j). Then the solution y of* (2.1) *satisfies* $|y^{(i)}(1)| \leq c\epsilon^{-i}$, $1 \leq i \leq j + 2$, *where $c > 0$ does not depend on ϵ.*

Proof. From (2.1), $-\epsilon y'' + py' = h$ where $h = g - qy$. Let $P(x)$ be an indefinite integral of p. Then we obtain

$$y(x) = y_p(x) + K_1 + K_2 \int_x^1 \exp(-\epsilon^{-1}(P(1) - P(t)))\,dt, \tag{2.3}$$

where

$$y_p(x) = -\int_x^1 z(t)\,dt, \qquad z(x) = \int_x^1 \epsilon^{-1}h(t)\exp(-\epsilon^{-1}(P(t) - P(x)))\,dt.$$

Using the inequality

$$\exp(-\epsilon^{-1}(P(t) - P(x))) \leq \exp(-a\epsilon^{-1}(t - x)), \qquad x \leq t, \tag{2.4}$$

and (2.2),

$$\begin{aligned}|z(x)| &\leq c\epsilon^{-1}\int_x^1 [\exp(-\epsilon^{-1}a(t - x)) + K\epsilon^{-1}\exp(-\epsilon^{-1}a(1 - x))]\,dt \\ &\leq c[1 + \epsilon^{-2}(1 - x)\exp(-\epsilon^{-1}a(1 - x))].\end{aligned}$$

Hence $|y_p(x)| \leq c$. The constants K_1 and K_2 must satisfy

$$K_1 + K_2\int_0^1 \exp\{-\epsilon^{-1}(P(1) - P(t))\}\,dt = \alpha - y_p(0), \qquad K_1 = \beta.$$

Since $p(x)$ is bounded on $(0, 1)$, $P(1) - P(t) \leq c(1 - t)$. Hence

$$\int_0^1 \exp\{(-\epsilon^{-1}(P(1) - P(t))\}\,dt \geq c\epsilon,$$

and we find that $K_2 \leq c\epsilon^{-1}$. Hence, $|y'(1)| = |K_2| \leq c\epsilon^{-1}$, so the inequality is proved with $i = 1$. If $i > 1$, the result is obtained by induction and repeated differentiations of (2.1).

LEMMA 2.3. *Let g be of class (K, j). Then the solution y of* (2.1) *satisfies*

$$|y^{(i)}(x)| \leq c\{1 + \epsilon^{-i}\exp(-a\epsilon^{-1}(1 - x))\}, \qquad 0 \leq i \leq j + 1, \tag{2.5}$$

where $c > 0$ does not depend on ϵ.

Proof. The proof is by induction. From Lemma 2.1, the inequality holds for $i = 0$. Differentiating both sides of (2.1) $i - 1$ times and setting $z = y^{(i)}$, we have $-\epsilon z' + pz = h$, where h depends on y, p, q, g, and their derivatives of order up to and including $i - 1$. Using (2.2) and the inductive hypothesis,

$$h(x) \leq c\{1 + \epsilon^{-i}\exp(-a\epsilon^{-1}(1 - x))\}. \tag{2.6}$$

Let P be an indefinite integral of p. Then

$$\begin{aligned}z(x) = {}& z(1)\exp(-\epsilon^{-1}(P(1) - P(x))) \\ &+ \epsilon^{-1}\int_x^1 h(t)\exp(-\epsilon^{-1}(P(t) - P(x)))\,dt.\end{aligned}$$

From (2.4), (2.6), and Lemma 2.2,

$$|z(x)| \leqslant c\epsilon^{-i} \exp(-a\epsilon^{-1}(1-x)) + c\epsilon^{-1} \int_x^1 \{\exp(-a\epsilon^{-1}(t-x)) + \epsilon^{-i} \exp(-a\epsilon^{-1}(1-x))\}\, dt,$$

and the desired inequality follows from this.

Remark 1. In particular, (2.5) holds when y is the solution of (1.1). This is used in the analysis of the difference schemes L_h^k, $k = 1, 2$.

LEMMA 2.4. *Let y satisfy* (1.1). *Then*

$$y(x) = \gamma \exp(-p(1)\epsilon^{-1}(1-x)) + z(x),$$

where $|\gamma| \leqslant c_1$, *and*

$$|z^{(i)}(x)| \leqslant c_2\{1 + \epsilon^{-i+1} \exp(-a\epsilon^{-1}(1-x))\}, \tag{2.7}$$

with $c_1 > 0$ *and* $c_2 > 0$ *independent of* ϵ.

Proof. Set $v(x) = \exp(-p(1)\epsilon^{-1}(1-x))$, and set $\gamma = \epsilon y'(1)/p(1)$. Then from Lemma 2.2, we see that $|\gamma| \leqslant c_1$ where $c_1 > 0$ does not depend on ϵ. Set $z(x) = y(x) - \gamma v(x)$. Then $z'(1) = 0$. Differentiating both sides of (2.3) and setting $x = 0$, we find that $|y'(0)| \leqslant c$. Calculating

$$Lz = f - qy + \gamma[p(1) - p(x)]v' + qz = g$$

and differentiating once, we get $Lz' = g' - p'z' - q'z$. Using (2.5), we see that y' is of class (K, j). Hence, $z' = y' - \gamma v'$ is of class (K, j), and we find that the function Lz' is of class (K, j). Using Lemma 2.3 with y replaced by z', we find that z satisfies (2.7), and the lemma is proved.

