Some Interior Estimates for Semidiscrete Galerkin Approximations for Parabolic Equations

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Abstract. Consider a solution u of the parabolic equation

$$u_t + Au = f \quad \text{in} \quad \Omega \times [0, T],$$

where A is a second order elliptic differential operator. Let $\{S_h; h \text{ small}\}$ denote a family of finite element subspaces of $H^1(\Omega)$ which permits approximation of a smooth function to order $O(h^r)$. Let $\Omega_0 \subset \Omega$ and assume that $u_h: [0, T] \to S_h$ is an approximate solution which satisfies the semidiscrete interior equation

$$(u_{h,t}, \chi) + A(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h^0(\Omega_0) = \{\chi \in S_h, \text{ supp } \chi \subset \Omega_0\},$$

where $A(\cdot, \cdot)$ denotes the bilinear form on $H^1(\Omega)$ associated with A. It is shown that if the finite element spaces are based on uniform partitions in a specific sense in Ω_0, then difference quotients of u_h may be used to approximate derivatives of u in the interior of Ω_0 to order $O(h^r)$ provided certain weak global error estimates for $u_h - u$ to this order are available. This generalizes results proved for elliptic problems by Nitsche and Schatz [9] and Bramble, Nitsche and Schatz [1].

1. Introduction. Let A be a second order elliptic differential operator in a smooth domain $\Omega \subset R^N$ and let u be a solution of

$$(1.1) \qquad\qquad Au = f \quad \text{in} \quad \Omega,$$

with some boundary condition on $\partial\Omega$. Let $\{S_h; h \text{ small}\}$ denote a family of finite element subspaces of $H^1(\Omega)$ which permits approximation of a smooth function to order $O(h^r)$ in L_2 or L_∞, say. Let $\Omega_0 \subset \Omega$ and assume that $u_h \in S_h$ is an approximate solution of (1.1) which satisfies the interior equation

$$A(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h^0(\Omega_0) = \{\chi \in S_h; \text{ supp } \chi \subset \Omega_0\},$$

where $A(\cdot, \cdot)$ denotes the bilinear form on $H^1(\Omega)$ associated with the operator A. It was then proved in Nitsche and Schatz [9], Bramble, Nitsche and Schatz [1] that if the finite element spaces are based on uniform partitions in a specific sense in the interior domain Ω_0, then difference quotients of u may be used to approximate derivatives of u in the interior of Ω_0 to order $O(h^r)$. More precisely, it was shown in [1] that if Q_h is a finite difference operator approximating $D^\alpha (|\alpha| = m)$ with accuracy r, if $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$ and if u is sufficiently smooth, then for each p,

(1.2) $|Q_h u_h - D^\alpha u|_{\Omega_2} \leqslant Ch^r \|u\|_{m+r+N_0, \Omega_1} + C \|u_h - u\|_{-p, \Omega_1}$,

$$N_0 = [N/2] + 1.$$

Combined with known global error estimates for u_h this yields uniform interior estimates for approximations to arbitrary derivatives. Here and below we denote for $\Omega_1 \subset \Omega$,

$$|\cdot|_{m, \Omega_1} = \|\cdot\|_{W_\infty^m(\Omega_1)} \quad \text{and} \quad \|\cdot\|_{m, \Omega_1} = \|\cdot\|_{H^m(\Omega_1)},$$

with m and Ω_1 suppressed when $m = 0$ or $\Omega_1 = \Omega$, respectively, and where for $m > 0$ with $(\cdot, \cdot)$ the inner product in $L_2(\Omega)$,

$$\|u\|_{H^{-m}(\Omega_1)} = \sup_{\varphi \in C_0^\infty(\Omega_1)} \frac{(u, \varphi)}{\|\varphi\|_{H^m(\Omega_1)}}.$$

The purpose of the present paper is to derive similar results for the solution u of a parabolic equation ($u_t = \partial u / \partial t$)

(1.3) $u_t + Au = f \quad \text{in} \quad \Omega \times [0, T]$,

and an approximate solution $u_h \colon [0, T] \longrightarrow S_h$ satisfying the semidiscrete interior equation

$$(u_{h,t}, \chi) + A(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h^0(\Omega_0).$$

These results are stated and proved in Section 4. Our first main result (Theorem 4.1 below) shows that similarly to (1.2) above, if the initial-values satisfy, with $A_0(\cdot, \cdot)$ some elliptic bilinear form on $H^1(\Omega)$,

$$A_0(u_h(0) - u(0), \chi) = 0 \quad \forall \chi \in S_h^0(\Omega_0),$$

then for $t \in [0, T]$, and with $e_h = u_h - u$,

(1.4) $|Q_h u_h(t) - D^\alpha u(t)|_{\Omega_2} \leqslant C \{ h^r B(u) + R(e_h) \}$,

where $B(u)$ depends on a number of derivatives of u in $\Omega_1 \times [0, t]$ and $R(e_h)$ consists of weak norms on this set of the error; with p, q arbitrary numbers,

$$R(e_h) = \|e_h(0)\|_{-p, \Omega_1} + \left(\int_0^t (\|e_h\|^2 + h^q \|e_{h,t}\|^2) \, d\tau \right)^{1/2}.$$

In our second main result (Theorem 4.2) we show that if we are content with an estimate for time bounded away from zero, then an error estimate analogous to (1.4) holds, with $B(u)$ and $R(e_h)$ now only depending on u and e_h over a short interval preceding t and without any restriction on the initial-values. The first of these two results may be thought of as interior in space only whereas the second is then interior with respect to both space and time. In the same way as for the elliptic problem quoted above, these estimates may be combined with known estimates for e_h to yield spatially interior uniform $O(h^r)$ bounds for $Q_h u_h - D^\alpha u$. Several such error estimates are available in the literature, cf. e.g. [3], [5], [6], [7], [10], [11], and further

estimates, specially tailored to the present situation, are derived in Section 5 below.

An estimate for $Q_h u_h - D^\alpha u$ of the type just described was obtained in Bramble, Schatz, Thomée and Wahlbin [2] as a part of a rather general theory treating the homogeneous case ($f = 0$) of the equation (1.3) under Dirichlet boundary conditions, with A selfadjoint, nonnegative and time-independent.

This particular estimate was derived using the corresponding known estimates in the elliptic case quoted above, by considering u_h as an approximate solution of a non-homogeneous elliptic problem with u_t as a right-hand side, first trading difference quotients in space for time derivatives and then applying global error estimates for time derivatives. These global estimates were obtained by a method using spectral representations and allowing for considerable generality in treating the approximation of the boundary conditions. We shall review the framework of these estimates in Section 5 below. The constant in the resulting $O(h^r)$ estimate for $Q_h u_h - D^\alpha u$ here behaves like a negative power of t and the estimate, therefore, is not valid uniformly for small t. Also, since it uses global error estimates in space for the time derivatives, restrictive regularity assumptions have to be imposed on the solution which are global in space. The technique of estimating $Q_h u_h - D^\alpha u$ in terms of time derivatives of e_h generalizes to the nonhomogeneous equation as stated in Theorem 4.3.

Besides being valid for the nonhomogeneous equation the error estimates of the present paper differ from the corresponding ones of [2] in that they require severe regularity assumptions only in a neighborhood of the domain over which the error is sought, and one of the estimates is uniform down to $t = 0$. They will be derived by the energy method, using the parabolic character directly rather than considering u_t as an inhomogeneity in an elliptic equation. This method of proof will allow us to treat without complications operators which are nonselfadjoint and which depend on time. The major step is to derive a spatially interior estimate for the semidiscrete parabolic equation corresponding to the following estimate for the continuous problem, namely

$$\|u(t)\|_{m,\Omega_2} \leqslant C\left\{\|u(0)\|_{m,\Omega_1} + \left(\int_0^t (\|f\|_{m-1,\Omega_1}^2 + \|u\|_{\Omega_1}^2)\,d\tau\right)^{1/2}\right\}.$$

This result will be shown in Section 3 below. One of the tools in deriving it is a super-approximation result for the approximating spaces which is based on the following super-approximation property assumed in [1] and [9]: for $\omega \in C_0^\infty(\Omega_1)$,

$$(1.5) \qquad \inf_{\chi \in S_h^0(\Omega_1)} \|\omega U - \chi\|_{1,\Omega_1} \leqslant Ch\|U\|_{1,\Omega_1} \quad \forall U \in S_h.$$

The result we shall need (Lemma 2.4 below) is essentially that if $\widetilde{P}\colon H^1(\Omega_1) \to S_h$ denotes the elliptic projection over Ω_1 with respect to $A(\,\cdot\,,\,\cdot\,)$, without boundary conditions, then

$$(1.6) \qquad \inf_{\chi \in S_h^0(\Omega_1)} \|\widetilde{P}(\omega U) - \chi\|_{1,\Omega_1} \leqslant Ch\|U\|_{\Omega_1}.$$

Notice that the norm on the right in (1.6) is weaker than that in (1.5).

As indicated above, Section 5 is devoted to deriving, within the framework of the nonhomogeneous analogue of the theory developed in [2] the global $O(h^r)$ estimates needed for appraising to this order the terms in e_h in the results of Section 4. The first such result, Theorem 5.1, corresponding to the situation of Theorem 4.1, estimates the term $R(e_h)$ in (1.4) with $u_h(0)$ chosen as the elliptic projection of $u(0)$ and under certain regularity assumptions on u in $\Omega_1 \times [0, t]$. In Theorem 5.2 we then show, using the results of [2] that if, as in Theorem 4.2, we are content with error estimates for t bounded away from zero, then those regularity assumptions only have to be made for the time immediately preceding t. We shall also see that in this case we have considerable freedom in the choice of the discrete initial-values. For completeness we finally deduce in Theorem 5.3 the global estimates for the time derivatives of the error needed to generalize the argument of [2] to the nonhomogeneous equation for time bounded away from zero. Again, this method uses more than necessary regularity, globally in space.

Throughout this paper C will denote different positive constants, independent of h, t and the functions involved.

2. Interior Approximation Properties and Elliptic Estimates. In this section we shall first briefly recall the local regularity assumptions on the finite element spaces S_h employed in Nitsche and Schatz [9] and Bramble, Nitsche and Schatz [1], and review some of the interior estimates of these papers which will be needed below. We shall then introduce a local elliptic projection which when applied to the exact solution of the parabolic problem will be shown in Section 4, as a major step towards our final results, to yield a function in S_h close to the semidiscrete solution. We shall finally show a super-approximation result for the local elliptic projection of a localization of a function in S_h.

