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Scheme-Independent Stability Criteria for Difference Approximations of Hyperbolic Initial-Boundary Value Problems. I

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Abstract. Easily checkable sufficient stability criteria are obtained for explicit dissipative approximations to mixed initial-boundary value problems associated with the system $u_t = Au_x$ in the quarter plane $x \geqslant 0$, $t \geqslant 0$. The criteria are given entirely in terms of the boundary conditions for the outflow unknowns. The results imply that certain well-known boundary conditions, when used in combination with any (stable) dissipative scheme, always maintain stability.

0. Introduction. This paper provides a stability study of a wide class of difference approximations for a hyperbolic mixed initial-boundary value problem in the quarter plane $x \geqslant 0$, $t \geqslant 0$. The approximated differential system is $u_t = Au_x$ where A is diagonal, and the inflow and outflow unknowns interact at the boundary. For the difference approximation we consider general dissipative explicit two-level schemes, with general boundary conditions which determine the boundary values in terms of outflow values at interior points. We show that such boundary conditions may have arbitrary degree of accuracy. This is included in Section 1.

In Section 2 we begin to discuss stability. We show that the *entire approximation is stable if and only if the scalar components of its outflow part are stable*; thus reducing the stability question to that of a scalar outflow problem. From that point on, our aim will be to obtain *easily checkable* sufficient stability criteria for the reduced problem. All our results are *scheme-independent* and are given exclusively in terms of the outflow boundary conditions. The only such result that we know of, is due to Kreiss ([3]; see also [1]) who proved that for dissipative schemes, boundary extrapolation always maintains stability.

In the remainder of Section 2 we state our main result and consider several examples. The main result is for the scalar outflow case where the boundary conditions are *translatory*, i.e., determined at all boundary points by the same procedure. The result states that *if the boundary conditions are generated by a solvable stable scheme, then the entire approximation is stable, independently of the interior scheme.* The examples considered show that *if the outflow boundary conditions are generated by oblique extrapolation, by explicit or implicit Euler schemes, or by the Box-scheme, then overall stability is assured.*

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In Section 3 we derive a preliminary scheme-independent result, which is an explicit interpretation of Kreiss' determinantal stability criteria [4]. This result, which seems to be of independent interest, is used in Section 4 to derive our stability criteria for the case of translatory boundary conditions.

The theoretical basis for our work was given by Kreiss [4] and by Gustafsson, Kreiss and Sundström [2]. We assume that the reader is familiar with these papers.

In a forthcoming paper we discuss the extension of our results to *nondissipative* and *multi-level* schemes.

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1. The Difference Approximation. Consider a first order hyperbolic system of partial differential equations

$$\partial u/\partial t = A\partial u/\partial x, \qquad x \geqslant 0,\ t \geqslant 0, \tag{1.1a}$$

where $u = (u^{(1)}(x, t), \ldots, u^{(n)}(x, t))'$ is the transposed vector of unknowns, and A is a constant $n \times n$ Hermitian matrix of the form

$$A = \begin{pmatrix} A^{\mathrm{I}} & 0 \\ 0 & A^{\mathrm{II}} \end{pmatrix}, \qquad A^{\mathrm{I}} < 0,\ A^{\mathrm{II}} > 0.$$

Without restriction we may assume that A is *diagonal.*

The solution of (1.1a) is uniquely determined if initial values

$$u(x, 0) = f(x), \qquad x \geqslant 0, \tag{1.1b}$$

and boundary conditions

$$u^{\mathrm{I}}(0, t) = Su^{\mathrm{II}}(0, t) + g(t), \qquad t \geqslant 0, \tag{1.1c}$$

are prescribed. Here the partition

$$u^{\mathrm{I}} = (u^{(1)}, \ldots, u^{(l)})', \quad u^{\mathrm{II}} = (u^{(l+1)}, \ldots, u^{(n)})'$$

corresponds to that of A, and S is an $l \times (n - l)$ rectangular matrix.

To solve the initial-boundary value problem (1.1) by a difference approximation we introduce a mesh-size $h = \Delta x > 0$, $k = \Delta t > 0$, such that $\lambda = k/h =$ constant. Using the standard notation $v_\nu(t) = v(\nu h, t)$, we approximate (1.1a) by a consistent two-sided difference scheme of the form

$$v_\nu(t + k) = Qv_\nu(t), \qquad \nu = r, r + 1, \ldots,$$

$$Q = \sum_{j=-r}^{p} A_j E^j, \qquad Ev_\nu = v_{\nu+1}, \tag{1.2a}$$

with initial values

$$v_\nu(0) = f_\nu, \qquad \nu = 0, 1, 2, \ldots. \tag{1.2b}$$

The A_j are fixed *diagonal* $n \times n$ matrices depending on A and on λ such that A_{-r}, A_p are nonsingular.

