21 [3.10, 6.15].—A. N. TIKHONOV & V. Y. ARSENIN, *Solutions of Ill-Posed Problems,* Halsted Press, New York, 1977, xiii + 258 pp., 22½cm. Price $19.75.

A recent trend in applied mathematics can best be described as a disavowal of the statement: "The only problems which merit investigation are those which are properly posed." The study of improperly-posed problems has grown dramatically in the past 10 years. Unfortunately, as with most emerging research areas, an introduction to the subject is not easily acquired; the literature is spread throughout journal articles except for a few monographs [1], [2], [3], [4]*. However, these books deal almost exclusively with ill-posed problems involving partial differential equations. The resulting gap in the literature has been filled by the recent translation of *Solutions of Ill-Posed Problems* by A. N. Tikhonov and V. Y. Arsenin.

The emphasis of the authors is on the development of methods for obtaining stable approximations to ill-posed problems and, thereby, rendering their numerical treatment possible.

We may cast the problem in an appropriate setting by considering the equation

$$Az = u, \tag{1}$$

where A is a continuous map from a metric space F with metric ρ_F to a metric space U with metric ρ_U. In this context, ill-posed problems arise when we try to invert A; i.e., given $u_T \in U$ find $z_T \in F$ such that

$$Az_T = u_T.$$

Specifically, suppose we have only approximate data u_δ such that $\rho_U(u_\delta, u_T) \leqslant \delta$. We wish to find an approximate solution z_δ corresponding to u_δ in some sense, so that as $\delta \longrightarrow 0$, $z_\delta \longrightarrow z_T$. We cannot in general set $z_\delta = A^{-1}u_\delta$ since

(i) u_δ need not lie in Range of A.

(ii) Even when A^{-1} is well defined it need not be continuous; i.e., for $u_\delta \in$ Range(A) and $z_\delta \in F$ such that $Az_\delta = u_\delta$ $\rho_U(u_\delta, u_T) \longrightarrow 0 \not\Rightarrow \rho_F(z_\delta, z_T) \longrightarrow 0$.

Thus, in our abstract setting, the study of ill-posed problems consists of trying to determine z_δ from u_δ in a stable manner when A^{-1} is not continuous (or even well defined). To stabilize (1) we must utilize additional a priori information about the solution z_T. This information can be of two forms: 1) quantitative information about the solution; or, 2) qualitative information about the solution. For instance, we may have an explicit bound on the size of the solution or we may know that the solution must satisfy certain regularity conditions.

The first part of the book is a development of stabilization methods, the emphasis being on the regularization method which uses information of type 2 to stabilize the problem.

The first chapter briefly surveys other methods of stabilization; in particular, the authors consider the selection method, the quasisolution method, and two techniques

*Contains an excellent bibliography.

of perturbing the equation. The selection method restricts consideration of possible solutions z of (1) to a subset $M \subset F$. If A is 1-1, M is compact, and $u_\delta \in M$, then we have stabilized the problem since $A^{-1}: AM \longrightarrow M$ is continuous.

Problems arise when the measured data $u_\delta \notin D(A^{-1})$. This difficulty motivates the notion of a quasisolution of (1) on M. A quasisolution is simply a $z_\delta \in M$ such that

$$\rho_U(Az_\delta, u_\delta) = \inf_{z \in M} \rho_U(Az, u_\delta).$$

Under certain restrictions on $N = AM$ the quasisolution is unique and depends continuously on the data.

The two perturbation methods discussed are quasireversibility and a technique of Lavrentiev. In both cases the equation is perturbed to a well-posed problem. In the latter (1) is replaced by

$$(A + \alpha I)z = u,$$

and the behavior is examined as $\alpha \longrightarrow 0$.

The quasireversibility method is a technique for dealing with unstable evolution equations such as the backward heat equation. The idea is to perturb the unstable differential operator to a stable operator by adding on a spatial operator depending on ϵ. As in Lavrentiev technique, the behavior is examined as $\epsilon \longrightarrow 0$.