Remark 2. Lemma 2.4 is used in the analysis of the Il'in scheme, L_h^3.

3. The Difference Equations. Let (0, 1) be divided into N uniformly spaced mesh intervals, with mesh spacing $h = N^{-1}$ and with mesh points $x_i = ih$, $0 \leqslant i \leqslant N$. Using the usual notations for divided differences,

$$D_+u_i = (u_{i+1} - u_i)/h, \qquad D_-u_i = (u_i - u_{i-1})/h,$$

$$D_0u_i = (u_{i+1} - u_{i-1})/2h, \qquad D_+D_-u_i = (u_{i+1} - 2u_i + u_{i-1})/h^2,$$

we define our difference operators by

$$L_h^1u_i = -\epsilon D_+D_-u_i + p_iD_-u_i + q_iu_i,$$

$$L_h^2u_i = \frac{-\epsilon}{1 + hp_i/2\epsilon} D_+D_-u_i + p_iD_-u_i + q_iu_i,$$

$$L_h^3u_i = -\frac{1}{2} p_ih \left(\coth \frac{p_ih}{2\epsilon}\right) D_+D_-u_i + p_iD_0u_i + q_iu_i,$$

where $p_i = p(x_i)$, $q_i = q(x_i)$. We shall also write $f_i = f(x_i)$.

In this section we give some elementary facts concerning the positivity and the truncation errors of these operators.

LEMMA 3.1. *For $k = 1, 2, 3$, the system $L_h^ku_i = f_i$, $1 \leqslant i \leqslant N-1$, with u_0 and u_N specified, has a solution. If $L_h^ku_i \leqslant L_h^kv_i$, $1 \leqslant i \leqslant N-1$, and if $u_0 \leqslant v_0$, $u_N \leqslant v_N$, then $u_i \leqslant v_i$, $1 \leqslant i \leqslant N-1$.*

Proof. The equations $L_h^k u_i = f_i$, $1 \leqslant i \leqslant N-1$, may be regarded as a system of $N-1$ linear equations in the unknowns u_i, $1 \leqslant i \leqslant N-1$, where for $i=1$ and $i = N-1$, the terms involving u_0 and u_N have been moved to the right-hand side. It is easy to see that the matrix of coefficients is diagonally dominant and has nonpositive off diagonal entries. Hence, the matrix is an irreducible M matrix [7], and so has a positive inverse. Hence, the solution u_i, $1 \leqslant i \leqslant N-1$, exists and, if the v_i are as described in the lemma, $u_i \leqslant v_i$, $1 \leqslant i \leqslant N-1$.

The following lemma, whose proof is a computation, enables one to give a bound, that is uniform in ϵ and h, for the norm of the inverse of L_h^k.

LEMMA 3.2. *Let* $z_i = (1 + x_i)$, $0 \leqslant i \leqslant N$. *Then* $L_h^k z_i \leqslant c$, $k = 1, 2, 3$, *where* $c > 0$ *does not depend on* ϵ.

We now consider the truncation error associated with the operators L_h^k. If $y(x)$ is a smooth function, we define $\tau_i^k = L_h^k y_i - Ly(x_i)$. We require estimates for τ_i^k that are in integrable form.

LEMMA 3.3. *There is a constant* $c > 0$ *depending only on* $p(x)$ *such that*

$$|\tau_i^k| \leqslant c \int_{x_{i-1}}^{x_{i+1}} [\epsilon |y^{(3)}(t)| + |y^{(2)}(t)|]\, dt, \qquad k = 1, 3, \tag{3.1}$$

$$|\tau_i^2| \leqslant \frac{ch}{h+\epsilon} \int_{x_{i-1}}^{x_{i+1}} [\epsilon^2 |y^{(4)}(t)| + \epsilon |y^{(3)}(t)| + |y^{(2)}(t)|]\, dt, \tag{3.2}$$

$$|\tau_i^3| \leqslant \text{ch} \int_{x_{i-1}}^{x_{i+1}} [\epsilon |y^{(4)}(t)| + |y^{(3)}(t)| + (h+\epsilon)^{-1} |y^{(2)}(t)|\, dt. \tag{3.3}$$

Proof. By repeated use of the fundamental theorem of calculus, or by Peano's theorem [8, p. 70] we obtain the formulas

$$\begin{aligned} D_- y(x) - y^{(1)}(x) &= -h^{-1} \int_{x-h}^{x} (s - x + h) y^{(2)}(s)\, ds \\ &= -\frac{1}{2} h y^{(2)}(x) + \frac{1}{2} h^{-1} \int_{x-h}^{x} (s + h - x)^2 y^{(3)}(s)\, ds, \end{aligned} \tag{3.4}$$

$$\begin{aligned} D_0 y(x) - y^{(1)}(x) &= -\frac{1}{2} h^{-1} \int_{x-h}^{x} (s - x + h) y^{(2)}(s)\, ds \\ &\quad + \frac{1}{2} h^{-1} \int_{x}^{x+h} (x + h - s) y^{(2)}(s)\, ds \\ &= \frac{1}{4} h^{-1} \int_{x-h}^{x} (s + h - x)^2 y^{(3)}(s)\, ds \\ &\quad + \frac{1}{4} h^{-1} \int_{x}^{x+h} (x + h - s)^2 y^{(3)}(s)\, ds, \end{aligned} \tag{3.5}$$

$$\begin{aligned} D_+ D_- y(x) - y^2(x) &= \frac{-1}{2} h^{-2} \int_{x-h}^{x} (s + h - x)^2 y^{(3)}(s)\, ds \\ &\quad + \frac{1}{2} h^{-2} \int_{x}^{x+h} (x + h - s)^2 y^{(3)}(s)\, ds \\ &= \frac{1}{6} h^{-2} \int_{x-h}^{x} (s + h - x)^3 y^{(4)}(s)\, ds \\ &\quad + \frac{1}{6} h^{-2} \int_{x}^{x+h} (x + h - s)^3 y^{(4)}(s)\, ds. \end{aligned} \tag{3.6}$$