Throughout the paper we assume that scheme (1.2a) is *dissipative*, i.e., for some $\delta > 0$ and positive integer ω, the eigenvalues $\theta(\xi)$ of the amplification matrix

$$\hat{Q}(\xi) = \sum_{j=-r}^{p} A_j e^{ij\xi}, \qquad -\pi < \xi \leqslant \pi,$$

satisfy

$$|\theta(\xi)| \leqslant 1 - \delta|\xi|^{2\omega}.$$

Since the A_j are diagonal, dissipativity guarantees that the scheme (1.2a) is *stable.*

In order to uniquely determine the solution of (1.2), we must specify, at each time step, r boundary values $v_\mu(t)$, $\mu = 0, 1, \ldots, r-1$. For the outflow unknowns we do it by boundary conditions of the form

$$\sum_{j=0}^{s} C_{\mu j}^{(1)} v_{\mu+j}^{\mathrm{II}}(t+k) = \sum_{j=0}^{s} C_{\mu j}^{(0)} v_{\mu+j}^{\mathrm{II}}(t), \qquad \mu = 0, \ldots, r-1, \tag{1.3}$$

$C_{\mu j}$ being fixed diagonal $(n-l) \times (n-l)$ matrices. For the inflow part we use the physical boundary condition

$$v_0^{\mathrm{I}}(t) = S v_0^{\mathrm{II}}(t) + g(t) \tag{1.4a}$$

together with $r-1$ additional conditions of the form

$$v_\mu^{\mathrm{I}}(t) = \sum_{j=0}^{q} D_{\mu j} v_j^{\mathrm{II}}(t) + g_\mu(t), \qquad \mu = 1, \ldots, r-1, \tag{1.4b}$$

where $D_{\mu j}$ are fixed $l \times (n-l)$ matrices and the $g_\mu(t)$ are vectors depending on h and on $g(t)$.

It is well known that using conditions of the general form (1.3), one can achieve at the boundary arbitrary degree of accuracy. We note that this is true also for conditions of type (1.4b). In fact, if accuracy of order d is desired, then using the differential system and (1.1c), we find that a Taylor expansion of a smooth solution of (1.1) yields

$$\begin{aligned}
u_\mu^{\mathrm{I}}(t) &= \sum_{j=0}^{d} \frac{(\mu h)^j}{j!} \frac{\partial^j}{\partial x^j} u^{\mathrm{I}}(0, t) + O(h^{d+1}) \\
&= \sum_{j=0}^{d} \frac{(\mu h)^j}{j!} (A^{\mathrm{I}})^{-j} \frac{\partial^j}{\partial t^j} u^{\mathrm{I}}(0, t) + O(h^{d+1}) \\
&= \sum_{j=0}^{d} \frac{(\mu h)^j}{j!} (A^{\mathrm{I}})^{-j} \left[S \frac{\partial^j}{\partial t^j} u^{\mathrm{II}}(0, t) + \frac{d^j}{dt^j} g(t) \right] + O(h^{d+1}) \\
&= \sum_{j=0}^{d} \frac{(\mu h)^j}{j!} (A^{\mathrm{I}})^{-j} \left[S (A^{\mathrm{II}})^j \frac{\partial^j}{\partial x^j} u^{\mathrm{II}}(0, t) + \frac{d^j}{dt^j} g(t) \right] + O(h^{d+1}).
\end{aligned}$$

Thus, (1.4b) follows upon approximating $\partial^j / \partial x^j u^{\mathrm{II}}(0, t)$ by linear combinations of $u_0^{\mathrm{II}}(t), \ldots, u_q^{\mathrm{II}}(t)$ of the right accuracy.