The above methods are severely limited since they presuppose the class of possible solution can be a priori restricted to a compact set M. However, in many applied problems this is impossible. To remove this restrictive assumption the authors introduce the regularization method. Specifically, let u_T denote the true data; and suppose we have measured data u_δ such that $\rho_U(u_\delta, u_T) \leqslant \delta$. We shall introduce regularizing operators to define z_δ's corresponding to u_δ's such that $z_\delta \longrightarrow z_T$ as $u_\delta \longrightarrow u_T$. To be precise, a regularizing operator $R(u, \delta)$ of (1) in a neighborhood of u_T is an operator with the properties

1) $\exists \delta_i > 0$ such that $R(u, \delta)$ is defined $\forall \delta\ 0 \leqslant \delta \leqslant \delta_1$ and $\forall u\, \rho_U(u, u_T) \leqslant \delta$ and

2) $\forall \epsilon > 0\ \exists \delta_0 = \delta_0(\epsilon, u_T) \leqslant \delta_1$ such that

$$\rho_U(u_\delta, u_T) \leqslant \delta \leqslant \delta_0 \Rightarrow \rho_F(z_\delta, z_T) \leqslant \epsilon,$$

where $z_\delta = R(u_\delta, \delta)$.

Regularizing operators, by definition, stabilize (1). Thus, the problem of finding approximate solutions of (1) is reduced to the construction of regularizing operators.

The construction can be accomplished by a variational principle. First, we say $\Omega[z]: F_1 \longrightarrow \overline{\mathbf{R}^+}$, where $\overline{F}_1 = F$ is a stabilizing functional if

a) $z_T \in D(\Omega)$,

b) $\forall d > 0\ \{z \in F_1 \ni \Omega[z] \leqslant d\}$ is compact in F_1.

Thus, for $Q_\delta = \{z \ni \rho_U(Az, u_\delta) \leqslant \delta\}$, if $\widetilde{R}(u_\delta, \delta)$ is defined as a z minimizing $\Omega[z]$ over $Q_\delta \cap F_1$, then $\widetilde{R}$ is a regularizing operator. Under certain conditions this minimization is equivalent to choosing z_δ to minimize the functional

$$M^\alpha[z, u_\delta] = \rho_U^2(Az, u_\delta) + \alpha\Omega[z].$$

(α is called the regularization parameter. It depends on δ and u_δ.)

Having thus developed the regularization method, the remainder of the book details its application to a wide variety of ill-posed problems. Problems discussed include:

1) Singular and ill-conditioned systems of linear algebraic equations;
2) Fredholm integral equations of the first kind (with emphasis on kernels of convolution type);
3) Stable methods (in the space of continuous functions) for summing Fourier series with approximate coefficients in l_2; and
4) Problems in optimal control and mathematical programming.

In each case, the authors construct the regularization operator for the problem in detail. Methods for determining the optimal regularization parameter α under assumptions on the distribution of noise are also discussed.

The book also has an unexpected but particularly welcome feature: an extensive bibliography of the Russian literature. The reference list will be extremely valuable both to the active researcher and the student surveying the ill-posed problems literature.

In conclusion, it should be noted that although the book presents results not commonly included in an applied mathematics education, the techniques are accessible to graduate students and the engineering community. Thus, the book will serve as an excellent reference to anyone whose research leads him into the realm of ill-posed problems.

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1. M. M. LAVRENTIEV, *Some Improperly Posed Problems of Mathematical Physics*, Springer-Verlag, New York, 1967.

2. M. M. LAVRENTIEV, V. G. ROMANOV & V. G. VASILIEV, *Multi-dimensional Inverse Problems for Differential Equations*, Lecture Notes in Math., vol. 167, Springer-Verlag, Berlin, 1967.

3. R. LATTES & J. L. LIONS, *Methods of Quasireversibility: Applications to Partial Differential Equations*, American Elsevier, New York, 1969.

4. L. E. PAYNE, *Improperly Posed Problems in Partial Differential Equations*, Regional Conference Series in Applied Mathematics, No. 22, SIAM, Philadelphia, 1975.

22 [4.00, 12.00].—G. HALL & J. M. WATT, Editors, *Modern Numerical Methods for Ordinary Differential Equations*, Clarendon Press, Oxford, 1976, ix + 336 pp, 24cm. Price $21.50.

This volume is intended as an up-to-date account of theoretical questions and practical questions and methods in the numerical solution of ordinary differential equations. It consists of twenty-one chapters (eight on general initial value problems, six on stiff problems, five on boundary value problems, and two on functional differential equations); these chapters are written by thirteen scholars from England, New Zealand, and Scotland.