MATHEMATICS OF COMPUTATION, VOLUME 32, NUMBER 144
OCTOBER 1978, PAGES 1271–1273

A Sum of Binomial Coefficients

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Abstract. An explicit expression is derived for the sum of the $(k + 1)$st binomial coefficients in the nth, $(n - m)$th, $(n - 2m)$th, . . . row of the arithmetic triangle.

In combinatorial analysis and in probability theory we occasionally encounter the problem of calculating the sum

$$(1) \qquad Q(n, k, m) = \sum_{0 \leqslant j \leqslant n/m} \binom{n - jm}{k}$$

for $n = 0, 1, 2, \ldots$ where k and m are given positive integers. If n is large, the summation in (1) is time-consuming and it is desirable to derive some simple formulas for $Q(n, k, m)$ which make it possible to determine $Q(n, k, m)$ for any n in an easy way. For $m = 1$ and $m = 2$ such formulas are

$$(2) \qquad Q(n, k, 1) = \binom{n + 1}{k + 1}$$

and

$$(3) \qquad Q(n, k, 2) = \sum_{j=0}^{k} \binom{n + 2}{k + 1 - j} \frac{(-1)^j}{2^{j+1}} - \left[\frac{1 - (-1)^n}{2}\right] \frac{(-1)^k}{2^{k+1}}$$

Our aim is to derive analogous expressions for any m.

We shall prove that if $n \equiv r \pmod{m}$ where $0 \leqslant r < m$, then $Q(n, k, m)$ is a polynomial of degree $k + 1$ in the variable n. In this polynomial every term is independent of r except the constant term which does depend on r.

More specifically, we have the following result.

THEOREM. *If* $n \equiv r \pmod{m}$ *where* $0 \leqslant r < m$, *then*

$$(4) \qquad Q(n, k, m) = P(n + m, k, m) - P(r, k, m)$$

for $n \geqslant 0$, $k \geqslant 1$, $m \geqslant 1$ *where*

$$(5) \qquad P(x, k, m) = \frac{1}{m} \sum_{j=1}^{k+1} \binom{x}{j} A(m, k + 1 - j)$$

and $A(m, j)$ $(j = 0, 1, \ldots, k + 1)$ *are determined by the generating function*

$$(6) \qquad \frac{mx}{(1 + x)^m - 1} = \sum_{j=0}^{\infty} A(m, j) x^j,$$

Received September 1, 1977.
AMS (MOS) subject classifications (1970). Primary 39A05.

which is convergent if $|x| < 2\sin(\pi/m)$. *In another form we have*

$$P(x, k, m) = \sum_{j=0}^{k} \binom{x/m}{j+1} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \binom{mi}{k}. \tag{7}$$

Note. We define $\binom{x}{0} = 1$ for any x and $\binom{x}{j} = x(x-1)\ldots(x-j+1)/j!$ for any x and $j = 1, 2, \ldots$.

Proof. We observe that $Q(n, k, m)$ is the coefficient of x^{k+1} in the polynomial

$$[(1+x)^r + (1+x)^{r+m} + \cdots + (1+x)^n]x$$

$$= \left[\frac{(1+x)^{n+m} - (1+x)^r}{m}\right]\left[\frac{mx}{(1+x)^m - 1}\right]. \tag{8}$$

Consequently, (4) is true if $P(x, k, m)$ is defined by (5). It remains to determine $A(m, j)$ for $j = 0, 1, 2, \ldots$. By expanding (6) into partial fractions, we get

$$\frac{mx}{(1+x)^m - 1} = 1 + \sum_{r=1}^{m-1} \frac{\epsilon_r x}{x + 1 - \epsilon_r}, \tag{9}$$

where $\epsilon_r = e^{2r\pi i/m}$ for $1 \leqslant r \leqslant m-1$. Therefore, $A(m, 0) = 1$ and

$$A(m, j) = (-1)^{j-1} \sum_{r=1}^{m-1} \frac{\epsilon_r}{(1-\epsilon_r)^j} = -\sum_{r=1}^{m-1} \frac{\cos((2rj + mj - 4r)\pi/2m)}{(2\sin(r\pi/m))^j} \tag{10}$$

for $j = 1, 2, \ldots$. Formula (10) is an explicit expression for $A(m, j)$; however, it is more convenient to determine $A(m, j)$ for $j = 1, 2, \ldots$ by the recurrence formula

$$\sum_{i=1}^{j+1} \binom{m}{i} A(m, j+1-i) = 0, \tag{11}$$

starting from the initial condition $A(m, 0) = 1$. To prove (11) we multiply both sides of (6) by $(1+x)^m - 1$ and form the coefficient of x^{j+1}.