For example, if $v_1^{\mathrm{I}}(t)$ is required to second order of accuracy, we may use

$$\frac{\partial u^{\mathrm{II}}(0, t)}{\partial x} \sim \frac{-3u_0^{\mathrm{II}}(t) + 4u_1^{\mathrm{II}}(t) - u_2^{\mathrm{II}}(t)}{2h},$$

$$\frac{\partial^2 u^{\mathrm{II}}(0,t)}{\partial x^2} \sim \frac{u_0^{\mathrm{II}}(t) - 2u_1^{\mathrm{II}}(t) + u_2^{\mathrm{II}}(t)}{2h^2},$$

to obtain a second order accurate boundary condition of the form

$$v_1^{\mathrm{I}}(t) = D_0 v_0^{\mathrm{II}}(t) + D_1 v_1^{\mathrm{II}}(t) + D_2 v_2^{\mathrm{II}}(t) + g_1(t),$$

where

$$D_0 = M_0 - 3M_1/2 + \tfrac{1}{4}M_2, \quad D_1 = 2M_1 - \tfrac{1}{2}M_2, \quad D_2 = -\tfrac{1}{2}M_1 + \tfrac{1}{4}M_2,$$

$$M_j = (A^{\mathrm{I}})^{-j} S (A^{\mathrm{II}})^j, \qquad j = 0, 1, 2,$$

$$g_1(t) = g(t) + h(A^{\mathrm{I}})^{-1} \frac{dg}{dt}(t) + \frac{h^2}{2}(A^{\mathrm{I}})^{-2} \frac{d^2 g}{dt}.$$

2. Statement of Main Result and Examples. The difference approximation is completely defined now by the dissipative (stable) scheme (1.2) together with the boundary conditions (1.3) (1.4), and we raise the question of *overall stability* in the sense of Gustafsson et al. [2, Definition 3.3].

Since the A_j are diagonal, we can split the scheme (1.2a) into its inflow and outflow parts,

$$v_\nu^{\mathrm{I}}(t+k) = \sum_{j=-r}^{p} A_j^{\mathrm{I}} v_{\nu+j}^{\mathrm{I}}(t), \qquad \nu = r, r+1, \ldots, \tag{2.1}$$

and

$$v_\nu^{\mathrm{II}}(t+k) = \sum_{j=-r}^{p} A_j^{\mathrm{II}} v_{\nu+j}^{\mathrm{II}}(t), \qquad \nu = r, r+1, \ldots, \tag{2.2}$$

where

$$A_j = \begin{pmatrix} A_j^{\mathrm{I}} & 0 \\ 0 & A_j^{\mathrm{II}} \end{pmatrix}, \qquad -r \leqslant j \leqslant p,$$

correspond to the partition of A. We immediately see that the outflow problem (2.2) (1.3) is self contained, while the inflow problem (2.1) (1.4) depends on the outflow part to the extent that the outflow computations provide the inhomogeneous boundary values in (1.4). Thus, stability of the entire approximation is equivalent to the following two separate questions:

(a) Stability of the inflow problem (2.1) with inhomogeneous boundary values.

(b) Stability of the outflow problem (2.2) (1.3).

Since the stability definition 3.3 of [2] gives bounds for inhomogeneous boundary values, it suffices, for the inflow problem, to consider homogeneous boundary values. But then, since the A_j^{I} are diagonal, the problem splits into l independent dissipative approximations with homogeneous boundary values, which were shown by Kreiss [3, Theorem 3] to be stable independently of the basic scheme. Thus the inflow problem is stable, and it remains to consider (b). Since the A_j^{II} of (2.2) and the $C_{\mu j}$ of (1.3) are diagonal, the outflow problem in (b) splits as well and we have,

THEOREM 2.1. *The entire approximation* (1.2)–(1.4) *is stable if and only if the* $n - l$ *scalar components of its outflow part are stable.*

So, from now on, we may restrict the stability discussion to the following scalar

case where we approximate an initial-value problem of the form

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x}, \qquad a > 0,\ x \geqslant 0,\ t \geqslant 0,\ u(x, 0) = f(x), \tag{2.3}$$

by a consistent dissipative scheme

$$v_\nu(t + k) = Qv_\nu(t), \qquad \nu = r, r + 1, \ldots,$$

$$Q = \sum_{j=-r}^{p} a_j E^j, \qquad Ev_\nu = v_{\nu+1}, \tag{2.4a}$$

together with initial values

$$v_\nu(0) = f_\nu, \qquad \nu = 0, 1, 2, \ldots, \tag{2.4b}$$

and boundary conditions

$$\sum_{j=0}^{s} c_{\mu j}^{(1)} v_{\mu+j}(t + k) = \sum_{j=0}^{s} c_{\mu j}^{(0)} v_{\mu+j}(t), \qquad \mu = 0, \ldots, r - 1. \tag{2.5}$$

Here the a_j and the $c_{\mu j}$ are constants depending on a and on λ.