Let us designate $R_{m,n}$ to be a quantity which satisfies

$$|R_{m,n}| \leqslant ch^m \int_{x_{i-1}}^{x_{i+1}} |y^{(n)}(s)|\,ds.$$

Using (3.4) and (3.6) and the formula

$$L_h^1 y(x_i) - Ly(x_i) = -\epsilon[D_+D_-y(x_i) - y^{(2)}(x_i)] + p_i[D_-y(x_i) - y^{(1)}(x_i)], \tag{3.7}$$

we obtain

$$\tau_i^1 = \epsilon R_{0,3} + R_{0,2}.$$

This proves (3.1) with $k = 1$. Next, using (3.7) with the higher order error estimates of (3.4) and (3.6), we get

$$L_h^1 y(x_i) - Ly(x_i) = \epsilon R_{1,4} - \tfrac{1}{2} p_i h y^{(2)}(x_i) + R_{1,3}.$$

Using the differential equation to eliminate $y^{(2)}(x_i)$, and using (3.4) again, we obtain

$$y^{(2)}(x_i) = \epsilon^{-1}[p_i D_- y(x_i) + q_i y(x_i) - f_i] + \epsilon^{-1} R_{0,2},$$

so

$$\begin{aligned} -\epsilon D_+D_-y(x_i) + (1 + p_i h/2\epsilon)\{p_i D_-y(x_i) + q_i y(x_i) - Ly(x_i)\} \\ = \epsilon R_{1,4} + R_{1,3} + \epsilon^{-1} h R_{0,2}. \end{aligned}$$

Dividing both sides by $(1 + p_i h/2\epsilon)$, we see that the left side gives τ_i^2, and the right side gives the bound in (3.2). To analyse τ_i^3, we start with the formula

$$\begin{aligned} \tau_i^3 = -\epsilon\left[\frac{p_i h}{2\epsilon}\coth\frac{p_i h}{2\epsilon} - 1\right] D_+D_-y(x_i) - \epsilon[D_+D_-y(x_i) - y^{(2)}(x_i)] \\ + p_i[D_0 y(x_i) - y^{(1)}(x_i)]. \end{aligned} \tag{3.8}$$

Since $g(t) = t \coth t$ satisfies $g(0) = 1$, $g(t) = g(-t)$, we have $|g(t) - 1| \leqslant ct^2$ for $t \leqslant 1$. Since $\coth t \longrightarrow 1$ as $t \longrightarrow \infty$, $|g(t) - 1| \leqslant ct$ for $t \geqslant 1$. Hence

$$|t \coth t - 1| \leqslant ct^2/(1 + t), \qquad t \geqslant 0.$$

It is easily seen that

$$|D_+D_-y(x_i)| \leqslant h^{-1} \int_{x_{i-1}}^{x_{i+1}} |y^{(2)}(s)|\,ds.$$

Using these inequalities to estimate the first term of (3.8), and using (3.5) and (3.6) to estimate the two remaining terms, we easily obtain (3.1) with $k = 3$ and (3.3), completing the proof.

4. Error Bounds. In this section we derive error bounds for our difference schemes L_h^k, $k = 1, 2, 3$. We set $r_1 = 1 + ah\epsilon^{-1}$, $r_2 = r_1 + \tfrac{1}{2}a^2h^2\epsilon^{-2}$, $r_3 = \exp(ah\epsilon^{-1})$. We first require some inequalities.

LEMMA 4.1. (a) $(h^k/\epsilon^k) r_3^{-(N-i)} \leqslant c r_2^{-(N-i)} \leqslant c r_1^{-(N-i)}$, $0 \leqslant i < N$, *or*, $0 \leqslant i \leqslant N$ *if* $k = 0$, *where k is a nonnegative integer and c depends only on k;*

(b) $r_2^{-(N-i)} \leqslant r_1^{-(N-i)} \leqslant \exp\{-a(1 - x_i)/(ah + \epsilon)\}$;

(c) $r_2^{-(N-i)} \leqslant r_1^{-(N-i)} \leqslant c \exp\{-\epsilon^{-1}\bar{a}(1 - x_i)\}$, *where $h \leqslant \epsilon$, and $\bar{a} \in (0, a)$ is a constant depending only on a.*

Proof. (a) Since $e^t \geqslant 1 + t + t^2/2$ and $t^k(1 + t + t^2/2)e^{-t} \leqslant c$, for all $t \geqslant 0$ where c is some constant, these imply that $t \geqslant \ln(1 + t + t^2/2)$ and $k \ln t \leqslant t - \ln(1 + t + t^2/2) + \ln c$. It follows that

$$k \ln t \leqslant \frac{1 - x_i}{h}\left[t - \ln\left(1 + t + \frac{t^2}{2}\right)\right] + \ln c.$$

Let $t = ah/\epsilon$, then

$$\ln \frac{a^k h^k}{\epsilon^k} - \frac{1 - x_i}{h} \cdot \frac{ah}{\epsilon} \leqslant - \frac{1 - x_i}{h} \ln\left(1 + \frac{ah}{\epsilon} + \frac{a^2h^2}{2\epsilon^2}\right) + \ln c.$$

Taking the exponential of both sides, we get the first inequality. The second inequality is easy to prove.