We begin with the assumptions on $\{S_h\}$ and postulate first that for some interior subdomain Ω_0 of Ω, the functions in S_h are piecewise polynomials on a uniform partition. In order to make this more precise, let $Q_1, \ldots, Q_l$ be disjoint bounded domains in R^N such that their translates $Q_j^\nu = Q_j + \nu, j = 1, \ldots, l, \nu \in Z^N$, are disjoint and their closures cover R^N. Let r be an integer ≥ 2 and let $\psi_1, \ldots, \psi_k$ be continuous functions with compact supports which reduce to polynomials on the sets Q_j^ν and which are such that for each Q_j^ν the set of restrictions to Q_j^ν of $\{\psi_s(\cdot - \alpha); s = 1, \ldots, k, \alpha \in Z^N\}$ contains all polynomials of degree less than r. We shall assume then that $S_h(\Omega_0)$, the set of restrictions to Ω_0 of the functions in S_h, is spanned by the translates of the ψ_j, scaled to mesh-size h, or $\chi \in S_h(\Omega_0)$ if and only if it can be written in the form

$$\chi(x) = \sum_{j,\alpha} a_{j\alpha} \psi_j(h^{-1}x - \alpha) \quad \text{for} \quad x \in \Omega_0.$$

We note that as a consequence of this assumption, if $\Omega_1 \subset\subset \Omega_0$, then for small h any finite difference quotient with step-size h of a function in $S_h(\Omega_0)$ is in $S_h(\Omega_1)$. Further, for any $\Omega_h \subset \Omega_0$ which is a mesh-domain (the interior of the closure of a union of some sets hQ_j^ν) we have the inverse estimate (clearly $S_h(\Omega_0) \subset H^1(\Omega_0)$)

(a) $\qquad\qquad \|\chi\|_{1,\Omega_h} \leqslant Ch^{-1}\|\chi\|_{\Omega_h} \quad \forall \chi \in S_h(\Omega_0).$

We shall make the following two local approximability assumptions, where for $\Omega_1 \subset \Omega_0$,

$$S_h^0(\Omega_1) = \{\chi \in S_h; \text{supp } \chi \subset \Omega_1\},$$

and where $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$. The first assumption concerns the possibility of approximating functions with compact support in Ω_1 by functions in $S_h^0(\Omega_1)$:

(b) If $1 \leqslant l \leqslant r$ and if $w \in H^l(\Omega_1)$ and vanishes outside Ω_2, then

$$\inf_{\chi \in S_h^0(\Omega_1)} \|w - \chi\|_{1,\Omega_1} \leqslant Ch^{l-1}\|w\|_{l,\Omega_1}.$$

The second assumption is essential in order to be able to localize functions in S_h:

(c) If $\omega \in C^\infty(\overline{\Omega}_1)$ with supp $\nabla\omega \subset \Omega_2$, then for $U \in S_h(\Omega_1)$,

$$\inf_{\substack{\chi \in S_h(\Omega_1) \\ \chi = \omega U \text{ in } \Omega_1 \setminus \overline{\Omega}_2}} \|\omega U - \chi\|_{1,\Omega_1} \leqslant C\|U\|_{\Omega_1}.$$

For a more detailed discussion of these properties we refer to [9] and [1]. It is in fact only the properties (a) and (c) which will be used explicitly below; the others are made in order to permit application of the elliptic interior estimates of [9] and [1] In these papers, the latter property was stated only for $\omega \in C_0^\infty(\Omega_1)$ and in the form

(2.1) $\qquad\qquad \inf_{\chi \in S_h^0(\Omega_1)} \|\omega U - \chi\|_{1,\Omega_1} \leqslant Ch\|U\|_{1,\Omega_1},$

which then implies the estimate in (c) in view of (a). In the examples discussed in [9] and [1] one finds easily that (2.1) is valid for the more general ω used in (c). In many cases, an interpolant of ωU may be used to show (c).

Let now A be the elliptic operator

$$Au = -\sum_{j,k=1}^N \frac{\partial}{\partial x_j}\left(a_{jk}\frac{\partial u}{\partial x_k}\right) + \sum_{j=1}^N a_j\frac{\partial u}{\partial x_j} + a_0 u,$$

where we assume that the coefficients are in $C^\infty(\overline{\Omega} \times [0, T])$ and (a_{jk}) is uniformly positive definite in $\overline{\Omega} \times [0, T]$. We introduce the corresponding bilinear form

$$A(v, w) = A(t; v, w) = \int_\Omega \left(\sum_{j,k=1}^N a_{jk}\frac{\partial v}{\partial x_k}\frac{\partial w}{\partial x_j} + \sum_{j=1}^N a_j\frac{\partial v}{\partial x_j}w + a_0 vw\right) dx.$$

We shall quote some interior estimates for discrete elliptic equations which we shall need. We shall always assume that our above assumptions on S_h hold in $\Omega_0 \subset\subset \Omega$, although the full force of these are not required in each instance. Notice in particular that the constants below are independent of t.

LEMMA 2.1. *Let $\Omega_1 \subset \Omega_0$ and assume that $w_h \in S_h$ satisfies*

$$A(t; w_h, \chi) = 0 \quad \forall \chi \in S_h^0(\Omega_1).$$

Then for any $\Omega_2 \subset\subset \Omega_1$,

$$\|w_h\|_{1,\Omega_2} \leqslant C\|w_h\|_{\Omega_1}.$$

Proof. See [9, Lemma 5.2].

In the next lemma we denote by ∂_h^α the forward difference quotient

$$\partial_h^\alpha = \partial_{h,1}^{\alpha_1} \cdots \partial_{h,N}^{\alpha_N} \quad \text{with} \quad \partial_{h,j} w(x) = h^{-1}(w(x + he_j) - w(x)),$$

where e_j is the jth unit vector in R^N.

LEMMA 2.2. *Let* $\Omega_1 \subset \Omega_0$ *and assume that* $w_h \in S_h$ *satisfies*

$$A(t; w_h - w, \chi) = 0 \quad \forall \chi \in S_h^0(\Omega_1).$$

Then for any $\Omega_2 \subset\subset \Omega_1$ *and* $p \geqslant 0$, $|\alpha| = m$,

$$(2.2) \qquad \|\partial_h^\alpha(w_h - w)\|_{\Omega_2} \leqslant C\{h^r\|w\|_{r+m,\Omega_1} + \|w_h - w\|_{-p,\Omega_1}\},$$

and with $N_0 = [N/2] + 1$,

$$(2.3) \qquad |\partial_h^\alpha(w_h - w)|_{\Omega_2} \leqslant C\{h^r\|w\|_{r+m+N_0,\Omega_1} + \|w_h - w\|_{-p,\Omega_1}\}.$$

Further, for the time derivative of the error,

$$(2.4) \qquad \begin{aligned} \|\partial_h^\alpha(w_h - w)_t\|_{\Omega_2} \leqslant{} & C\{h^r(\|w\|_{r+m,\Omega_1} + \|w_t\|_{r+m,\Omega_1}) \\ & + \|w_h - w\|_{-p,\Omega_1} + \|(w_h - w)_t\|_{-p,\Omega_1}\}. \end{aligned}$$

Proof. The estimates (2.2) and (2.3) are contained in [9, Theorem 6.1] and [1, Theorem 1], respectively, and (2.4) follows in the same way as (2.2) using [9, Theorem 5.2] and the fact that

$$A(t; (w_h - w)_t, \chi) = - A'(t; w_h - w, \chi) \quad \forall \chi \in S_h^0(\Omega_1),$$

with A' the bilinear form obtained from A by time-differentiation of the coefficients. When A is independent of t, (2.4) is of course an immediate consequence of (2.2) and the first and third terms on the right may then be omitted.

We shall have reason to work below with a local elliptic projection corresponding to a Neumann problem on a subdomain Ω_1 of Ω_0 which we may assume to be smooth. For this purpose we denote

$$\widetilde{A}_{\Omega_1}(t; v, w) = \int_{\Omega_1} \left(\sum_{j,k=1}^N a_{jk} \frac{\partial v}{\partial x_k} \frac{\partial w}{\partial x_j} + vw \right) dx.$$

Note that this modified symmetric bilinear form is $H^1(\Omega_1)$-elliptic, uniformly in t, so that

$$\|v\|_{1,\Omega_1}^2 \leqslant C\widetilde{A}_{\Omega_1}(t; v, v) \quad \forall v \in H^1(\Omega_1), 0 \leqslant t \leqslant T.$$

We then define $\widetilde{P}_{\Omega_1}(t)$: $H^1(\Omega_1) \longrightarrow S_h(\Omega_1)$ by

$$(2.5) \qquad \widetilde{A}_{\Omega_1}(t; w - \widetilde{P}_{\Omega_1}(t)w, \chi) = 0 \quad \forall \chi \in S_h(\Omega_1).$$

It is well known that

$$(2.6) \quad \|(I - \widetilde{P}_{\Omega_1}(t))w\|_{j,\Omega_1} \leqslant Ch^{l-j}\|w\|_{l,\Omega_1}, \quad -(r-2) \leqslant j \leqslant 1 \leqslant l \leqslant r,$$

and similarly for the time derivative, in particular (cf. e.g. Douglas and Dupont [4]),

$$(2.7) \quad \|((I - \widetilde{P}_{\Omega_1}(t))w)_t\|_{\Omega_1} \leqslant Ch^l(\|w\|_{l,\Omega_1} + \|w_t\|_{l,\Omega_1}), \quad 1 \leqslant l \leqslant r.$$

Combining these estimates with Lemma 2.2, we have

LEMMA 2.3. *Let* $\Omega_1 \subset \Omega_0$ *and let* $\widetilde{P} = \widetilde{P}_{\Omega_1}(t)$ *be defined by* (2.5). *Then for* $\Omega_2 \subset\subset \Omega_1$ *and* $|\alpha| = m$,

$$\|\partial_h^\alpha(I - \widetilde{P})w\|_{\Omega_2} \leqslant Ch^r\|w\|_{r+m,\Omega_1},$$

$$|\partial_h^\alpha(I - \widetilde{P})w|_{\Omega_2} \leqslant Ch^r\|w\|_{r+m+N_0,\Omega_1},$$

and

$$\|\partial_h^\alpha((I - \widetilde{P})w)_t\|_{\Omega_2} \leqslant Ch^r(\|w\|_{r+m,\Omega_1} + \|w_t\|_{r+m,\Omega_1}).$$

We conclude this section by establishing our super-approximation estimate for the elliptic projection defined above of a localization of a function in the subspace. We emphasize that this estimate shows one degree better approximation than the corresponding estimates (c) and (2.1).

