Note on the Kelvin Phase Functions

By J. R. Philip

Abstract. Both theoretical consistency and practical convenience demand that the values
of the Kelvin phase functions $\theta_n(x)$ and $\phi_n(x)$ be defined uniquely and be well-ordered
in n. This is achieved by taking, for $n \geqslant 0$, $\theta_n(0) = -\phi_n(0) = 3n\pi/4$. The necessary
amendments to extant tables are indicated.

1. Introduction. The Kelvin phase functions $\theta_n(x)$ and $\phi_n(x)$ are customarily de-
fined [1, p. 122], [2, Eqs. 6.03 a, b] through the equations

$$(1) \qquad \tan \theta_n(x) = \mathrm{bei}_n\, x/\mathrm{ber}_n\, x, \qquad \tan \phi_n(x) = \mathrm{kei}_n\, x/\mathrm{ker}_n\, x,$$

or equivalent relations [3, p. 211], [4, p. 438], [5, Eqs. 9.10.8 and 9.10.18], the mod-
ulus functions being nonnegative. The values of the functions are left arbitrary by the
quantity $2k\pi$, with k any integer. (Partial exceptions are the definitions involving arc tar
given by Olver [5]; these, presumably, carry the implications $|\theta_n| \leqslant \frac{1}{2}\pi$, $|\phi_n| \leqslant \frac{1}{2}\pi$.
Olver, correctly, disregards these restrictions of values in the remainder of the article.)

Various authors seem to have accepted the arbitrariness in (1) as inherent. As we
shall show, this has led to needless confusion in the literature and, in particular, in the
tabulations of $\theta_n(x)$ and $\phi_n(x)$. In this note we present means of fixing the values of
$\theta_n(x)$ and $\phi_n(x)$ definitely and uniquely; we point out the theoretical and practical ad-
vantages of doing so; and we indicate the required amendments to extant tables of
$\theta_n(x)$ and $\phi_n(x)$. The present study is limited to all real nonnegative values of x and n.

2. Fixing the Values of $\theta_n(0)$, $\phi_n(0)$. It follows from the various ascending series
for the Kelvin functions [2, Eqs. 2.1–2.4], [5, Eqs. 9.9.9–9.9.12] that, for $n \geqslant 0$,

$$\tan \theta_n(0) = \tan (3n\pi/4), \qquad \tan \phi_n(0) = \tan (-3n\pi/4);$$

that is

$$(2) \qquad n \geqslant 0, \qquad \theta_n(0) = 3n\pi/4 + 2k_1\pi, \qquad \phi_n(0) = -3n\pi/4 + 2k_2\pi,$$

with k_1 and k_2 integers.

Since Bessel functions are functions of the order as well as of the argument, $\theta_n(x)$
and $\phi_n(x)$ are continuous functions of n. We conclude that we must take k_1 constant,
independent of n, since otherwise $\theta_n(0)$ cannot be a continuous function of n; and, by
the same argument, that we must also take k_2 constant, independent of n.

We now fix k_1, k_2 by requiring that the values of $\theta_n(x)$, $\phi_n(x)$ be consistent with
the asymptotic expansions [2, Eqs. 10.22, 10.25b, 10.20, 10.26b], [5, Eqs. 9.10.23
and 9.10.26]

(3) $$\theta_n(x) \sim \frac{x}{\sqrt{2}} + \left(\frac{1}{2}n - \frac{1}{8}\right)\pi + \frac{(4n^2 - 1)}{8\sqrt{2x}} + \frac{(4n^2 - 1)}{16x^2} - \cdots,$$

(4) $$\phi_n(x) \sim -\frac{x}{\sqrt{2}} - \left(\frac{1}{2}n + \frac{1}{8}\right)\pi - \frac{(4n^2 - 1)}{8\sqrt{2x}} + \frac{(4n^2 - 1)}{16x^2} + \cdots.$$

It is readily shown, e.g. by comparisons for $n = 0$, that consistency with (3) and (4) requires that

$$k_1 = 0, \qquad k_2 = 0.$$

We, therefore, have the required basis for fixing values of $\theta_n(x)$, $\phi_n(x)$, namely

(5) $$n \geqslant 0, \qquad \theta_n(0) = 3n\pi/4, \qquad \phi_n(0) = -3n\pi/4.$$

There are strong reasons why tabulations of $\theta_n(x)$ and $\phi_n(x)$ should be made to satisfy Eqs. (5). Firstly, conformity with (5) is highly desirable (if not absolutely essential) if the tables are to be used in conjunction with evaluation of asymptotic expansions (3) and (4) for large x. Secondly, it ensures that $\theta_n(x)$ and $\phi_n(x)$ are continuous functions of n, with functional values varying regularly and systematically as n increases. As well as yielding intellectual and aesthetic advantages, making $\theta_n(x)$ and $\phi_n(x)$ continuous well-behaved functions of n opens up the possibility of interpolation with respect to n. Thirdly, in physical applications, conformity with (5) greatly simplifies elucidation of phase relations among various physical quantities (e.g. between concentration and flux density, and between incident and reflected waves, in periodic diffusion with spatially-dependent diffusivity [6]).