(b) We only prove the second inequality. We have

$$B = \left(1 + \frac{ah}{\epsilon}\right)^{-(1-x_i)/h} = \left(1 - \frac{ah}{ah + \epsilon}\right)^{(1-x_i)/h},$$

$$\ln B = \frac{1 - x_i}{h} \ln\left(1 - \frac{ah}{ah + \epsilon}\right) < \frac{1 - x_i}{h}\left[-\frac{ah}{ah + \epsilon}\right] = -\frac{(1 - x_i)a}{ah + \epsilon}.$$

Taking the exponential of both sides, we get the results.

(c) To improve the upper bound in (b) when $h \leqslant \epsilon$, we start with the inequality

$$\exp(\bar{a}t) \leqslant 1 + at, \qquad 0 \leqslant t \leqslant 1, \tag{4.1}$$

where $\bar{a} \in (0, a)$ depends only on a. Setting $t = h/\epsilon$, we obtain $r_1^{-1} \leqslant \exp(-\bar{a}h\epsilon^{-1})$, and raising both sides to the power $N - i$, we get the result.

The next lemma will be used, with Lemmas 3.2 and 3.1, to convert bounds for the truncation error into bounds for the discretization error.

LEMMA 4.2. *There is a* $c > 0$ *depending only on* $p(x)$ *and* a *such that, for* $k = 1, 2, 3$,

$$L_h^k r_k^{-(N-i)} \geqslant \frac{c}{\max(h, \epsilon)} r_k^{-(N-i)}. \tag{4.2}$$

Proof. A computation shows that

$$L_h^1 r_1^i \geqslant h^{-1}(r_1 - 1)(p(x_i) - a)r_1^{i-1}.$$

If $h \geqslant \epsilon$, then there is a constant c_1 such that $r_1 - 1 \geqslant c_1 r_1$, so $L_h^1 r_1^i \geqslant c_2 h^{-1} r_1^i$, and we obtain (4.2) in this case. If $h \leqslant \epsilon$, since $r_1 - 1 = ah\epsilon^{-1}$ we have

$$L_h^1 r_1^i \geqslant c_3 \epsilon^{-1} r_1^{i-1} = c_3 r_1^i/(\epsilon + ah) \geqslant c_4 \epsilon^{-1} r_1^i,$$

and we obtain (4.2) in this case. A similar argument is used when $k = 2$. For $k = 3$, a computation gives

$$L_h^3 r_3^i \geqslant (p_i/2hr_3)(r_3 - 1)^2 A r_3^i,$$

where

$$A = \frac{r_3 + 1}{r_3 - 1} - \coth\frac{p_i h}{2\epsilon} = \coth\frac{ah}{2\epsilon} - \coth\frac{p_i h}{2\epsilon} = \sinh\frac{(p_i - a)h}{2\epsilon} \Big/ \sinh\frac{ah}{2\epsilon}\sinh\frac{p_i h}{2\epsilon}.$$

If $h \leqslant \epsilon$, since $c_1 t \leqslant \sinh t \leqslant c_2 t$ for $0 \leqslant t \leqslant c$, we have $A \geqslant c\epsilon/h$. Since, in this case,

$$(r_3 - 1)^2 r_3^{-1} = 4\sinh^2\frac{ah}{2\epsilon} \geqslant c\frac{h^2}{\epsilon^2},$$

we have

$$L_h^3 r_3^i \geqslant c r_3^i/\epsilon.$$

If $h \geqslant \epsilon$, since $c_1 e^t \leqslant \sinh t \leqslant c_2 e^t$ for $c \leqslant t < \infty$, we have $A \geqslant c e^{-ah/\epsilon} = c r_3^{-1}$. Since, in this case, $r_3 - 1 \geqslant c r_3$, we have

$$L_h^3 r_3^i \geqslant c r_3^i/\epsilon,$$

and the proof is complete.

Remark. The qualities r_k arise in the following way. If, in the definition of L_h^k, we set $q(x) \equiv 0$, $p(x) \equiv a$, then $L_h^k r_k^i = 0$, $k = 1, 2, 3$. We also note that r_k^{-1}, $k = 1, 2$, is the Padé approximation of type $(0, k)$ to $r_3^{-1} = \exp(-ah\epsilon^{-1})$.

We now prove the main theorems of the paper. Let $y(x)$ be the solution of (1.1), and let $y = \gamma v + z$ be the decomposition of Lemma 2.4, where we have set $v(x) = \exp(-p(1)\epsilon^{-1}(1 - x))$. We let y_{hi}^k be the solution of the system $L_h^k y_{hi}^k = Ly(x_i)$, $1 \leqslant i \leqslant N - 1$, $y_{h0}^k = y(0)$, $y_{hN}^k = y(1)$. We define the mesh functions v_{hi}^k and z_{hi}^k in a similar manner. Our first result is

THEOREM 4.1. *There is a $c > 0$, independent of h and ϵ such that*

$$|y(x_i) - y_{hi}^1| \leqslant ch[1 + \epsilon^{-1}\exp(-\bar{a}\epsilon^{-1}(1 - x_i))], \qquad h \leqslant \epsilon,$$

$$|y(x_i) - y_{hi}^1| \leqslant c[h + \exp(-a(1 - x_i)/(ah + \epsilon))], \qquad h \geqslant \epsilon,$$

where $\bar{a}$ is as in Lemma 4.1.