LEMMA 2.4. *Let* $\Omega_1 \subset \Omega_0$ *and* $\omega \in C_0^\infty(\Omega_1)$. *Then for* $U \in S_h(\Omega_1)$,

$$(2.8) \quad \inf_{\chi \in S_h^0(\Omega_1)} \{\|\widetilde{P}_{\Omega_1}(t)(\omega U) - \chi\|_{1,\Omega_1} + \|\omega U - \chi\|_{\Omega_1}\} \leqslant Ch\|U\|_{\Omega_1}.$$

Proof. Set for brevity $\widetilde{U} = \widetilde{P}_{\Omega_1}(t)(\omega U)$ and $\widetilde{A}(\cdot, \cdot) = \widetilde{A}_{\Omega_1}(t; \cdot, \cdot)$. We shall prove below that

$$(2.9) \quad \inf_{\chi \in S_h^0(\Omega_1)} \|\widetilde{U} - \chi\|_{1,\Omega_1} \leqslant Ch\|U\|_{\Omega_1}.$$

Let us first show that this implies (2.8). In fact,

$$\inf_{\chi \in S_h^0(\Omega_1)} (\|\widetilde{U} - \chi\|_{1,\Omega_1} + \|\omega U - \chi\|_{\Omega_1})$$

$$\leqslant \|\omega U - \widetilde{U}\|_{\Omega_1} + C \inf_{\chi \in S_h^0(\Omega_1)} \|\widetilde{U} - \chi\|_{1,\Omega_1},$$

so that with (2.9) proven it suffices to show

$$(2.10) \quad \|\omega U - \widetilde{U}\|_{\Omega_1} \leqslant Ch\|U\|_{\Omega_1}.$$

In order to prove (2.10) we note that duality yields

$$\|\omega U - \widetilde{U}\|_{\Omega_1} \leqslant Ch\|\omega U - \widetilde{U}\|_{1,\Omega_1} \leqslant Ch\,\widetilde{A}(\omega U - \widetilde{U}, \omega U - \widetilde{U})^{1/2},$$

and hence by the definition of $\widetilde{U}$ as the best approximation of ωU with respect to $\widetilde{A}(\cdot , \cdot)^{\frac{1}{2}}$, and by (c),

$$\|\omega U - \widetilde{U}\|_{\Omega_1} \leqslant Ch \inf_{\chi \in S_h^0(\Omega_1)} \|\omega U - \chi\|_{1,\Omega_1} \leqslant Ch \|U\|_{\Omega_1}.$$

In order to show (2.9) we shall only need to show that

$$(2.11) \qquad \inf_{\chi \in S_h^0(\Omega_1)} \|\widetilde{U} - \chi\|_{1,\Omega_1} \leqslant C \|\widetilde{U}\|_{\Omega_1 \backslash \text{supp } \omega}.$$

For by (2.10) we have

$$(2.12) \qquad \|\widetilde{U}\|_{\Omega_1 \backslash \text{supp } \omega} \leqslant \|\widetilde{U} - \omega U\|_{\Omega_1} \leqslant Ch \|U\|_{\Omega_1},$$

and together (2.11) and (2.12) prove (2.9).

For the purpose of showing (2.11), let Ω_2 and Ω_3 be such that supp $\omega \subset \Omega_3$ $\subset\subset \Omega_2 \subset\subset \Omega_1$. We shall then first prove that

$$(2.13) \qquad \inf_{\chi \in S_h^0(\Omega_1)} \|\widetilde{U} - \chi\|_{1,\Omega_1} \leqslant C \|\widetilde{U}\|_{1,\Omega_1 \backslash \bar{\Omega}_2}.$$

Since $\widetilde{U}$ is the elliptic projection of a function vanishing on $\Omega_1 \backslash \bar{\Omega}_2$, we shall be able to estimate its norm in $H^1(\Omega_1 \backslash \bar{\Omega}_2)$ by a norm in L_2. In order, however, to be able to apply the interior estimate of Lemma 2.1 we shall first shift the problem into the interior of Ω_1 by showing below that

$$(2.14) \qquad \|\widetilde{U}\|_{1,\Omega_1 \backslash \bar{\Omega}_2} \leqslant C \|\widetilde{U}\|_{1,\Omega_2 \backslash \bar{\Omega}_3}.$$

We now note that by the definition of $\widetilde{U}$ we have

$$\widetilde{A}(\widetilde{U}, \chi) = \widetilde{A}(\omega U, \chi) = 0 \quad \forall \chi \in S_h^0(\Omega_1 \backslash \text{supp } \omega),$$

so that Lemma 2.1 yields

$$(2.15) \qquad \|\widetilde{U}\|_{1,\Omega_2 \backslash \bar{\Omega}_3} \leqslant C \|\widetilde{U}\|_{\Omega_1 \backslash \text{supp } \omega}.$$

Together, (2.13), (2.14) and (2.15) imply (2.11) and thus complete the proof of (2.9) and hence of the lemma.

It remains to show (2.13) and (2.14). In order to show (2.13), let $\omega_1 \in C_0^\infty(\Omega_1)$ with $\omega_1 = 1$ in a neighborhood of Ω_2. Then

$$\inf_{\chi \in S_h^0(\Omega_1)} \|\widetilde{U} - \chi\|_{1,\Omega_1} \leqslant \|(1 - \omega_1)\widetilde{U}\|_{1,\Omega_1} + \inf_{\chi \in S_h^0(\Omega_1)} \|\omega_1 \widetilde{U} - \chi\|_{1,\Omega_1}.$$

Since supp$(1 - \omega_1) \subset \Omega_1 \backslash \bar{\Omega}_2$ we have for the first term,

$$\|(1 - \omega_1)\widetilde{U}\|_{1,\Omega_1} \leqslant C \|\widetilde{U}\|_{1,\Omega_1 \backslash \bar{\Omega}_2}.$$

For the second term, letting Ω_4 be such that supp $\nabla \omega_1 \subset \Omega_4 \subset\subset \Omega_1 \backslash \bar{\Omega}_2$ we obtain, using in the last step (c) applied to $\Omega_1 \backslash \bar{\Omega}_2$,

$$\inf_{\substack{\chi \in S_h^0(\Omega_1)}} \|\omega_1 \widetilde{U} - \chi\|_{1, \Omega_1} \leqslant \inf_{\substack{\chi \in S_h(\Omega_1) \\ \chi = \omega_1 \widetilde{U} \text{ in } \Omega_1 \backslash \overline{\Omega}_4}} \|\omega_1 \widetilde{U} - \chi\|_{1, \Omega_1}$$

$$= \inf_{\substack{\chi \in S_h(\Omega_1 \backslash \overline{\Omega}_2) \\ \chi = \omega_1 \widetilde{U} \text{ in } (\Omega_1 \backslash \overline{\Omega}_2) \backslash \overline{\Omega}_4}} \|\omega_1 \widetilde{U} - \chi\|_{1, \Omega_1 \backslash \overline{\Omega}_2} \leqslant C \|\widetilde{U}\|_{\Omega_1 \backslash \overline{\Omega}_2}.$$

Together these estimates show (2.13).

We finally turn to (2.14). Let $\varphi \in C^\infty(\Omega_1)$ with $\varphi = 1$ in a neighborhood of $\Omega_1 \backslash \Omega_2$ in Ω_1 and $\varphi = 0$ in a neighborhood of $\overline{\Omega}_3$. Let also Ω_5 be such that $\text{supp } \nabla\varphi \subset \Omega_5 \subset\subset \Omega_2 \backslash \overline{\Omega}_3$. We have

(2.16)
$$\|\widetilde{U}\|_{1, \Omega_1 \backslash \Omega_2}^2 \leqslant \|\varphi\widetilde{U}\|_{1, \Omega_1}^2 \leqslant C\widetilde{A}(\varphi\widetilde{U}, \varphi\widetilde{U})$$

$$= C\left\{\widetilde{A}(\widetilde{U}, \varphi^2\widetilde{U}) + [A(\varphi\widetilde{U}, \varphi\widetilde{U}) - \widetilde{A}(\widetilde{U}, \varphi^2\widetilde{U})]\right\}.$$

Here, for $\chi \in S_h(\Omega_1)$ with $\chi = \varphi^2\widetilde{U}$ outside Ω_5 we have

$$\widetilde{A}(\widetilde{U}, \varphi^2\widetilde{U}) = \widetilde{A}(\widetilde{U}, \varphi^2\widetilde{U} - \chi) \leqslant C\|\widetilde{U}\|_{1, \Omega_2 \backslash \overline{\Omega}_3} \|\varphi^2\widetilde{U} - \chi\|_{1, \Omega_2 \backslash \overline{\Omega}_3},$$

so that by (c),

(2.17)
$$\widetilde{A}(\widetilde{U}, \varphi^2\widetilde{U}) \leqslant C\|\widetilde{U}\|_{1, \Omega_2 \backslash \overline{\Omega}_3} \inf_{\substack{\chi \in S_h(\Omega_2 \backslash \overline{\Omega}_3) \\ \chi = \varphi^2\widetilde{U} \text{ in } (\Omega_2 \backslash \overline{\Omega}_3) \backslash \Omega_5}} \|\varphi^2\widetilde{U} - \chi\|_{1, \Omega_2 \backslash \overline{\Omega}_3}$$

$$\leqslant C\|\widetilde{U}\|_{1, \Omega_2 \backslash \overline{\Omega}_3}^2.$$

Further, by an easy calculation,

(2.18)
$$|\widetilde{A}(\varphi\widetilde{U}, \varphi\widetilde{U}) - \widetilde{A}(\widetilde{U}, \varphi^2\widetilde{U})| = \left| \int_{\Omega_1} \sum_{j,k} a_{jk} \frac{\partial\varphi}{\partial x_j} \frac{\partial\varphi}{\partial x_k} \widetilde{U}^2 \, dx \right|$$

$$\leqslant C\|\widetilde{U}\|_{\text{supp}(\nabla\varphi)}^2 \leqslant C\|\widetilde{U}\|_{1, \Omega_2 \backslash \overline{\Omega}_3}^2.$$

Together, (2.16), (2.17) and (2.18) yield (2.14).

The proof of Lemma 2.4 is now complete.

3. Interior Estimates for the Semidiscrete Problem. In this section we shall first derive, in Lemma 3.1, a discrete interior version of the energy type inequality

(3.1)
$$\|u(t)\|^2 + \int_0^t \|u\|_1^2 \, d\tau \leqslant C\left\{\|u(0)\|^2 + \int_0^t \|f\|_{-1}^2 \, d\tau\right\},$$

valid for a solution u of the continuous equation (1.1), vanishing on $\partial\Omega$. We shall then apply this inequality to difference quotients to obtain our basic a priori interior estimate, which is given in Lemma 3.3 below. The proof of the discrete interior counterpart of (3.1) will depend on the super-approximation property, described in Lemma 2.4, of the projection $\widetilde{P}_{\Omega_1'}(t)$ defined by (2.5), with Ω_1' a subdomain of Ω_0.