Our purpose is to provide easily checkable sufficient stability criteria for approximation (2.4)(2.5) which depend entirely on the boundary conditions (2.5). Particularly effective criteria of that nature will be obtained when the boundary conditions are *translatory*, i.e., of the form

$$\sum_{j=0}^{s} c_j^{(1)} v_{\mu+j}(t + k) = \sum_{j=0}^{s} c_j^{(0)} v_{\mu+j}(t), \qquad \mu = 0, \ldots, r - 1, \tag{2.6}$$

where the c_j are independent of μ. In this case we introduce the *boundary-scheme*

$$T^{(1)}v_\nu(t + k) = T^{(0)}v_\nu(t), \qquad \nu = 0, \pm 1, \pm 2, \ldots,$$

$$T^{(\alpha)} \equiv T^{(\alpha)}(E) = \sum_{j=0}^{s} c_j^{(\alpha)} E^j, \qquad \alpha = 0, 1, \qquad Ev_\nu = v_{\nu+1}, \tag{2.7}$$

which generates (2.6) upon restriction of ν to the values $0, 1, \ldots, r - 1$.

We call the boundary scheme *stable* if it is stable when applied to all grid points νh, $-\infty < \nu < \infty$. We state

THEOREM 2.2 (THE MAIN THEOREM). *Approximation* (2.4)(2.6) *is stable if the boundary-scheme* (2.7) *is stable and if*

$$T^{(1)}(\kappa) \equiv \sum_{j=0}^{s} c_j^{(1)} \kappa^j \neq 0 \quad \forall\, |\kappa| \leqslant 1. \tag{2.8}$$

The stability criterion in this theorem is *independent* of the basic scheme. The proof, as well as other scheme-independent results for the translatory case, are given in Section 4.

Often, the boundary-scheme is known in advance to be stable. Thus, in applying Theorem 2.2, it only remains to verify the solvability-condition (2.8).

If the boundary conditions are explicit, i.e., of the form

$$v_\mu(t+k) = \sum_{j=0}^{s} c_j v_{\mu+j}(t), \qquad \mu = 0, \ldots, r-1, \tag{2.9}$$

then $T^{(1)}(\kappa) \equiv 1$. Hence (2.8) is automatically fulfilled and Theorem 2.2 reduces to

COROLLARY 2.3. *If the boundary conditions* (2.9) *are generated by a stable scheme, then approximation* (2.4)(2.9) *is stable.*

Example 1. Determine the boundary conditions by oblique extrapolation:

$$v_\mu(t+k) = v_{\mu+1}(t), \qquad \mu = 0, \ldots, r-1. \tag{2.10}$$

Clearly, the generating boundary-scheme is explicit and unconditionally stable; so by Corollary 2.3 the stability of (2.4)(2.10) is assured.

Note that (2.10) is inconsistent with the differential equation (2.3), unless $\lambda a = 1$, in which case (2.10) coincides with our next example.

Example 2. Let the boundary conditions be generated by the right-sided explicit Euler scheme, i.e.,

$$v_\mu(t+k) = v_\mu(t) + \lambda a[v_{\mu+1}(t) - v_\mu(t)], \qquad \mu = 0, \ldots, r-1. \tag{2.11}$$

Since the basic scheme (2.4a) must satisfy the Courant-Friedrichs-Levi condition $\lambda a \leqslant 1$, Euler's scheme is stable too, and by Corollary 2.3 the stability of (2.4)(2.11) follows.

Example 3. Take

$$\begin{aligned} v_\mu(t+k) - \lambda a[v_{\mu+1}(t+k) - v_\mu(t+k)] &= v_\mu(t), \\ \mu &= 0, \ldots, r-1, \end{aligned} \tag{2.12}$$

which is generated by the right-sided unconditionally stable implicit Euler scheme. To comply with Theorem 2.2, we must have

$$T^{(1)}(\kappa) \equiv 1 - \lambda a(\kappa - 1) \neq 0 \quad \forall\, |\kappa| \leqslant 1.$$

Since $\lambda a > 0$, then κ with $|\kappa| \leqslant 1$ gives

$$\operatorname{Re} T^{(1)}(\kappa) = 1 + \lambda a(1 - \operatorname{Re} \kappa) \geqslant 1,$$

so (2.4)(2.12) is stable.