For $m \leqslant 12$ and $j \leqslant 10$ the following table contains the numbers $A(m, j)\Pi_{p|m} p^{[j/(p-1)]}$ where $p = 2, 3, 5, 7, \ldots$ are prime numbers. A Texas SR 52 calculator was programmed to obtain the entries of this table.

j	0	1	2	3	4	5	6	7	8	9	10
$A(1, j)$	1	0	0	0	0	0	0	0	0	0	0
$A(2, j)2^j$	1	−1	1	−1	1	−1	1	−1	1	−1	1
$A(3, j)3^{[j/2]}$	1	−1	2	−1	1	0	−1	1	−2	1	−1
$A(4, j)2^j$	1	−3	5	−5	1	7	−15	15	1	−33	65
$A(5, j)5^{[j/4]}$	1	−2	2	−1	−1	4	−3	0	11	−11	3
$A(6, j)2^j3^{[j/2]}$	1	−5	35	−35	−119	567	−1765	−3355	41041	−41041	−249613
$A(7, j)7^{[j/6]}$	1	−3	4	−2	−2	4	−8	−29	39	0	−52
$A(8, j)2^j$	1	−7	21	−21	−63	231	−15	−1521	3073	4319	−29631
$A(9, j)3^{[j/2]}$	1	−4	20	−10	−62	108	80	−755	1699	3160	−20332
$A(10, j)2^j5^{[j/4]}$	1	−9	33	−33	−891	3003	3333	−37125	188441	1568743	−5091303
$A(11, j)11^{[j/10]}$	1	−5	10	−5	−17	28	25	−110	29	317	−4467
$A(12, j)2^j3^{[j/2]}$	1	−11	143	−143	−3575	11583	87659	−673387	41041	29982095	−180388429

Now we are going to prove (7). Let us denote the right-hand side of (7) by $R(x, k, m)$. By Newton expansion we obtain

$$\binom{mx + r}{k} = \sum_{j=0}^{k} \binom{x}{j} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \binom{mi + r}{k} \tag{12}$$

for any x. If we add (12) for $x = 0, 1, \ldots, s$, we get

$$Q(ms + r, k, m) = \sum_{j=0}^{k} \binom{s + 1}{j + 1} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \binom{mi + r}{k} \tag{13}$$

for $0 \leqslant r < m$. If, in particular, $n \equiv 0 \pmod{m}$, that is, $r = 0$, then by (13) we have $Q(n, k, m) = R(n + m, k, m)$. We have demonstrated that $Q(n, k, m)$ is a polynomial in the variable n and only the constant term depends on r. Accordingly, for any $n \equiv r \pmod{m}$, $Q(n, k, m)$ differs from $R(n + m, k, m)$ by a constant. If we put first $r = 0$ and then $x = r/m$ in (12), then we obtain that $Q(r, k, m) = R(r + m, k, m) - R(r, k, m)$. This implies that if $n \equiv r \pmod{m}$ and $0 \leqslant r < m$, then

$$Q(n, k, m) = R(n + m, k, m) - R(r, k, m), \tag{14}$$

where $R(x, k, m)$ is the right-hand side of (7). Since both $P(0, k, m)$ and $R(0, k, m)$ are 0, therefore $R(x, k, m) = P(x, k, m)$ for all x. This completes the proof of the theorem.

We note that formulas (4) and (7) are more advantageous than (13) because in (7) the second sum does not depend on r. If we use (13), the second sum should be calculated for every $r = 0, 1, \ldots, m - 1$. Actually, in (7) the second sum is $\Delta^j\binom{mx}{k}$ taken at $x = 0$. This can be determined from the sequence $\{\binom{mx}{k}, x = 0, 1, 2, \ldots\}$ by forming repeated differences.

By using (7) we can derive another formula for $A(m, j)$. By (5) we have

$$A(m, j) = mP(1, j, m), \tag{15}$$

where the right-hand side is given by (7).

We remark also that from (4) and (7) it follows that

$$\lim_{n\to\infty} Q(n, m, k)/n^{k+1} = 1/(k + 1)!m. \tag{16}$$

Finally, I would like to thank the referee for calling my attention to the paper of L. Carlitz [1]. In this paper Carlitz introduced the polynomials $\beta_m(\lambda)$ defined by

$$\frac{x}{(1 + \lambda x)^{1/\lambda} - 1} = \sum_{m=0}^{\infty} \beta_m(\lambda)\frac{x^m}{m!} \, . \tag{17}$$

A comparison with (6) shows that

$$A(m, j) = \beta_j\left(\frac{1}{m}\right)\frac{m^j}{j!} \, . \tag{18}$$

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1. L. CARLITZ, "A degenerate Staudt-Clausen theorem," *Arch. Math.*, v. 7, 1956, pp. 28–33.