We define for U: $[0, T] \longrightarrow S_h(\Omega_0)$,

$$(L_h U)(\chi) = (U_t, \chi) + A(t; U, \chi) \quad \text{for} \quad \chi \in S_h(\Omega_0),$$

and set for $\Omega_1 \subset \Omega_0$,

$$\|L_h U\|_{h, -j, \Omega_1} = \sup_{\chi \in S_h^0(\Omega_1)} \frac{|(L_h U)(\chi)|}{\|\chi\|_{j, \Omega_1}}, \quad j = 0, 1.$$

We have then

LEMMA 3.1. *Let* $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$ *and let* q *be an arbitrary number. We then have for* U: $[0, T] \longrightarrow S_h(\Omega_0)$,

$$\|U(t)\|_{\Omega_2}^2 + \int_0^t \|U\|_{1, \Omega_2}^2 \, d\tau$$

$$\leqslant C \left\{ \|U(0)\|_{\Omega_1}^2 + \int_0^t [\|U\|_{\Omega_1}^2 + h^q \|U_t\|_{\Omega_1}^2 + \|L_h U\|_{h, -1, \Omega_1}^2] \, d\tau \right\}.$$

Proof. We shall first show this result for $q = 2$. We shall then derive in Lemma 3.2 an estimate for U_t which will allow us to improve the result to the case of a general q.

Let $\Omega_2 \subset\subset \Omega_1' \subset\subset \Omega_1$, take $\omega \in C_0^\infty(\Omega_1')$ with $\omega \equiv 1$ on Ω_2 and let $\widetilde{P} = \widetilde{P}_{\Omega_1'}(t)$ be the projection defined above with respect to $\widetilde{A}(\cdot, \cdot) = \widetilde{A}_{\Omega_1'}(t; \cdot, \cdot)$. Then using the definition of $L_h U$ we have for any $\chi \in S_h^0(\Omega_1')$,

$$(U_t, \omega^2 U) + \widetilde{A}(U, \omega^2 U)$$

$$= (U_t, \omega^2 U - \chi) + \widetilde{A}(U, \widetilde{P}(\omega^2 U) - \chi) + (L_h U)(\chi) + \widetilde{A}(U, \chi) - A(t; U, \chi)$$

or

$$\frac{1}{2} \frac{d}{dt} \|\omega U\|^2 + \widetilde{A}(\omega U, \omega U)$$

$$\leqslant |\widetilde{A}(\omega U, \omega U) - \widetilde{A}(U, \omega^2 U)| + \|U_t\|_{\Omega_1} \|\omega^2 U - \chi\|_{\Omega_1'}$$

$$+ \|U\|_{1, \Omega_1'} \|\widetilde{P}(\omega^2 U) - \chi\|_{1, \Omega_1'} + (\|L_h U\|_{h, -1, \Omega_1} + C \|U\|_{\Omega_1}) \|\chi\|_{1, \Omega_1'}.$$

Here we find easily

$$|\widetilde{A}(\omega U, \omega U) - \widetilde{A}(U, \omega^2 U)| = \left| \int_{\Omega_1'} \sum_{j,k} a_{jk} \frac{\partial \omega}{\partial x_j} \frac{\partial \omega}{\partial x_k} U^2 \, dx \right| \leqslant C \|U\|_{\Omega_1}^2,$$

and choosing χ as in Lemma 2.4, we have

$$\|U_t\|_{\Omega_1} \|\omega^2 U - \chi\|_{\Omega_1'} + \|U\|_{1, \Omega_1'} \|\widetilde{P}(\omega^2 U) - \chi\|_{1, \Omega_1'}$$

$$\leqslant Ch \|U\|_{\Omega_1} (\|U_t\|_{\Omega_1} + \|U\|_{1, \Omega_1'}).$$

Further,

$$\|\chi\|_{1,\Omega_1'} \leqslant \|\widetilde{P}(\omega^2 U) - \chi\|_{1,\Omega_1'} + \|\widetilde{P}(\omega^2 U)\|_{1,\Omega_1'}$$

$$\leqslant C(h\|U\|_{\Omega_1} + \|\omega^2 U\|_{1,\Omega_1'}) \leqslant C(\|U\|_{\Omega_1} + \widetilde{A}(\omega U, \omega U)^{1/2}).$$

Together, these estimates yield

$$\frac{d}{dt}\|\omega U\|^2 + 2\widetilde{A}(\omega U, \omega U)$$

$$\leqslant C\Big\{\|U\|_{\Omega_1}^2 + h\|U_t\|_{\Omega_1}\|U\|_{\Omega_1} + h\|U\|_{1,\Omega_1'}\|U\|_{\Omega_1}$$

$$+ (\|L_h U\|_{h,-1,\Omega_1} + \|U\|_{\Omega_1})(\widetilde{A}(\omega U, \omega U)^{1/2} + \|U\|_{\Omega_1})\Big\}.$$

Using the inverse estimate (a) on the third term on the right, we obtain easily

$$\frac{d}{dt}\|\omega U\|^2 + \widetilde{A}(\omega U, \omega U) \leqslant C\Big\{\|U\|_{\Omega_1}^2 + h^2\|U_t\|_{\Omega_1}^2 + \|L_h U\|_{h,-1,\Omega_1}^2\Big\}.$$

Integration over $(0, t)$ now completes the proof for $q = 2$.

For the purpose of further estimating the term in U_t above we shall now prove

LEMMA 3.2.   *Let* $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$ *and let* q *be an arbitrary number. We then have for* $U: [0, T] \longrightarrow S_h(\Omega_0)$,

$$\|U(t)\|_{1,\Omega_2}^2 + \int_0^t \|U_t\|_{\Omega_2}^2 \, d\tau$$

$$\leqslant C\Big\{\|U(0)\|_{1,\Omega_1}^2 + \int_0^t (\|U\|_{1,\Omega_1}^2 + h^q\|U_t\|_{\Omega_1}^2 + \|L_h U\|_{h,0,\Omega_1}^2) d\tau\Big\}.$$

Proof.   With the notation of the proof of Lemma 3.1, we have now for $\chi \in S_h^0(\Omega_1')$,

$$(U_t, \omega^2 U_t) + \widetilde{A}(U, \omega^2 U_t)$$

$$= (U_t, \omega^2 U_t - \chi) + \widetilde{A}(U, \widetilde{P}(\omega^2 U_t) - \chi) + (L_h U)(\chi) + (\widetilde{A}(U, \chi) - A(t; U, \chi)).$$

Hence

$$\|\omega U_t\|^2 + \widetilde{A}(\omega U, \omega U_t) \leqslant |\widetilde{A}(\omega U, \omega U_t) - \widetilde{A}(U, \omega^2 U_t)|$$

$$+ \|U_t\|_{\Omega_1}\|\omega^2 U_t - \chi\|_{\Omega_1'} + \|U\|_{1,\Omega_1}\|\widetilde{P}(\omega^2 U_t) - \chi\|_{1,\Omega_1'}$$

$$+ (\|L_h U\|_{h,0,\Omega_1} + C\|U\|_{1,\Omega_1})\|\chi\|_{\Omega_1'}.$$

We find here

$$|\widetilde{A}(\omega U, \omega U_t) - \widetilde{A}(U, \omega^2 U_t)|$$

$$= \left|\int_{\Omega_1} \sum_{j,k} \omega U_t\left[a_{jk}\frac{\partial \omega}{\partial x_j}\frac{\partial U}{\partial x_k} + \frac{\partial}{\partial x_j}\left(a_{jk}\frac{\partial \omega}{\partial x_k} U\right)\right] dx\right|$$

$$\leqslant C\|\omega U_t\|\,\|U\|_{1,\Omega_1} \leqslant \tfrac{1}{4}\|\omega U_t\|^2 + C\|U\|_{1,\Omega_1}^2.$$

Further, choosing χ by Lemma 2.4,

$$\|U_t\|_{\Omega_1}\|\omega^2 U_t - \chi\|_{\Omega_1'} + \|U\|_{1,\Omega_1}\|\widetilde{P}(\omega^2 U_t) - \chi\|_{1,\Omega_1'}$$

$$\leqslant Ch\|U_t\|_{\Omega_1}(\|U_t\|_{\Omega_1} + \|U\|_{1,\Omega_1})$$

and

$$\|\chi\|_{\Omega_1'} \leqslant \|\omega^2 U_t - \chi\|_{\Omega_1'} + \|\omega^2 U_t\| \leqslant C(h\|U_t\|_{\Omega_1} + \|\omega U_t\|).$$

Hence, since

$$\left| \widetilde{A}(\omega U, \omega U_t) - \frac{1}{2}\frac{d}{dt}\widetilde{A}(\omega U, \omega U) \right| \leqslant C\|U\|_{1,\Omega_1}^2,$$

we obtain

$$\|\omega U_t\|^2 + \frac{1}{2}\frac{d}{dt}\widetilde{A}(\omega U, \omega U)$$

$$\leqslant \frac{1}{4}\|\omega U_t\|^2 + C\Big\{\|U\|_{1,\Omega_1}^2 + h\|U_t\|_{\Omega_1}^2 + h\|U\|_{1,\Omega_1}\|U_t\|_{\Omega_1}$$

$$+ (\|L_h U\|_{h,0,\Omega_1} + \|U\|_{1,\Omega_1})(h\|U_t\|_{\Omega_1} + \|\omega U_t\|)\Big\},$$

or by obvious estimates

$$\|\omega U_t\|^2 + \frac{d}{dt}\widetilde{A}(\omega U, \omega U) \leqslant C\Big\{\|U\|_{1,\Omega_1}^2 + h\|U_t\|_{\Omega_1}^2 + \|L_h U\|_{h,0,\Omega_1}^2\Big\}.$$

By integration over $(0, t)$ we obtain

$$\int_0^t \|U_t\|_{\Omega_2}^2\, d\tau + \|U\|_{1,\Omega_2}^2$$

$$\leqslant C\Big\{\|U(0)\|_{1,\Omega_1}^2 + \int_0^t [\|U\|_{1,\Omega_1}^2 + h\|U_t\|_{\Omega_1}^2 + \|L_h U\|_{h,0,\Omega_1}^2]\, d\tau\Big\},$$

that is, the desired estimate with $q = 1$. Repeated application of this estimate, using a sequence of intermediate domains, allows us to increase the power of h to any order, and thus completes the proof of the lemma.

We may now complete the proof of Lemma 3.1. Let $\Omega_2 \subset\subset \Omega_1'' \subset\subset \Omega_1' \subset\subset \Omega_1$ and note that by the inverse estimate (a),

$$h\|U\|_{1,\Omega_1'} \leqslant C\|U\|_{\Omega_1}$$

and

$$h\|L_h U\|_{h,0,\Omega_1'} \leqslant C\|L_h U\|_{h,-1,\Omega_1}.$$

We may, therefore, conclude from Lemma 3.2 that

$$h^2 \int_0^t \|U_t\|_{\Omega_1''}^2\, d\tau$$

$$\leqslant C\Big\{h^2\|U(0)\|_{1,\Omega_1'}^2 + \int_0^t [h^2\|U\|_{1,\Omega_1'}^2 + h^q\|U_t\|_{\Omega_1}^2 + h^2\|L_h U\|_{h,0,\Omega_1'}^2]\, d\tau\Big\}$$

$$\leqslant C\Big\{\|U(0)\|_{\Omega_1}^2 + \int_0^t [\|U\|_{\Omega_1}^2 + h^q\|U_t\|_{\Omega_1}^2 + \|L_h U\|_{h,-1,\Omega_1}^2]\, d\tau\Big\}.$$

The desired result is now a consequence of the case $q = 2$ of Lemma 3.1 which we have already established.