Example 4. We use the unconditionally stable Box-scheme to generate

$$\begin{aligned} &v_\mu(t+k) + v_{\mu+1}(t+k) - \lambda a[v_{\mu+1}(t+k) - v_\mu(t+k)] \\ &\quad = v_\mu(t) + v_{\mu+1}(t) + \lambda a[v_{\mu+1}(t) - v_\mu(t)], \qquad \mu = 0, \ldots, r-1. \end{aligned} \tag{2.13}$$

Since

$$\operatorname{Re} T^{(1)}(\kappa) = 1 + \operatorname{Re} \kappa + \lambda a(1 - \operatorname{Re} \kappa) > 0 \qquad \forall\, |\kappa| \leqslant 1,$$

then by Theorem 2.1 again, (2.4)(2.13) is stable.

In view of Theorem 2.1, Examples 1–4 imply that *if the boundary conditions* (1.3) *are generated by oblique extrapolation, explicit or implicit right-sided Euler schemes, or by the Box-scheme, then the entire approximation* (1.2)–(1.4) *is stable.*

The boundary conditions in Examples 1–4 were studied by Gustafsson et al. [2] and by Sköllermo [5] in combination with *specific 3-point basic schemes.*

3. A Preliminary Determinantal Criterion. In order to investigate the stability of (2.4)(2.5) we introduce the space $l_2(h)$ of all grid functions

$$w = \{w_\nu\}_{\nu=0}^{\infty} \quad \text{with } \|w\|^2 \equiv h \sum_{\nu=0}^{\infty} |w_\nu|^2 < \infty.$$

We write the approximation in operator form

$$v(t + k) = Gv(t),$$

where G: $l_2(h) \longrightarrow l_2(h)$ is uniquely determined by (2.4a) and (2.5). Following Kreiss [4, Main Theorem], we will show that G has no eigenvalues z with $|z| \geqslant 1$. This will suffice to assure stability both in the sense of [4] and in the sense of Definition 3.3 of [2].

To check the eigenvalues of G we must adopt Kreiss' recipe in [4]: If z with $|z| \geqslant 1$ is an eigenvalue, then for some nontrivial $g \in l_2(h)$, $Gg = zg$. Thus, by (2.4a) and (2.5), g must satisfy the *resolvent equation*

$$zg_\nu = \sum_{\nu=-r}^{p} a_j g_{\nu+j}, \qquad \nu = r, r+1, \ldots, \tag{3.1}$$

and the boundary relations

$$z \sum_{j=0}^{s} c_{\mu j}^{(1)} g_{\mu+j} = \sum_{j=0}^{s} c_{\mu j}^{(0)} g_{\mu+j}, \qquad \mu = 0, \ldots, r-1. \tag{3.2}$$

The most general solution of (3.1) in $l_2(h)$ can be written as

$$g_\nu = \sum_{\alpha=1}^{k} \sum_{\beta=0}^{m_\alpha - 1} \sigma_{\alpha\beta} P_{\alpha\beta}(\nu) \kappa_\alpha^\nu, \qquad \nu \geqslant 0, \tag{3.3}$$

where $\kappa_\alpha = \kappa_\alpha(z)$, $1 \leqslant \alpha \leqslant k$, are the distinct roots of the *characteristic equation*

$$\sum_{j=-r}^{p} a_j \kappa^j - z = 0, \tag{3.4}$$

which satisfy $0 < |\kappa_\alpha| < 1$, each with multiplicity $m_\alpha = m_\alpha(z)$; $P_{\alpha\beta}(\nu)$ are arbitrary polynomials in ν with $\deg[P_{\alpha\beta}(\nu)] = \beta$; and $\sigma_{\alpha\beta}$ are free parameters yet to be determined. Since (2.4a) is dissipative and consistent with (2.3), we use Lemma 2 and part of the proof of Lemma 7 of [4] to find that for z with $|z| \geqslant 1$, Eq. (3.4) has precisely r roots κ with $0 < |\kappa| < 1$. Thus, g_ν of (3.3) depends on r parameters. Substituting (3.3) in (3.2), we obtain a linear homogeneous system of r equations with the r unknowns $\sigma_{\alpha\beta}$. The system may take the form $J\sigma' = 0$ where $J = J(z)$ is an $r \times r$ matrix and σ is the unknown vector. This yields

LEMMA 3.1 [4, LEMMA 3]. *z with $|z| \geqslant 1$ is an eigenvalue of G if and only if* $\det J(z) \neq 0$.