Proof. We first suppose that $h \leqslant \epsilon$. We obtain from Lemma 3.3 and 2.3,

$$|\tau_i^1| \leqslant c\left\{\epsilon^{-2}\int_{x_{i-1}}^{x_{i+1}} \exp(-\epsilon^{-1}a(1 - t))\,dt + h\right\}$$

$$\leqslant c\epsilon^{-1}\sinh(ah\epsilon^{-1})\exp(-\epsilon^{-1}a(1 - x_i)) + ch.$$

Since $\sinh t \leqslant ct$ for t bounded, we obtain, using Lemma 4.1(a)

$$|\tau_i^1| \leqslant ch\{\epsilon^{-2} r_3^{-(N-i)} + 1\} \leqslant ch\{\epsilon^{-2} r_1^{-(N-i)} + 1\}.$$

Since $L_h^1(y(x_i) - y_{hi}^1) = \tau_i^1$, we may use Lemmas 3.2, 4.2, and 3.1 to obtain

$$|y(x_i) - y_{hi}^1| \leqslant ch\{\epsilon^{-1} r_1^{-(N-i)} + 1\}.$$

We obtain the desired inequality from Lemma 4.1(c). To treat the case $h \geqslant \epsilon$, we use the decomposition $y = \gamma v + z$, $y_{hi}^1 = \gamma v_{hi}^1 + z_{hi}^1$. We have

$$|y(x_i) - y^1_{hi}| \leqslant c\{\,|v(x_i) - v^1_{hi}| + |z(x_i) - z^1_{hi}|\,\}. \tag{4.4}$$

To estimate the z term, we use Lemmas 3.3 and 2.4 to obtain

$$\begin{aligned} |L^1_h(z(x_i) - z^1_{hi})| &= |L^1_h z(x_i) - Lz(x_i)| \\ &\leqslant c\int_{x_{i-1}}^{x_{i+1}} \{\epsilon|z^{(3)}(t)| + |z^{(2)}(t)|\}\,dt \\ &\leqslant c\left\{\epsilon^{-1}\int_{x_{i-1}}^{x_{i+1}} \exp(-\epsilon^{-1}a(1-t))\,dt + h\right\} \\ &\leqslant c\sinh(ah\epsilon^{-1})\exp(-\epsilon^{-1}a(1-x_i)) + ch. \end{aligned}$$

Since $\sinh t \leqslant ce^t$ for $t \geqslant c_1$, we have, using Lemma 4.1(a),

$$|L^1_h(z(x_i) - z^1_{hi}))| \leqslant cr_3^{-(N-(i+1))} + ch \leqslant cr_1 r_1^{-(N-i)} + ch.$$

Hence, we obtain from Lemmas 3.2, 4.2, and 3.1,

$$|z(x_i) - z^1_{hi}| \leqslant chr_1^{-(N-(i+1))} + ch \leqslant ch.$$

It remains to bound the v term on the right side of (4.4) in the case $h \geqslant \epsilon$. From the definition of $v(x)$, $|Lv(x)| \leqslant c\epsilon^{-1}v(x)$. Since $v(x_i) \leqslant r_3^{-(N-i)}$, we have

$$|L^1_h v^1_{hi}| = |Lv(x_i)| \leqslant c\epsilon^{-1} r_3^{-(N-i)} \leqslant ch^{-1} r_1^{-(N-i)},$$

where we have used Lemma 4.1(a) with $k = 1$. Hence, from Lemmas 4.2 and 3.1,

$$|v^1_{hi}| \leqslant cr_1^{-(N-i)},$$

so

$$|v(x_i) - v^1_{hi}| \leqslant |v(x_i)| + |v^1_{hi}| \leqslant cr_1^{-(N-i)}, \qquad 1 \leqslant i \leqslant N-1.$$

The desired estimate then follows from Lemma 4.1(b), and the proof is complete.

We next have, for the operator L^2_h,

THEOREM 4.2. *There is a constant $c > 0$, independent of h and ϵ, such that*

$$|y(x_i) - y^2_{hi}| \leqslant \frac{ch^2}{h+\epsilon}\,[1 + \epsilon^{-1}\exp(-\bar{a}\epsilon^{-1}(1-x_i))], \qquad h \leqslant \epsilon,$$

$$|y(x_i) - y^2_{hi}| \leqslant c\left[\frac{h^2}{h+\epsilon} + \exp(-a(1-x_i)/(ah+\epsilon))\right], \qquad h \geqslant \epsilon,$$

where $\bar{a}$ is as in Lemma 4.1.

Proof. The proof is the same as that of Theorem 4.1, except that (3.2) is used instead of (3.1) to estimate the truncation error, and r_2 is used in place of r_1.

To analyse the Il'in scheme, we shall use the decomposition $y = \gamma v + z$ both when $h \leqslant \epsilon$ and when $h \geqslant \epsilon$. In the next lemma, we give a bound for $v - v^3_h$. Note that if $p(x)$ is a constant, then $v_h = v$ and the lemma is not needed.