We can now state and prove our main a priori interior estimate for the semidiscrete problem.

LEMMA 3.3. *Let* $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$ *and let q be arbitrary. Assume that* $U: [0, T] \rightarrow S_h(\Omega_0)$ *satisfies*

$$(L_h U)(\chi) \equiv (U_t, \chi) + A(t; U, \chi) = (F, \chi) \qquad \forall \chi \in S_h^0(\Omega_0), U(0) = V.$$

Then for any $m \geq 0$,

$$\sum_{|\alpha| \leq m} \|\partial_h^\alpha U(t)\|_{\Omega_2}^2$$

$$\leq C\left\{ \sum_{|\alpha| \leq m} \|\partial_h^\alpha V\|_{\Omega_1}^2 + \int_0^t \left[\|U\|_{\Omega_1}^2 + h^q \|U_t\|_{\Omega_1}^2 + \sum_{|\alpha| \leq m} \|\partial_h^\alpha F\|_{-1, \Omega_1}^2 \right] d\tau \right\}.$$

Proof. For $m = 0$ this is contained in Lemma 3.1. In order to apply Lemma 3.1 to $\partial_h^\alpha U$ for $|\alpha| = m > 0$, let $\Omega_2 \subset\subset \Omega_1' \subset\subset \Omega_1$ and notice that for $\chi \in S_h^0(\Omega_1')$ and small h,

$$(L_h(\partial_h^\alpha U))(\chi) = (\partial_h^\alpha U_t, \chi) + A(t; \partial_h^\alpha U, \chi) = (\partial_h^\alpha F, \chi) + R_h^{(\alpha)},$$

where (with $E_h^\alpha v = v(\cdot + \alpha h)$)

$$R_h^{(\alpha)} = -\int_{\Omega_1'} \sum_{\beta < \alpha} \binom{\alpha}{\alpha - \beta} \left\{ \sum_{j,k=1}^N \partial_h^{\alpha-\beta} a_{jk} E_h^{\alpha-\beta} \partial_h^\beta \frac{\partial U}{\partial x_j} \frac{\partial \chi}{\partial x_k} \right.$$
$$\left. + \sum_{j=1}^N \partial_h^{\alpha-\beta} a_j E_h^{\alpha-\beta} \partial_h^\beta \frac{\partial U}{\partial x_j} \chi + \partial_h^{\alpha-\beta} a_0 E_h^{\alpha-\beta} \partial_h^\beta U \chi \right\} dx.$$

We find immediately

$$|R_h^{(\alpha)}| \leq C \sum_{|\beta| < |\alpha|} \|\partial_h^\beta U\|_{1, \Omega_1} \|\chi\|_{1, \Omega_1'},$$

so that

$$\|L_h(\partial_h^\alpha U)\|_{h, -1, \Omega_1'} \leq C\left(\|\partial_h^\alpha F\|_{-1, \Omega_1} + \sum_{|\beta| < |\alpha|} \|\partial_h^\beta U\|_{1, \Omega_1} \right).$$

Noticing that

$$h^{q + 2|\alpha|} \|\partial_h^\alpha U_t\|_{\Omega_1'}^2 \leq C h^q \|U_t\|_{\Omega_1}^2$$

and

$$\sum_{|\alpha| \leq m} \|\partial_h^\alpha U\|_{\Omega_1'}^2 \leq \sum_{|\alpha| \leq m-1} \|\partial_h^\alpha U\|_{1, \Omega_1}^2,$$

we now obtain by application of Lemma 3.1 to $\partial_h^\alpha U$ for $|\alpha| \leq m$, with q replaced by

$q + 2|\alpha|$ and Ω_1 by Ω_1',

$$\sum_{|\alpha| \leqslant m} \left[\|\partial_h^\alpha U\|_{\Omega_2}^2 + \int_0^t \|\partial_h^\alpha U\|_{1,\Omega_2}^2 \, d\tau \right]$$

$$\leqslant C \left\{ \sum_{|\alpha| \leqslant m} \|\partial_h^\alpha V\|_{\Omega_1}^2 + \int_0^t \left[\sum_{|\alpha| \leqslant m-1} \|\partial_h^\alpha U\|_{1,\Omega_1}^2 + h^q \|U_t\|_{\Omega_1}^2 \right. \right.$$

$$\left. \left. + \sum_{|\alpha| \leqslant m} \|\partial_h^\alpha F\|_{-1,\Omega_1}^2 \right] d\tau \right\}.$$

By iteration of this inequality we may further reduce the first term of the integrand on the right (using intermediate sets Ω_1' with $\Omega_2 \subset\subset \Omega_1' \subset\subset \Omega_1$) to obtain

$$\sum_{|\alpha| \leqslant m} \|\partial_h^\alpha U\|_{\Omega_1}^2$$

$$\leqslant C \left\{ \sum_{|\alpha| \leqslant m} \|\partial_h^\alpha V\|_{\Omega_1}^2 + \int_0^t \left[\|U\|_{1,\Omega_1}^2 + h^q \|U_t\|_{\Omega_1}^2 + \sum_{|\alpha| \leqslant m} \|\partial_h F\|_{-1,\Omega_1}^2 \right] d\tau \right\}.$$

The result now follows by a final application of Lemma 3.1 (again using an intermediate set between Ω_2 and Ω_1).

We shall complete this section by deriving a version of the above inequality valid for t bounded away from zero, and with the bound on the right using only function values over a short interval preceding t.

LEMMA 3.4. *Let* $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$ *and let* q *be arbitrary. Assume that* U: $[0, T] \longrightarrow S_h(\Omega_0)$ *satisfies*

$$(L_h U)(\chi) = (U_t, \chi) + A(t; U, \chi) = (F, \chi) \quad \forall \chi \in S_h^0(\Omega_0).$$

Then for $0 < \delta \leqslant t \leqslant T$ *and any* $m \geqslant 0$,

$$\sum_{|\alpha| \leqslant m} \|\partial_h^\alpha U(t)\|_{\Omega_2}^2 \leqslant C \int_{t-\delta}^t \left[\|U\|_{\Omega_1}^2 + h^q \|U_t\|_{\Omega_1}^2 + \sum_{|\alpha| \leqslant m} \|\partial_h F\|_{-1,\Omega_1}^2 \right] d\tau.$$

Proof. Let $\varphi \in C^1$ be such that $\varphi(\tau) = 1$ for $\tau \geqslant 0$, $\varphi(\tau) = 0$ for $\tau \leqslant -\delta/2$ and set $\varphi_0(\tau) = \varphi(\tau - t)$. We have at time τ,

$$L_h(\varphi_0 U)(\chi) = \varphi_0 (L_h U)(\chi) + \varphi_0'(U, \chi) = (\varphi_0 F + \varphi_0' U, \chi) \quad \forall \chi \in S_h^0(\Omega_0).$$

Application of Lemma 3.3 hence gives

$$\sum_{|\alpha| \leqslant m} \|\partial_h^\alpha U(t)\|_{\Omega_2}^2$$

$$\leqslant C \int_{t-\delta/2}^t \left[\sum_{|\alpha| \leqslant \max(m-1,0)} \|\partial_h^\alpha U\|_{\Omega_1}^2 + h^q \|U_t\|_{\Omega_1}^2 + \sum_{|\alpha| \leqslant m} \|\partial_h^\alpha F\|_{-1,\Omega_1}^2 \right] d\tau.$$

The desired result now follows by an easy induction over m.

4. The Basic Interior Error Estimates. In this section we shall show the basic results of this paper, the interior maximum-norm estimates for the error in approximating a derivative of the exact solution of the parabolic equation by an appropriate

finite difference quotient of the semidiscrete solution. The first such estimate below, Theorem 4.1, is interior with respect to the space variables but uniform for t in $[0, T]$ whereas the second result, Theorem 4.2, is interior in both space and time, thus valid for t bounded away from zero. Both error estimates contain one term which is $O(h^r)$ under the appropriate local regularity assumptions and one term containing weak norm of the error in a larger domain. In our applications in Section 5 these latter terms will be majorized by global error bounds. For comparison, we shall finally state in Theorem 4.3 an interior in space estimate derived by a technique used in [2]. The error bound now contains time derivatives of the error.

We shall assume throughout this section without explicit mention that $\{S_h\}$ satisfies the regularity assumptions of Section 2 in the interior domain Ω_0. We shall consider the interior equations

$$(4.1) \qquad Lu = u_t + Au = f \quad \text{in} \quad \Omega_0 \times [0, T],$$

and with $u_h \colon [0, T] \longrightarrow S_h$,

$$(4.2) \qquad (L_h u_h)(\chi) = (u_{h,t}, \chi) + A(t; u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h^0(\Omega_0),$$

respectively.

We shall begin by deriving an interior L_2-estimate for difference quotients of the error.

LEMMA 4.1. *Assume that u and u_h are solutions of (4.1) and (4.2), respectively, and that $v_h = u_h(0)$ and $v = u(0)$ satisfy the interior equation*

$$(4.3) \qquad A_0(v_h - v, \chi) = 0 \quad \forall \chi \in S_h^0(\Omega_0),$$

where $A_0(\cdot, \cdot)$ is a bilinear form corresponding to a second order elliptic differential operator with smooth coefficients. Let $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$ and let p, q be arbitrary and m positive. Then for $e_h = u_h - u$,

$$(4.4) \qquad \sum_{|\alpha| \leqslant m} \|\partial_h^\alpha e_h(t)\|_{\Omega_2} \leqslant C\left\{h^r B_{r+m, \Omega_1}(t; u) + R_{\Omega_1}(t; e_h)\right\},$$

where

$$B_{s, \Omega_1}(t; u) = \sup_{0 \leqslant \tau \leqslant t} \|u(\tau)\|_{s, \Omega_1} + \left(\int_0^t \|u_t\|_{s-1, \Omega_1}^2 \, d\tau\right)^{1/2}$$

and

$$(4.5) \qquad R_{\Omega_1}(t; e_h) = \|e_h(0)\|_{-p, \Omega_1} + \left(\int_0^t (\|e_h\|_{\Omega_1}^2 + h^q \|e_{h,t}\|_{\Omega_1}^2) \, d\tau\right)^{1/2}.$$