Going through the above process, we make a particular choice of the polynomials $P_{\alpha\beta}(\nu)$ that leads to an explicit expression of J which later proves useful. We choose

$$P_{\alpha\beta}(\nu) = \kappa_\alpha^{-\beta} \beta! \binom{\nu}{\beta}. \tag{3.5}$$

Inserting (3.5) in (3.3) and then in (3.2), we obtain

$$(3.6)\qquad \sum_{\alpha=1}^{k}\sum_{\beta=0}^{m_\alpha-1}\sum_{j=0}^{s}[zc_{\mu j}^{(1)}-c_{\mu j}^{(0)}]\,\beta!\binom{\mu+j}{\beta}\kappa_\alpha^{\mu+j-\beta}\,\sigma_{\alpha\beta}=0,\qquad \mu=0,\ldots,r-1,$$

which constitutes the system $J\sigma'=0$.

At this point we associate with the boundary conditions (2.5) a set of polynomial *boundary-functions*

$$(3.7)\qquad b_\mu(z,\kappa)=\sum_{j=0}^{s}[zc_{\mu j}^{(1)}-c_{\mu j}^{(0)}]\,\kappa^{\mu+j},\qquad \mu=0,\ldots,r-1.$$

Since

$$\frac{\partial^\beta b_\mu(z,\kappa)}{\partial\kappa^\beta}=\sum_{j=0}^{s}[zc_{\mu j}^{(1)}-c_{\mu j}^{(0)}]\,\beta!\binom{\mu+j}{\beta}\kappa^{\mu+j-\beta},$$

then system (3.6) may be written as

$$(3.8)\qquad \sum_{\alpha=1}^{k}\sum_{\beta=0}^{m_\alpha-1}\frac{\partial^\beta b_\mu(z,\kappa_\alpha)}{\partial\kappa_\alpha^\beta}\sigma_{\alpha\beta}=0,\qquad \mu=0,\ldots,r-1.$$

Thus, the coefficient matrix J takes the form

$$(3.9a)\qquad J=[B(z,\kappa_1,m_1),\ldots,B(z,\kappa_k,m_k)],$$

where $B(z,\kappa_\alpha,m_\alpha)$, $1\leqslant\alpha\leqslant k$, are $r\times m_\alpha$ blocks given by

$$(3.9b)\qquad B(z,\kappa_\alpha,m_\alpha)=\left[\begin{bmatrix}b_0(z,\kappa)\\ b_1(z,\kappa)\\ \vdots\\ b_{r-1}(z,\kappa)\end{bmatrix},\frac{\partial}{\partial\kappa}\begin{bmatrix}b_0(z,\kappa)\\ b_1(z,\kappa)\\ \vdots\\ b_{r-1}(z,\kappa)\end{bmatrix},\ldots,\frac{\partial^{m_\alpha-1}}{\partial\kappa^{m_\alpha-1}}\begin{bmatrix}b_0(z,\kappa)\\ b_1(z,\kappa)\\ \vdots\\ b_{r-1}(z,\kappa)\end{bmatrix}\right]_{\kappa=\kappa_\alpha}$$

Defining a *partition* of r to be any set of positive integers $\{m_i\}_{i=1}^N$ which satisfies $m_1+\cdots+m_N=r$, we state

THEOREM 3.2. *Approximation* (2.4)(2.5) *is stable if for every* z *with* $|z|\geqslant 1$, *every partition* $\{m_i\}_{i=1}^N$ *of* r, *and every set of distinct values* $\{\kappa_i\}_{i=1}^N$ *with* $0<|\kappa_i|<1$,

$$(3.10)\qquad \det[B(z,\kappa_1,m_1),\ldots,B(z,\kappa_N,m_N)]\neq 0.$$

Proof. Take an arbitrary z_0 with $|z_0|\geqslant 1$ and let $\kappa_\alpha(z_0)$, $1\leqslant\alpha\leqslant k$, be the distinct roots of (3.4), each with multiplicity $m_\alpha(z_0)$. Since $\{m_\alpha(z_0)\}_{\alpha=1}^k$ is a partition of r, then (3.10) holds for our z_0, $\kappa_\alpha(z_0)$ and $m_\alpha(z_0)$. Thus, by Lemma 3.1, z_0 is not an eigenvalue of G, and by Kreiss' Main Theorem in [4], stability follows.