LEMMA 4.3. $|v(x_i) - v_{hi}| \leqslant ch^2/(h+\epsilon)$, *where $c > 0$ is independent of ϵ.*

Proof. A computation gives

$$Lv(x) = -\epsilon^{-1}p(1)[p(1) - p(x)]\,v(x) + q(x)v(x),$$

$$L_h^3 v = -\frac{2p(x)\sinh(\tfrac12 p(1)h\epsilon^{-1})\sinh(\tfrac12 h\epsilon^{-1}[p(1) - p(x)])}{h\sinh(\tfrac12 p(x)h\epsilon^{-1})}v(x) + q(x)v(x).$$

We use the approximation $\sinh\xi = \xi + S$, where $|S| \leqslant c|\xi|^3(1 + \xi^2)^{-1}e^{|\xi|}$. We have

$$\tau(x) = Lv(x) - L_h^3 v(x)$$

$$= 2p(x)\frac{[\tfrac12 p(1)h\epsilon^{-1} + S_1][\tfrac12 h\epsilon^{-1}(p(1) - p(x)) + S_2]}{\tfrac12 h^2 p(x)\epsilon^{-1} + hS_3}v(x)$$

$$-\epsilon^{-1}p(1)[p(1) - p(x)]\,v(x),$$

$$\begin{aligned}\tau(x) = \{&p(x)h\epsilon^{-1}[p(1) - p(x)]S_1 \\ &+ p(x)p(1)h\epsilon^{-1}S_2 + 2p(x)S_1S_2 \\ &-h\epsilon^{-1}p(1)[p(1) - p(x)]S_3\}v(x)/h\sinh(\tfrac12 p(x)h\epsilon^{-1}),\end{aligned} \tag{4.5}$$

where

$$|S_1| \leqslant \frac{ch^3}{\epsilon(h^2 + \epsilon^2)}\exp(\tfrac12 p(1)h\epsilon^{-1}),$$

$$|S_2| \leqslant \frac{ch^3(1 - x)}{\epsilon(h + \epsilon)^2}\exp(c(1 - x)h\epsilon^{-1}),$$

$$|S_3| \leqslant \frac{ch^3}{\epsilon(h^2 + \epsilon^2)}\exp(\tfrac12 p(x)h\epsilon^{-1}),$$

and where we have used the inequality $|p(1) - p(x)| \leqslant c(1 - x)$. Using the inequality $\sinh\xi \geqslant c\xi(1 + \xi)^{-1}e^{\xi}$, $\xi > 0$, we see that the denominator in (4.5) is bounded from below by $ch^2(h + \epsilon)^{-1}\exp(\tfrac12 h\epsilon^{-1}p(x))$. The numerator in (4.5) consists of four terms. We bound each of these terms as follows:

$$\begin{aligned}|p(x)h\epsilon^{-1}[p(1) - p(x)]S_1| &\leqslant ch\epsilon^{-1}(1 - x)\cdot h^3\epsilon^{-1}(h^2 + \epsilon^2)^{-1}\exp(\tfrac12 p(1)h\epsilon^{-1}) \\ &\leqslant ch^4\epsilon^{-2}(1 - x)(h + \epsilon)^{-2}\exp(\tfrac12 p(1)h\epsilon^{-1});\end{aligned}$$

$$|p(x)p(1)h\epsilon^{-1}S_2| \leqslant ch^4\epsilon^{-2}(1 - x)(h + \epsilon)^{-2}\exp(c(1 - x)h\epsilon^{-1});$$

$$|2p(x)S_1S_2| \leqslant ch^4\epsilon^{-2}(1 - x)(h + \epsilon)^{-2}\exp[\tfrac12 p(1)h\epsilon^{-1} + c(1 - x)h\epsilon^{-1}];$$

$$|h\epsilon^{-1}p(1)[p(1) - p(x)]S_3| \leqslant ch^4\epsilon^{-2}(1 - x)(h^2 + \epsilon^2)^{-1}\exp(\tfrac12 p(x)h\epsilon^{-1}).$$

Using these inequalities in (4.5), we obtain

$$|\tau(x)| \leqslant \frac{ch^2(1 - x)}{\epsilon^2(h + \epsilon)}\exp(c(1 - x)h\epsilon^{-1})v(x).$$

Let $b = \tfrac12(p(1) - a)$, so $b > 0$. We may find a constant $c_1 > 0$ so that when $h \leqslant$

c_1, $p(1) - a - ch \geqslant b$. Then we have, for $h \leqslant c_1$,

$$(4.6)\qquad \begin{aligned} |\tau(x)| &\leqslant \frac{ch^2(1-x)}{\epsilon^2(h+\epsilon)} \exp(-a\epsilon^{-1}(1-x)) \exp(-b\epsilon^{-1}(1-x)), \\ |\tau(x)| &\leqslant \frac{ch^2}{\epsilon(h+\epsilon)} \exp(-a\epsilon^{-1}(1-x)). \end{aligned}$$

At the mesh point x_i, (4.6) yields

$$|\tau(x_i)| = |L_h^3(v_i - v(x_i))| \leqslant \frac{ch^2}{\epsilon(h+\epsilon)} r_3^{-(N-i)}.$$

We now use Lemma 4.2 and Lemma 3.1 to obtain a bound for $v_i - v(x_i)$. If $h \leqslant \epsilon$, $h \leqslant c_1$, we have

$$|v(x_i) - v_i| \leqslant \frac{ch^2}{h+\epsilon} r_3^{-(N-i)} \leqslant \frac{ch^2}{h+\epsilon}.$$

If $h \geqslant \epsilon$, $h \leqslant c_1$, we have

$$|v(x_i) - v_i| \leqslant \frac{ch^2}{h+\epsilon} \cdot h\epsilon^{-1} r_3^{-(N-i)}.$$

Since $i \leqslant N-1$, $1 - x_i \geqslant h$, and

$$h\epsilon^{-1} r_3^{-(N-i)} = h\epsilon^{-1}\exp(-a\epsilon^{-1}(1-x)) \leqslant h\epsilon^{-1} \exp(-ah\epsilon^{-1}) \leqslant c,$$

so the inequality is obtained in this case. There remains the case $h \geqslant c_1$. For this, it suffices to show that v and v_h are bounded for all $h \geqslant c_1$, $\epsilon \leqslant 1$. This is true by inspection for v. To bound v_h, we note that $L_h^3 v_h = Lv$ is bounded for all h and ϵ. Hence, from Lemmas 3.1 and 3.2, v_h is bounded for all h and ϵ. This completes the proof of the lemma.