Proof. Let $\widetilde{P} = \widetilde{P}_{\Omega_1}(t)$ denote the projection used in Section 2 with respect to $\widetilde{A}(\cdot, \cdot)$, and set $\theta = u_h - \widetilde{P}u$ and $\rho = (I - \widetilde{P})u$. We then have

$$(L_h\theta)(\chi) = \left(\rho_t + \sum_j a_j \frac{\partial \rho}{\partial x_j} + (a_0 - 1)\rho, \chi\right) \quad \forall \chi \in S_h^0(\Omega_1),$$

so that noticing that $\theta \in S_h(\Omega_0)$ and letting $\Omega_2 \subset\subset \Omega_1' \subset\subset \Omega_1$ we may apply Lemma 3.3 to obtain

$$\sum_{|\alpha| \leqslant m} \|\partial_h^\alpha \theta\|_{\Omega_2}^2$$

$$\leqslant C\left\{ \sum_{|\alpha| \leqslant m} \|\partial_h^\alpha \theta(0)\|_{\Omega_1'}^2 + \int_0^t \left[\|\theta\|_{\Omega_1'}^2 + h^q \|\theta_t\|_{\Omega_1'}^2 \right.\right.$$

$$\left.\left. + \sum_{|\alpha| \leqslant m} \left(\|\partial_h^\alpha \rho\|_{\Omega_1'}^2 + \|\partial_h^\alpha \rho_t\|_{-1,\Omega_1'}^2 \right) \right] d\tau \right\}.$$

Consider first the term with $\theta(0) = v_h - \widetilde{P}v$. We have for $|\alpha| \leqslant m$, applying Lemma 2.2 to $e_h(0)$ and Lemma 2.3 to $\rho(0)$,

$$\|\partial_h^\alpha \theta(0)\|_{\Omega_1'} \leqslant \|\partial_h e_h(0)\|_{\Omega_1'} + \|\partial_h^\alpha \rho(0)\|_{\Omega_1'}$$

$$\leqslant C\left\{ h^r \|u(0)\|_{r+m,\Omega_1} + \|e_h(0)\|_{-p,\Omega_1} \right\},$$

which is bounded by the right side of (4.4).

Further, assuming as we may, that $q \geqslant 2$, we have

$$\int_0^t \left[\|\theta\|_{\Omega_1'}^2 + h^q \|\theta_t\|_{\Omega_1'}^2 \right] d\tau$$

$$\leqslant C \int_0^t \left[\|e_h\|_{\Omega_1}^2 + h^q \|e_{h,t}\|_{\Omega_1}^2 + \|\rho\|_{\Omega_1}^2 + h^2 \|\rho_t\|_{\Omega_1}^2 \right] d\tau.$$

Here, the terms containing e_h are bounded by $C \cdot R_{\Omega_1}(t; e_h)^2$ and by (2.6) and (2.7) we have

$$\int_0^t (\|\rho\|_{\Omega_1}^2 + h^2 \|\rho_t\|_{\Omega_1}^2) d\tau \leqslant C h^{2r} \int_0^t (\|u\|_{r,\Omega_1}^2 + \|u_t\|_{r-1,\Omega_1}^2) d\tau.$$

Finally, by Lemma 2.3,

$$\int_0^t \sum_{|\alpha| \leqslant m} \left[\|\partial_h^\alpha \rho\|_{\Omega_1'}^2 + \|\partial_h^\alpha \rho_t\|_{-1,\Omega_1'}^2 \right] d\tau$$

$$\leqslant C h^{2r} \int_0^t \left[\|u\|_{r+m,\Omega_1}^2 + \|u_t\|_{r+m-1,\Omega_1}^2 \right] d\tau$$

so that altogether

$$(4.6) \qquad \sum_{|\alpha| \leqslant m} \|\partial_h^\alpha \theta\|_{\Omega_2} \leqslant C\left\{ h^r B_{r+m,\Omega_1}(t; u) + R_{\Omega_1}(t; e_h) \right\}.$$

Since by Lemma 2.3,

$$\sum_{|\alpha| \leqslant m} \|\partial_h^\alpha \rho(t)\|_{\Omega_2} \leqslant C h^r \|u(t)\|_{r+m,\Omega_1} \leqslant C h^r B_{r+m,\Omega_1}(t; u),$$

and since $e_h = \theta - \rho$ the desired result now follows by the triangle inequality.

We are now in a position to establish our first maximum-norm bound for the error between an arbitrary derivative $D^\alpha u$ ($|\alpha| = m$) and a corresponding finite difference

approximation $Q_h u_h$. Thus, let Q_h be defined, with finitely many constants $q_{\beta\gamma}$, by

$$Q_h w(x) = \sum_{|\beta|=m,\gamma} q_{\beta\gamma} \partial_h^\beta w(x - \gamma h).$$

We say that Q_h approximates D^α with order of accuracy r if

(4.7) $$|Q_h w - D^\alpha w|_{\Omega_2} \leqslant Ch^r |w|_{r+m,\Omega_1} \quad \text{if} \quad \Omega_2 \subset\subset \Omega_1.$$

We then have

THEOREM 4.1. *Assume that u, u_h, $v = u(0)$ and $v_h = u_h(0)$ satisfy (4.1), (4.2) and (4.3) and let Q_h be a finite difference operator approximating D^α ($|\alpha| = m$) with order of accuracy r. Let $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$ and let p, q be arbitrary. Then with the notation of Lemma 4.1 we have for $0 \leqslant t \leqslant T$ ($N_0 = [N/2] + 1$),*

$$|Q_h u_h(t) - D^\alpha u(t)| \leqslant C \left\{ h^r B_{r+m+N_0,\Omega_1}(t; u) + R_{\Omega_1}(t; e_h) \right\}.$$

Proof. Let Ω_1' and Ω_1'' be such that $\Omega_2 \subset\subset \Omega_1'' \subset\subset \Omega_1' \subset\subset \Omega_1$. We have by the form of Q_h and since $u_h - u = \theta - \rho$,

$$|Q_h u_h(t) - D^\alpha u(t)|_{\Omega_2} \leqslant C \sum_{|\alpha|=m} (|\partial_h^\alpha \theta(t)| + |\partial_h^\alpha \rho(t)|) + |(Q_h - D^\alpha) u(t)|_{\Omega_2}.$$

Here by (4.7),

$$|(Q_h - D^\alpha) u(t)|_{\Omega_2} \leqslant Ch^r |u(t)|_{r+m,\Omega_1} \leqslant Ch^r B_{r+m+N_0,\Omega_1}(t; u).$$

Using the discrete Sobolev inequality,

$$|\chi|_{\Omega_1''} \leqslant C \sum_{|\beta|\leqslant N_0} \|\partial_h^\beta \chi\|_{\Omega_1'} \quad \forall \chi \in S_h(\Omega_1),$$

we have by (4.6),

$$\sum_{|\alpha|\leqslant m} |\partial_h^\alpha \theta(t)|_{\Omega_1''} \leqslant C \sum_{|\alpha|\leqslant m+N_0} \|\partial_h^\alpha \theta(t)\|_{\Omega_1'}$$

$$\leqslant C \left\{ h^r B_{r+m+N_0,\Omega_1}(t; u) + R_{\Omega_1}(t; e_h) \right\}.$$

Since finally by Lemma 2.3,

$$\sum_{|\alpha|\leqslant m} |\partial_h^\alpha \rho(t)|_{\Omega_1''} \leqslant Ch^r \|u(t)\|_{r+m+N_0,\Omega_1} \leqslant Ch^r B_{r+m+N_0,\Omega_1}(t; u),$$

the proof is complete.

We shall now deduce an estimate which is interior also with respect to time. The result is now independent of the choice of the initial data and the low order norms of e_h in the error bound are taken only over a short interval preceding the time at which the estimate for $Q_h u_h - D^\alpha u$ is given.

THEOREM 4.2. *Assume that u and u_h satisfy (4.1) and (4.2), respectively, and let Q_h be a finite difference operator approximating D^α ($|\alpha| = m$) with order of accuracy r. Let $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$ and let p, q be arbitrary. Then for $0 < \delta \leqslant t \leqslant T$,*

with $e_h = u_h - u$,

$$|Q_h u_h(t) - D^\alpha u(t)|_{\Omega_2} \leqslant C\{h^r B_{r+m+N_0, \Omega_1}(t - \delta, t; u) + R_{\Omega_1}(t - \delta, t; e_h)\},$$

where

$$B_s(t - \delta, t; u) = \sup_{t-\delta \leqslant \tau \leqslant t} \|u(\tau)\|_{s, \Omega_1} + \left(\int_{t-\delta}^t \|u_t\|_{s-1, \Omega_1}^2 \, d\tau\right)^{1/2}$$

and

$$(4.8) \qquad R_{\Omega_1}(t - \delta, t; e_h) = \left(\int_{t-\delta}^t (\|e_h\|_{\Omega_1}^2 + h^q \|e_{h,t}\|_{\Omega_1}^2) \, d\tau\right)^{1/2}.$$

   Proof.   First, we prove in exactly the same way as in Lemma 4.1, using only Lemma 3.4 instead of Lemma 3.3, that for $m > 0$,

$$\sum_{|\alpha| \leqslant m} \|\partial_h^\alpha e_h(t)\|_{\Omega_2} \leqslant C\{h^r B_{r+m, \Omega_1}(t - \delta, t; u) + R_{\Omega_1}(t - \delta, t; e_h)\}.$$

The proof is then completed as in Theorem 4.1.

   We shall now state, for comparison, the result obtained by the technique used for the homogeneous equation in [2], consisting essentially in considering the time derivative in the parabolic equation as a forcing term in an elliptic equation. The result thus derived is weaker than the ones just proved in that the error bound now contains derivatives of e_h with respect to time. In our applications in Section 5 this will require stronger regularity properties of u than our previous results. In order to make the reference to the proof in [2] simple we shall therefore content ourselves with formulating the result in the generality concerning the operator A employed in [2], and in Section 5 below.

   THEOREM 4.3.   *Assume that A is time-independent, selfadjoint and nonnegative $(a_0 \geqslant 0)$ and let u and u_h be solutions of (4.1) and (4.2), respectively. Let Q_h be a finite difference operator approximating D^α ($|\alpha| = m$) with order of accuracy r. Then if $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$ we have for $0 \leqslant t \leqslant T$, with $e_h = u_h - u$,*

$$|Q_h u_h(t) - D^\alpha u(t)|_{\Omega_2}$$

$$\leqslant C\left\{h^r \sum_{2l \leqslant m + N_0} \|D_t^l u(t)\|_{r+m+N_0-2l, \Omega_1} + \sum_{2l \leqslant m+N_0+2} \|D_t^l e_h(t)\|_{\Omega_1}\right\}.$$

   Proof.   This is proved step by step as [2, Theorem 6.1] with the obvious modifications due to the fact that the equation may now be nonhomogeneous.

   5. Some Global Error Estimates for a Class of Time-Independent Problems. In this section we shall supply the global error estimates needed for bounding the terms in e_h in Theorems 4.1 through 4.3 in the special case of the generalization to non-homogeneous equations of the theory developed in [2] for homogeneous parabolic equations. These estimates will all be of order $O(h^r)$ for sufficiently smooth data, and

application of the results of Section 4 will thus show this order of interior convergence of difference quotients of the approximate solution to derivatives of the exact solution We shall leave to the reader to combine the regularity requirements of the global estimates below with those of our previous interior results.