Since the determinant in (3.10) depends entirely on the boundary-functions (3.7), Theorem 3.2 is scheme-independent. Thus, in applying the theorem, one avoids the inherent difficulty of solving the characteristic equation (3.4).

4. Translatory Boundary Conditions. In this section we return to consider approximation (2.4)(2.6) where the boundary conditions are translatory.

The boundary-functions associated with (2.6) are

$$b_\mu(z, \kappa) = \sum_{j=0}^{s} (zc_j^{(1)} - c_j^{(0)})\kappa^{\mu+j}, \qquad \mu = 0, \ldots, r-1.$$

Thus, writing $b(z, \kappa) \equiv b_0(z, \kappa)$, we have

$$b_\mu(z, \kappa) = \kappa^\mu b(z, \kappa), \qquad \nu = 0, \ldots, r-1.$$

Consequently, if $\{m_i\}_{i=1}^N$ is a partition of r, then the $r \times r$ matrix

$$[B(z, \kappa_1, m_1), \ldots, B(z, \kappa_N, m_N)] \tag{4.1}$$

in (3.10) is given by the $r \times m_i$ blocks

$$B(z, \kappa_i, m_i) = \left[\begin{bmatrix} b(z,\kappa) \\ \kappa\, b(z,\kappa) \\ \vdots \\ \kappa^{r-1} b(z,\kappa) \end{bmatrix}, \frac{\partial}{\partial\kappa}\begin{bmatrix} b(z,\kappa) \\ \kappa\, b(z,\kappa) \\ \vdots \\ \kappa^{r-1} b(z,\kappa) \end{bmatrix}, \cdots, \frac{\partial^{m_i-1}}{\partial\kappa^{m_i-1}}\begin{bmatrix} b(z,\kappa) \\ \kappa\, b(z,\kappa) \\ \vdots \\ \kappa^{r-1} b(z,\kappa) \end{bmatrix}\right]_{\kappa=\kappa_i}$$

The fact that (4.1) is determined now by the single boundary-function $b(z, \kappa)$, implies the following significant simplification of Theorem 3.2.

THEOREM 4.1. *Approximation* (2.4)(2.6) *is stable if for every* z *with* $|z| \geqslant 1$ *and* κ *with* $0 < |\kappa| < 1$, *we have*

$$b(z, \kappa) \equiv \sum_{j=0}^{s} (zc_j^{(1)} - c_j^{(0)})\kappa^j \neq 0. \tag{4.2}$$

Proof. Take an arbitrary z with $|z| \geqslant 1$, a partition $\{m_i\}_{i=1}^N$ of r, and distinct values κ_i, $1 \leqslant i \leqslant N$, with $0 < |\kappa_i| < 1$. In order to prove stability, it suffices, by Theorem 3.2, to verify (3.10). For this purpose, let

$$\sum_{\mu=0}^{r-1} \gamma_\mu \begin{bmatrix} \kappa_1^\mu b(z, \kappa_1) \\ \vdots \\ \partial^{m_N-1}[\kappa_N^\mu b(z, \kappa_N)]/\partial\kappa_N^{m_N-1} \end{bmatrix}' = 0 \tag{4.3}$$

be a vanishing linear combination of the rows of (4.1). The vector relation in (4.3) consists of r scalar equations

$$\left\{\sum_{\mu=0}^{r-1} \gamma_\mu \frac{\partial^j}{\partial\kappa^j}[\kappa^\mu b(z, \kappa)]\right\}_{\kappa=\kappa_i} = 0, \qquad 1 \leqslant i \leqslant N,\ 0 \leqslant j \leqslant m_i - 1,$$

which we write as

$$\frac{\partial^j}{\partial\kappa^j}\left\{b(z, \kappa)\left[\sum_{\mu=0}^{r-1} \gamma_\mu \kappa^\mu\right]\right\}_{\kappa=\kappa_i} = 0, \qquad 1 \leqslant i \leqslant N,\ 0 \leqslant j \leqslant m_i - 1. \tag{4.4}$$