We shall give two error bounds for the Il'in scheme. Our first bound was also given in [2].

THEOREM 4.3. *There is a constant $c > 0$, independent of h and ϵ, such that $|y(x_i) - y_{hi}^3| \leqslant ch$.*

Proof. Using the decomposition of Lemma 2.4, let $\tau_i = L_h^3(z - z_h) = L_h^3 z - Lz$. Using (3.1) with $k = 3$, and (2.7),

$$\begin{aligned} |\tau_i| &\leqslant c\int_{x_{i-1}}^{x_{i+1}} [\epsilon|z^{(3)}(t)| + |z^{(2)}(t)|]\,dt \\ &\leqslant ch + c\epsilon^{-1}\int_{x_{i-1}}^{x_{i+1}} \exp(-a\epsilon^{-1}(1-t))dt \\ &\leqslant ch + c \sinh ah\epsilon^{-1} \cdot \exp(-a\epsilon^{-1}(1-x_i)) \\ &= ch + c \sinh ah\epsilon^{-1} \cdot r_3^{-(N-i)}. \end{aligned}$$

Using Lemmas 3.2, 4.2, and 3.1, we obtain

$$|z(x_i) - z_{hi}| \leqslant ch + c \max(h, \epsilon) \sinh ah\epsilon^{-1} \cdot \exp(-a\epsilon^{-1}(1 - x_i))$$
$$\leqslant ch + c \max(h, \epsilon)[1 - \exp(-2ah\epsilon^{-1})].$$

For $h \leqslant \epsilon$, using the inequality $1 - e^{-t} \leqslant ct$, $t > 0$, we get $|z(x_i) - z_{hi}| \leqslant ch$. For $h \geqslant \epsilon$, we also obtain this inequality. Hence, using Lemma 4.3 and the triangle inequality, we obtain the result.

THEOREM 4.4. *There is a constant $c > 0$, independent of h and ϵ, such that*

$$|y(x_i) - y_{hi}^3| \leqslant \frac{ch^2}{h + \epsilon} + \frac{ch^2}{\epsilon} \exp(-a\epsilon^{-1}(1 - x_i)).$$

Proof. Again setting $\tau_i = L_h^3(z - z_h)$, we have from (3.3) and (2.7),

$$|\tau_i| \leqslant c \int_{x_{i-1}}^{x_{i+1}} \left[h\epsilon|z^{(4)}| + h|z^{(3)}| + \frac{h}{h + \epsilon} |z^{(2)}| \right] dt$$
$$\leqslant \frac{ch^2}{h + \epsilon} + \frac{ch}{h + \epsilon} \sinh ah\epsilon^{-1} \cdot \exp(-a\epsilon^{-1}(1 - x_i))$$
$$+ ch\epsilon^{-1} \sinh ah\epsilon^{-1} \exp(-a\epsilon^{-1}(1 - x_i))$$
$$\leqslant ch^2(h + \epsilon)^{-1} + ch\epsilon^{-1} \sinh ah\epsilon^{-1} \cdot r_3^{-(N-i)}.$$

Using Lemmas 3.2, 4.2, and 3.1, we obtain

$$(4.7) \quad |z(x_i) - z_{hi}| \leqslant \frac{ch^2}{h + \epsilon} + ch\epsilon^{-1} \max(h, \epsilon) \sinh ah\epsilon^{-1} \cdot \exp(-a\epsilon^{-1}(1 - x_i)).$$

For $h \leqslant \epsilon$, we use the inequality $\sinh at \leqslant ct$, $t \leqslant a$, to get

$$|z(x_i) - z_{hi}| \leqslant \frac{ch^2}{h + \epsilon} + ch^2\epsilon^{-1} \exp(-a\epsilon^{-1}(1 - x_i)).$$

Using the triangle inequality and Lemma 4.3, we obtain the bound for the error in this case. For $h \geqslant \epsilon$, $h \leqslant 2h^2(h + \epsilon)^{-1}$, and the result follows from Theorem 4.3.

The difference operators L_h^2 and L_h^3 have, for $\epsilon > 0$, a truncation error that is $O(h^2)$, whereas the reduced difference operators, obtained by letting $\epsilon \longrightarrow 0$, have a truncation error that is $O(h)$. We shall now show that this loss of an order of accuracy near $\epsilon = 0$ holds for all tridiagonal difference operators of positive type. For this, it suffices to consider the case of constant coefficients, $p(x) \equiv p > 0$, $q(x) \equiv q \geqslant 0$. We consider the difference operator

$$(L_h u)_i = r(h, \epsilon)u_{i+1} + s(h, \epsilon)u_i + t(h, \epsilon)u_{i-1}.$$