Consider thus the initial-boundary value problem

$$u_t + Au = f \quad \text{in} \quad \Omega \times [0, T],$$

(5.1) $$u = 0 \quad \text{on} \quad \partial\Omega \times [0, T],$$

$$u(0) = v,$$

where A is now a selfadjoint, nonnegative second order uniformly elliptic differential operator with smooth coefficients, independent of time. Let $\{S_h\}$ be a family of finite dimensional subspaces of $H^1(\Omega)$ and assume that corresponding to each S_h we are given an approximate solution operator T_h: $L_2(\Omega) \longrightarrow S_h$ of the corresponding elliptic problem

$$Au = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

such that

(i) T_h is selfadjoint, positive semidefinite on $L_2(\Omega)$ and positive definite on S_h; and with $T = A^{-1}$ and some $r \geqslant 2$,

(ii) There is a constant C such that

$$\|(T_h - T)f\| \leqslant Ch^{s+2}\|f\|_s, \quad f \in H^s(\Omega), 0 \leqslant s \leqslant r - 2.$$

Consider now the approximate semidiscrete problem of finding u_h: $[0, T] \longrightarrow S_h$ such that

$$T_h u_{h,t} + u_h = T_h f \quad \text{for} \quad t \in [0, T],$$

(5.2)

$$u_h(0) = v_h \in S_h,$$

where v_h is some approximation of v. Clearly, since T_h is positive definite on S_h, u_h is well defined by (5.2) for $t > 0$.

If $\{S_h\}$ satisfies certain standard approximation properties and if the elements of S_h vanish on $\partial\Omega$, operators satisfying conditions of types (i) and (ii) may be defined by the ordinary Galerkin equation

$$A(T_h f, \chi) = (f, \chi) \quad \forall \chi \in S_h;$$

the semidiscrete equation may then be written

$$(u_{h,t}, \chi) + A(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h.$$

An important feature of the present formulation is that it encompasses also several different other procedures for dealing with the homogeneous boundary conditions (cf. [2]).

If in addition to (i) and (ii) we demand that the family $\{S_h\}$ satisfies the regularity assumptions of Section 2 on the interior subdomain Ω_0 of Ω, and that the

operator T_h is such that

(iii) $$A(T_h f, \chi) = (f, \chi) \quad \forall \chi \in S_h^0(\Omega_0),$$

so that the semidiscrete interior equation

$$(u_{h,t}, \chi) + A(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h^0(\Omega_0)$$

holds for the solution of (5.2), then the results of Section 4 all apply in the present context. The property (iii) is shared by several of the methods referred to above satisfying (i) and (ii).

In the first part of this section we shall assume that $v \in \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$ and use for the initial-values v_h of u_h the elliptic projection of v defined by

$$v_h = P_1 v = T_h A v.$$

Recall that for $0 \leqslant s \leqslant r - 2$,

(5.3) $$\|(P_1 - I)v\|_{-s} = \|(T_h - T)Av\|_{-s} \leqslant Ch^r \|v\|_{r-s}.$$

In fact, we have by (ii),

$$\|(T_h - T)f\|_0 \leqslant Ch^r \|f\|_{r-2},$$

and hence, since $T_h - T$ is selfadjoint,

$$\|(T_h - T)f\|_{-(r-2)} = \sup_{\varphi \in C_0^\infty(\Omega)} \frac{|((T_h - T)f, \varphi)|}{\|\varphi\|_{r-2}} \leqslant Ch^r \|f\|_0.$$

We conclude by interpolation for $0 \leqslant s \leqslant r - 2$,

$$\|(T_h - T)f\|_{-s} \leqslant Ch^r \|f\|_{r-2-s},$$

from which (5.3) follows for $f = Av$.

We shall now derive the following global error estimate for the nonhomogeneous equation.

LEMMA 5.1.   *Under the above assumptions, we have for $e_h = u_h - u$,*

$$\left(\int_0^t \|e_h\|^2 \, d\tau \right)^{1/2} \leqslant Ch^r \left\{ \|v\|_{\max(r-1,2)} + \left(\int_0^t \|u\|_r^2 \, d\tau \right)^{1/2} \right\}.$$

Proof.   We notice (cf. [2]) that the error satisfies the equation

$$T_h e_{h,t} + e_h = \rho \equiv (T - T_h)Au = (I - P_1)u \quad \text{for} \quad t \geqslant 0.$$

This implies immediately

$$\frac{1}{2} \frac{d}{dt}(T_h e_h, e_h) + \|e_h\|^2 = (\rho, e_h) \leqslant \frac{1}{2} \|\rho\|^2 + \frac{1}{2} \|e_h\|^2,$$

and hence by integration

$$\int_0^t \|e_h\|^2 \, d\tau \leqslant (T_h e_h(0), e_h(0)) + \int_0^t \|\rho\|^2 \, d\tau.$$

Here, using the boundedness of T: $H^{-1}(\Omega) \longrightarrow \overset{\circ}{H}{}^1(\Omega)$, and the property (ii) we have

$$(5.4) \qquad (T_h w, w) = (Tw, w) + ((T_h - T)w, w) \leqslant C\|w\|^2_{-1} + Ch^2\|w\|^2,$$

and hence in view of (5.3),

$$(T_h e_h(0), e_h(0)) \leqslant Ch^{2r}\|v\|^2_{\max(r-1,2)}.$$

Further, by (ii),

$$\int_0^t \|\rho\|^2 \, d\tau = \int_0^t \|(T - T_h)Au\|^2 \, d\tau \leqslant Ch^{2r}\int_0^t \|u\|_r^2 \, d\tau.$$

Together these estimates prove the lemma.

We shall now consider the boundedness of $e_{h,t}$.

LEMMA 5.2. *Under the above assumptions, we have for $e_h = u_h - u$,*

$$\left(\int_0^t \|e_{h,t}\|^2 \, d\tau\right)^{1/2} \leqslant C\left\{\|v\|_2 + \left(\int_0^t \|f\|^2 \, d\tau\right)^{1/2}\right\}.$$

Proof. We have for the exact solution

$$\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}A(u, u) = (f, u_t) \leqslant \frac{1}{2}\|f\|^2 + \frac{1}{2}\|u_t\|^2,$$

and hence

$$(5.5) \qquad \int_0^t \|u_t\|^2 \, d\tau \leqslant A(v, v) + \int_0^t \|f\|^2 \, d\tau \leqslant C\left\{\|v\|_1^2 + \int_0^t \|f\|^2 \, d\tau\right\}.$$

Similarly, with $A_h = T_h^{-1}$ on S_h,

$$\|u_{h,t}\|^2 + \frac{1}{2}\frac{d}{dt}(A_h u_h, u_h) = (f, u_{h,t}) \leqslant \frac{1}{2}\|f\|^2 + \frac{1}{2}\|u_{h,t}\|^2,$$

so that

$$(5.6) \qquad \int_0^t \|u_{h,t}\|^2 \, d\tau \leqslant (A_h v_h, v_h) + \int_0^t \|f\|^2 \, d\tau.$$

Here $A_h v_h = A_h T_h Av = P_0 Av$ so that by the boundedness of T_h,

$$(5.7) \qquad (A_h v_h, v_h) = (Av, T_h Av) \leqslant C\|Av\|^2 \leqslant C\|v\|_2^2.$$

The result, therefore, follows by (5.5), (5.6), (5.7) and the triangle inequality.

Combination of these results now implies the following estimate suitable for combination with Theorem 4.1.

THEOREM 5.1. *Assume that (i) and (ii) hold and let u and u_h be the solutions of (5.1) and (5.2), respectively, with $v_h = P_1 v$. Then, with $p = r - 2$, $q = 2r$ in (4.5), we have for $e_h = u_h - u$,*

$$R_{\Omega_1}(t; e_h) \leqslant \|e_h(0)\|_{-(r-2)} + \left(\int_0^t (\|e_h\|^2 + h^{2r}\|e_{h,t}\|^2) \, d\tau\right)^{1/2}$$

$$\leqslant Ch^r\left\{\|v\|_{\max(r-1,2)} + \left(\int_0^t (\|u\|_r^2 + \|f\|^2) \, d\tau\right)^{1/2}\right\}.$$

Proof. We have by (5.3),

$$\|e_h(0)\|_{-(r-2)} = \|(P_1 - I)v\|_{-(r-2)} \leqslant Ch^r\|v\|_2,$$

and by Lemmas 5.1 and 5.2,

$$\int_0^t (\|e_h\|^2 + h^{2r}\|e_{h,t}\|^2)\, d\tau$$

$$\leqslant Ch^{2r}\left\{\|v\|_{\max(r-1,2)}^2 + \int_0^t(\|u\|_r^2 + \|f\|^2)\, d\tau\right\}.$$

Together, these estimates prove the theorem.

Combined with Theorem 4.1 we may conclude, under the appropriate assumptions, that for $\Omega_2 \subset\subset \Omega_1 \subset \Omega_0$,

$$|Q_h u_h(t) - D^\alpha u(t)|_{\Omega_2}$$

$$\leqslant Ch^r\left\{\|v\|_{\max(r-1,2)} + \left(\int_0^t(\|u\|_r^2 + \|f\|^2)\, d\tau\right)^{1/2} + B_{r+m+N_0,\Omega_1}(t;u)\right\}.$$

The regularity demands on u in order that the right-hand side of this inequality be finite can also be expressed exclusively in terms of the data f and v of the problem (cf. e.g. [8]). In addition to regularity of these functions one then also has to impose compatibility conditions between them and the differential operator at $\partial\Omega$ for $t = 0$.

We shall now show, using the results of [2] that if we are content with error estimates for time bounded away from zero, then the regularity demands for a $O(h^r)$ result reduce considerably. In such a case, in order to derive the estimate for $Q_h u_h - D^\alpha u$ at time t we shall only have to require strong regularity for a short time preceding t and the compatibility requirements at $\partial\Omega \times \{t = 0\}$ disappear. We shall also have more freedom in the choice of approximate initial-values v_h; we need to require only that $v - v_h = O(h^r)$ in some negative norm and that v_h is bounded in L_2. As examples, we notice

(5.8) $$\|v - P_0 v\|_{-r} + h^r\|P_0 v\| \leqslant Ch^r\|v\|,$$

and (cf. [2]),

$$\|v - P_1 v\|_{-(r-2)} + h^r\|P_1 v\| \leqslant Ch^r\|v\|_2.$$

We begin by recalling the relevant result from [2], the following global error estimate for the homgeneous equation [2, Theorem 3.3].