Since $0 < |\kappa_i| < 1$, then by hypothesis,

$$[b(z, \kappa)]_{\kappa=\kappa_i} \neq 0, \qquad 1 \leqslant i \leqslant N.$$

Thus, expanding (4.4) by Leibniz' rule and using induction on $j \geqslant 0$, we find that the sum in (4.4) has vanishing derivatives at $\kappa = \kappa_i$, i.e.,

$$\frac{d^j}{d\kappa^j}\left[\sum_{\mu=0}^{r-1} \gamma_\mu \kappa^\mu\right]_{\kappa=\kappa_i} = 0, \qquad 1 \leqslant i \leqslant N,\ 0 \leqslant j \leqslant m_i - 1.$$

We conclude that the polynomial

$$P(\kappa) \equiv \sum_{\mu=0}^{r-1} \gamma_\mu \kappa^\mu,$$

which is of degree $r-1$ at most, has r roots; κ_i, $1 \leqslant i \leqslant N$, each with multiplicity m_i. Hence, $P(\kappa) \equiv 0$ and the coefficients γ_μ must vanish. By (4.3), therefore, the rows of (4.1) are linearly independent, so (3.10) holds, and stability follows.

Before turning to the proof of the Main Theorem (Theorem 2.2) we note that since Theorem 4.1 applies also to single-leveled boundary conditions, we can immediately obtain the following result of Kreiss [3] (see also [1]): Let the boundary conditions (2.6) be determined by extrapolation of arbitrary degree $s-1$, i.e.,

$$\sum_{j=0}^{s} \binom{s}{j} (-1)^j v_{\mu+j}(t) = 0, \qquad \mu = 0, \ldots, r-1. \tag{4.5}$$

The associated boundary-function is

$$b(\kappa) = -\sum_{j=0}^{s} \binom{s}{j} (-1)^j \kappa^j = -(1-\kappa)^s.$$

Thus, $b(\kappa) \neq 0$ for $0 < |\kappa| < 1$, and by Theorem 4.1, (2.4)(4.5) is stable.

Proof of Theorem 2.2. The amplification factor $\hat{T}(\xi)$ of the boundary-scheme (2.7) is given by

$$\hat{T}(\xi) = \hat{T}^{(0)}(\xi)/\hat{T}^{(1)}(\xi), \qquad \hat{T}^{(\alpha)}(\xi) = T^{(\alpha)}(e^{i\xi}), \quad \alpha = 0, 1.$$

By (2.8) we have $\hat{T}^{(1)}(\xi) \neq 0$; thus, $\hat{T}(\xi)$ is well defined. Since the boundary-scheme is stable, then $|\hat{T}(\xi)| \leqslant 1$, hence

$$|\hat{T}^{(0)}(\xi)| \leqslant |\hat{T}^{(1)}(\xi)| \neq 0 \quad \forall\, \xi. \tag{4.6}$$

The boundary-function associated with (2.6) satisfies

$$b(z, \kappa) = \sum_{j=0}^{s} (z c_j^{(1)} - c_j^{(0)}) \kappa^j = z T^{(1)}(\kappa) - T^{(0)}(\kappa).$$

So, for $|z| > 1$, we use (4.6) to find that

$$|b(z, e^{i\xi})| = |z T^{(1)}(e^{i\xi}) - T^{(0)}(e^{i\xi})| = |z\hat{T}^{(1)}(\xi) - \hat{T}^{(0)}(\xi)|$$
$$\geqslant |z| \cdot |\hat{T}^{(1)}(\xi)| - |\hat{T}^{(0)}(\xi)| > 0.$$

That is, the equation

$$b(z, \kappa) = 0 \quad \text{with } |z| > 1, \tag{4.7}$$

has no roots κ with $|\kappa| = 1$. Since the roots κ of (4.7) are continuous functions of z, we conclude that the number of κ with $|\kappa| < 1$ is fixed for $|z| > 1$, and can be determined by considering large values of $|z|$. Writing (4.7) in the form

$$T^{(1)}(\kappa) - z^{-1}T^{(0)}(\kappa) = 0,$$

we let $|z| \longrightarrow \infty$ and use (2.8) to find that (4.7) has no roots in the unit disc. In other words, if $|z| > 1$ and $b(z, \kappa) = 0$, then κ must satisfy $|\kappa| > 1$. By continuity, therefore, if $|z| \geqslant 1$ and $b(z, \kappa) = 0$, then $|\kappa| \geqslant 1$; i.e.,

$$b(z, \kappa) \neq 0, \quad \forall\, |z| \geqslant 1, |\kappa| < 1.$$

This implies (4.2), and Theorem 4.1 completes the proof.

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