We suppose that r, s, and t are continuous functions of (h, ϵ) for $h > 0$, $\epsilon \leqslant 1$. We say that the difference operator is of positive type if $s(h, \epsilon) > 0$, $r(h, \epsilon) < 0$, $t(h, \epsilon) < 0$. If $y(x)$ is a smooth function, we denote the truncation error by

$$\tau(x, h, \epsilon) = r(h, \epsilon)y(x + h) + s(h, \epsilon)y(x) + t(h, \epsilon)y(x - h)$$
$$+ \epsilon y^{(2)}(x) - py^{(1)}(x) - qy(x).$$

With these notations we have

THEOREM 4.5. *Suppose that, for any smooth function $y(x)$, the truncation error satisfies $|\tau(x, h, \epsilon)| \leqslant ch^\delta$ where $\delta > 1$ and where c does not depend on x, h, or ϵ, then the difference operator is not of positive type for all ϵ and h sufficiently small.*

Proof. Letting y be a quadratic polynomial, calculating $\tau(x)$, and comparing coefficients, we obtain

$$r + s + t = q + \mu(x, h, \epsilon),$$

$$h(r - t) = p + \lambda(x, h, \epsilon),$$

$$\tfrac{1}{2}h^2(r + t) = -\epsilon + \eta(x, h, \epsilon),$$

where $|\eta|$, $|\lambda|$, $|\mu| \leqslant ch^\delta$, uniformly in (x, h, ϵ). Solving this system of equations, we obtain

$$r(h, \epsilon) = -\frac{\epsilon}{h^2} + \frac{p}{2h} + \frac{\eta(x, h, \epsilon)}{h^2} + \frac{\lambda(x, h, \epsilon)}{2h}.$$

Then $r(h, 0) = p/2h + O(h^{-1+\delta})$, so for h sufficiently small, $r(h, 0) > 0$. Hence, for each $h > 0$ sufficiently small, there is an $\epsilon > 0$ sufficiently small, such that for this h and ϵ, the difference approximation is not of positive type.

5. **Numerical Results.** We give some numerical results for our difference schemes, as applied to the problem $-\epsilon y'' + y' = 1$ with boundary conditions $y(0) = 0$, $y(1) = 0$. In addition to using the difference operators L_h^1 and L_h^2 we have used the centered difference operator

$$L_h^0 u_i = -D_+D_-u_i + D_0u_i, \qquad D_0u_i = (u_{i+1} - u_{i-1})/2h.$$

This difference operator gives an $O(h^2)$ approximation to the differential equation, but is known to give poor results for small ϵ [1]. The three figures give results of computations using $h = 0.02$ and for three different values of ϵ. In Figures 1 and 2 we have plotted the errors in the approximate solutions. Denoting the errors in using L_h^k by e^k, e^0, e^1, and e^2 are represented respectively by the solid line, the dashed line, and the long dashed line. For $\epsilon = 0.1$, Figure 1 shows that L_h^2 produces a solution that is almost as accurate as that produced by L_h^0, while the first order scheme gives a much larger error. For $\epsilon = 0.01 < h$, Figure 2 shows that $e^2(x)$ is smaller than $e^0(x)$, and $e^1(x)$ is the largest error. This indicates that L_h^2 gives the most accurate solution. In Figure 3 we present results for $\epsilon = 0.001$. $e^2(x)$ is very close to zero in the entire interval, and $e^1(x)$ is close to zero except near $x = 1$. The oscillating solid line is the centered difference solution $u^0(x)$ which we have shown superimposed on the true solution, $y(x)$. This figure indicates that the use of L_h^2 gives a very good approximation, while the centered scheme is worthless. For this problem, the scheme L_h^3 gives the exact answer. In conclusion, the difference operators L_h^2 and L_h^3 provide accurate solutions to a singular perturbation problem without turning points over a wide range of values of h and ϵ.

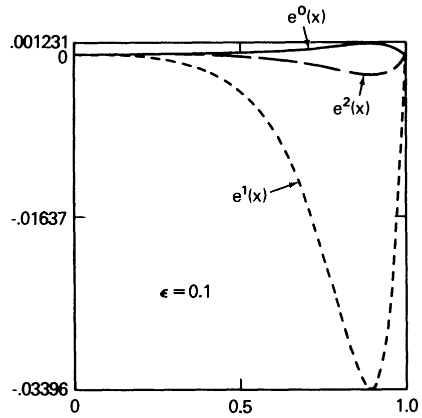

FIGURE 1

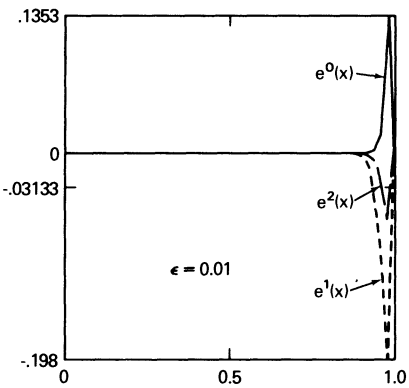

FIGURE 2

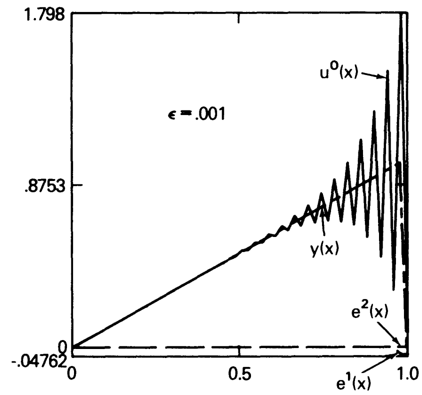

FIGURE 3

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