LEMMA 5.3. *Assume that* (i) *and* (ii) *hold and that u and u_h are the solutions of* (5.1) *and* (5.2) *with $f = 0$. Then for l, p nonnegative and arbitrary we have for $e_h = u_h - u$ (with $D_t = \partial/\partial t$),*

$$\|D_t^l e_h(t)\| \leqslant C(h^r t^{-r/2-l}\|v_h\| + t^{-p/2-l}\|v_h - v\|_{-p}).$$

We are now in a position to show the following global in space, interior in time error estimate, which is what is needed for application of Theorem 4.2.

THEOREM 5.2. *Assume that* (i) *and* (ii) *hold and let* u *and* u_h *be the solutions of* (5.1) *and* (5.2), *respectively, with* v_h *arbitrary. Then with* $q = 2r$ *in* (4.8), *we have for arbitrary* p, $0 < \delta \leqslant t \leqslant T$, *and* $e_h = u_h - u$,

$$R_{\Omega_1}(t - \delta/2, t; e_h) \leqslant \left(\int_{t-\delta/2}^t (\|e_h\|^2 + h^{2r}\|e_{h,t}\|^2)\, d\tau \right)^{1/2}$$

$$(5.9) \qquad \leqslant Ch^r \left\{ \|v_h\| + \|v\| + h^{-r}\|v - v_h\|_{-p} \right.$$

$$\left. + \left(\int_{t-\delta}^t (\|u\|_r^2 + \|f\|^2)\, d\tau \right)^{1/2} + \int_0^t \|f\|\, d\tau \right\}.$$

PROOF. We shall consider a fixed $t = t_1 \geqslant \delta$. Let $\varphi \in C^\infty$ be such that $\varphi(t) = 1$ for $t \geqslant -3\delta/4$, $\varphi(t) = 0$ for $t \leqslant -\delta$. Set $\varphi_1(t) = \varphi(t - t_1)$. We now write $u = u_1 + u_2 + u_3$ where $u_1 = u\varphi_1$ and u_2 is the solution of the homogeneous equation,

$$(5.10) \qquad Lu_2 = 0 \quad \text{for} \quad t \geqslant 0, u_2(0) = v.$$

Since

$$(5.11) \qquad Lu_1 = f_1 \equiv f\varphi_1 + u\varphi_1' \quad \text{for} \quad t \geqslant 0, u_1(0) = 0,$$

it follows that u_3 satisfies

$$(5.12) \qquad Lu_3 = f_3 \equiv f(1 - \varphi_1) - u\varphi_1' \quad \text{for} \quad t \geqslant 0, u_3(0) = 0.$$

We notice that f_1 and f_3 vanish for $t \leqslant t_1 - \delta$ and $t \geqslant t_1 - 3\delta/4$, respectively.

Let $u_{j,h}$, $j = 1, 2, 3$, be the semidiscrete approximations of problems (5.11), (5.10), (5.12) with $u_{1,h}(0) = u_{3,h}(0) = 0$, $u_{2,h}(0) = v_h$ and set $e_{j,h} = u_{j,h} - u_j$. Since by linearity $e_h = u_h - u = \Sigma_j e_{j,h}$ it hence suffices to estimate $R_{\Omega_1}(t_1 - \delta/2, t_1; e_{j,h})$, $j = 1, 2, 3$, by the right-hand side of (5.9) (with t replaced by t_1).

Consider first u_1. We have by Theorem 5.1, using the definitions of u_1 and f_1,

$$R_{\Omega_1}(t_1 - \delta/2, t_1; e_{1,h})^2 \leqslant R_{\Omega_1}(t_1; e_{1,h})^2 \leqslant C\int_0^{t_1} (\|u_1\|_r^2 + \|f_1\|^2)\, d\tau$$

$$\leqslant C\int_{t_1-\delta}^{t_1} (\|u\|_r^2 + \|f\|^2)\, d\tau.$$

For u_2, the solution of the homogeneous equation, we have by Lemma 5.3,

$$(5.13) \qquad \|D_t^l e_{2,h}(t_1)\| \leqslant C\{h^r\|v_h\| + \|v - v_h\|_{-p}\} \quad \text{for} \quad l \geqslant 0,$$

so that in particular

$$R_{\Omega_1}(t_1 - \delta/2, t_1; e_{2,h}) \leqslant C\{h^r\|v_h\| + \|v - v_h\|_{-p}\}.$$

For the purpose of dealing with u_3, finally, we introduce the solution operators $E(t)$ and $E_h(t)$ of the initial-value problems for the homogeneous exact and semidis-

crete equations. Setting $F_h(t) = E_h(t)P_0 - E(t)$, we notice that Lemma 5.3 implies, by choosing $v_h = P_0 v$ (cf. (5.8)),

$$\|D_t^l F_h(t)v\| \leqslant Ch^r\|v\| \quad \text{for} \quad t \geqslant \delta/4.$$

We observe now that by superposition we may write, for $t \geqslant t_1 - \delta/2$,

$$e_{3,h}(t) = \int_0^t F_h(t-\tau)f_3(\tau) \, d\tau = \int_0^{t_1 - 3\delta/4} F_h(t-\tau)f_3(\tau) \, d\tau$$

and also

$$D_t^l e_{3,h}(t) = \int_0^{t_1 - 3\delta/4} D_t^l F_h(t-\tau)f_3(\tau) \, d\tau \quad \text{for} \quad l \geqslant 0,$$

so that for $t \geqslant t_1 - \delta/2$,

$$\|D_t^l e_{3,h}(t)\| \leqslant Ch^r \int_0^{t_1 - 3\delta/4} \|f_3\| \, d\tau$$

(5.14)

$$\leqslant Ch^r \int_0^{t_1} (\|f\| + \|u\|) \, d\tau \leqslant Ch^r \left(\|v\| + \int_0^{t_1} \|f\| \, d\tau \right).$$

Here the last step follows by the fact that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + A(u, u) = (f, u) \leqslant \|f\| \, \|u\|$$

implies

$$\frac{d}{dt}\|u\| \leqslant \|f\|;$$

and hence for $t \geqslant 0$,

$$\|u(t)\| \leqslant \|v\| + \int_0^t \|f\| \, d\tau.$$

It follows in particular from (5.14) that

$$R_{\Omega_1}(t_1 - \delta/2, t_1; e_{3,h}) \leqslant Ch^r \left\{ \|v\| + \int_0^{t_1} \|f\| \, d\tau \right\},$$

which completes the proof of the theorem.

For the purpose of deriving finally the error estimates for time derivatives needed for application of Theorem 4.3 we shall first show the following error bound for the nonhomogeneous equation.

LEMMA 5.5. *Assume that* (i) *and* (ii) *hold and that u and u_h are the solutions of the problems* (5.1) *and* (5.2). *Let $v_h = P_1 v = T_h Av$. Then for $e_h = u_h - u$,*

$$\|e_h(t)\| \leqslant Ch^r \left\{ \|u(t)\| + \left(\int_0^t \|u_t\|_{\max(r-1,2)}^2 \, d\tau \right)^{1/2} \right\}.$$

Proof. Setting $\theta = e_h - \rho$, we have by the error equation (cf. Lemma 5.1),

$$T_h \theta_t + \theta = - T_h \rho_t.$$

After multiplication by θ_t we obtain by Schwarz's inequality

$$(T_h\theta_t, \theta_t) + \frac{1}{2}\frac{d}{dt}\|\theta\|^2 = -(T_h\rho_t, \theta_t) \leqslant \frac{1}{2}(T_h\rho_t, \rho_t) + \frac{1}{2}(T_h\theta_t, \theta_t),$$

so that after integration, since $\theta(0) = 0$,

$$\|\theta(t)\|^2 \leqslant \int_0^t (T_h\rho_t, \rho_t)\,d\tau.$$

As in (5.4), (5.3) we obtain, noticing that the elliptic projection commutes with time-differentiation,

$$(T_h\rho_t, \rho_t) \leqslant C\{\|\rho_t\|_{-1}^2 + h^2\|\rho_t\|^2\} \leqslant Ch^{2r}\|u_t\|_{\max(r-1,2)}^2,$$

implying

$$\|\theta(t)\|^2 \leqslant Ch^{2r}\int_0^t \|u_t\|_{\max(r-1,2)}^2\,d\tau.$$

This proves the lemma since

$$\|\rho(t)\| \leqslant Ch^r\|u(t)\|_r.$$

As a result, we may now establish the following global in space, interior in time error estimates for time derivatives.

THEOREM 5.3. *Assume that* (i) *and* (ii) *hold and let* u *and* u_h *be the solutions of* (5.1) *and* (5.2), *respectively, with* v_h *arbitrary. Then we have for arbitrary nonnegative* l, p, $0 < \delta \leqslant t \leqslant T$, *and* $e_h = u_h - u$,

$$\|D_t^l e_h(t)\| \leqslant Ch^r \left\{ \|v_h\| + \|v\| + h^{-r}\|v - v_h\|_{-p} + \|D_t^l u(t)\| \right.$$

$$\left. + \left(\int_{t-\delta}^t \sum_{j=0}^{l+1} \|D_t^j u\|_{\max(r-1,2)}^2\,d\tau\right)^{1/2} + \int_0^t \|f\|\,d\tau \right\}.$$

Proof. Consider again a fixed $t = t_1 \geqslant \delta$ and decompose the problem into problems (5.11), (5.10) and (5.12) as in the proof of Theorem 5.2. The latter two problems are treated exactly as before and we obtain by (5.13) and (5.14) for $t_1 \geqslant \delta > 0$, $l \geqslant 0$,

$$\|D_t^l e_{2,h}(t_1)\| \leqslant C\{h^r\|v_h\| + \|v - v_h\|_{-p}\},$$

and

$$\|D_t^l e_{3,h}(t_1)\| \leqslant Ch^r\left(\|v\| + \int_0^{t_1}\|f\|\,d\tau\right).$$

It remains to consider $u_1 = u\varphi_1$. We notice then that for any l, $D_t^l u_1$ and $D_t^l u_{1,h}$ satisfy equations of the form (5.1) and (5.2), respectively, with $D_t^l u_{1,h}(0) = P_1 D_t^l u_1(0) = 0$. We may, therefore, apply Lemma 5.4 to obtain

$$\|D_t^l e_{1,h}(t_1)\| \leqslant Ch^r\left\{\|D_t^l u_1(t_1)\| + \left(\int_0^{t_1}\|D_t^{l+1}u_1\|_{\max(r-1,2)}^2\,d\tau\right)^{1/2}\right\}$$

$$\leqslant Ch^r\left\{\|D_t^l u(t_1)\| + \left(\int_{t_1-\delta}^{t_1}\sum_{j=0}^{l+1}\|D_t^j u\|_{\max(r-1,2)}^2\,d\tau\right)^{1/2}\right\}.$$

Together these estimates prove the theorem.

Clearly, the estimate obtained by combination of Theorems 4.3 and 5.3 consumes more global regularity in space than the estimate derived by Theorems 4.2 and 5.2.

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