3. **The Young-Kirk Tables. Required Amendments.** Even though Young and Kirk [2] give asymptotic expansions equivalent to (3) and (4), their tabulated values of $\theta_n(x)$ and $\phi_n(x)$ are, in general, inconsistent with the asymptotic expansions. Their values of $\theta_n(0)$ and $\phi_n(0)$ were chosen so that

$$n = 0, 1, \ldots, 10, \qquad 0 \leqslant \theta_n(0) \leqslant 7\pi/4, \qquad -\pi/4 \leqslant \phi_n(0) \leqslant 3\pi/2.$$

In consequence their values fail in general to satisfy (5).
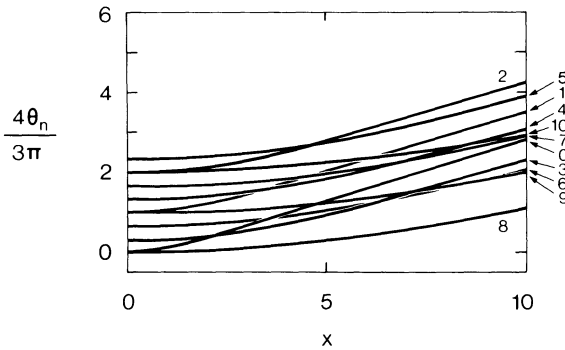

FIGURE 1

The set of $\theta_n(x)$ according to [2]. *Numerals on the curves denote values of n*
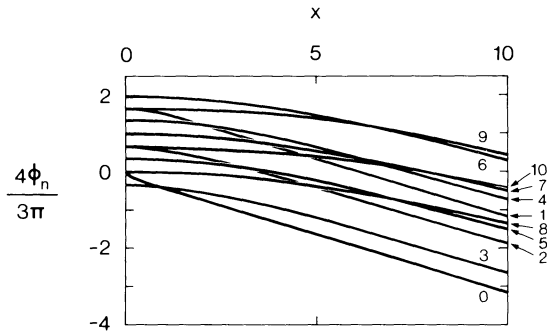
FIGURE 2

The set of $\phi_n(x)$ according to [2]. *Numerals on the curves denote values of n*

It follows that the Young-Kirk tables [2] display no orderly progression of values of $\theta_n(x)$ and $\phi_n(x)$ as n increases. Figures 1 and 2 show the confused picture of these functions according to these tables.

On the other hand, as we have already indicated, the inherent order of these func-
tions emerges when $\theta_n(0)$ and $\phi_n(0)$ are fixed by (5). Figures 3 and 4 display the orderly set of functions which follow. The comparison with Figures 1 and 2 is instructive.

TABLE 1

Required amendments to column headings of Young-Kirk tables [2]

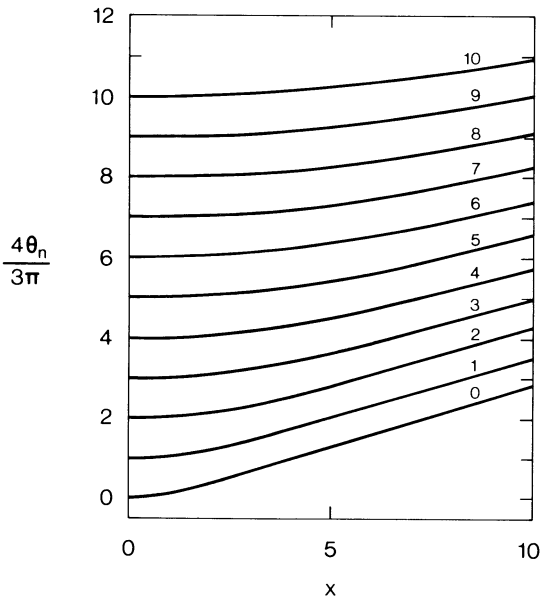| Pages	Column Heading	Change Column Heading to
48,50	$\theta_3(x)$	$\theta_3(x) - 2\pi$
52,54	$\theta_4(x) - \pi$	$\theta_4(x) - 3\pi$
56,58	$\theta_5(x) - \frac{3}{2}\pi$	$\theta_5(x) - \frac{7}{2}\pi$
60,62	$\theta_6(x) - \frac{1}{2}\pi$	$\theta_6(x) - \frac{9}{2}\pi$
64,66	$\theta_7(x) - \pi$	$\theta_7(x) - 5\pi$
68,70	$\theta_8(x)$	$\theta_8(x) - 6\pi$
72,74	$\theta_9(x) - \frac{1}{2}\pi$	$\theta_9(x) - \frac{13}{2}\pi$
76,78	$\theta_{10}(x) - \frac{3}{2}\pi$	$\theta_{10}(x) - \frac{15}{2}\pi$
17,19,21,23,25,41,43⁻	$\phi_1(x) - \frac{3}{2}\pi$	$\phi_1(x) + \frac{1}{2}\pi$
27,29,31,33,35,45,47	$\phi_2(x) - \frac{1}{2}\pi$	$\phi_2(x) + \frac{3}{2}\pi$
49,51	$\phi_3(x)$	$\phi_3(x) + 2\pi$
53,55	$\phi_4(x) - \pi$	$\phi_4(x) + 3\pi$
57,59	$\phi_5(x) - \frac{1}{2}\pi$	$\phi_5(x) + \frac{7}{2}\pi$
61,63	$\phi_6(x) - \frac{3}{2}\pi$	$\phi_6(x) + \frac{9}{2}\pi$
65,67	$\phi_7(x) - \pi$	$\phi_7(x) + 5\pi$
69,71	$\phi_8(x)$	$\phi_8(x) + 6\pi$
73,75	$\phi_9(x) - \frac{3}{2}\pi$	$\phi_9(x) + \frac{13}{2}\pi$
77,79	$\phi_{10}(x) - \frac{1}{2}\pi$	$\phi_{10}(x) + \frac{15}{2}\pi$

FIGURE 3

The set of $\theta_n(x)$ satisfying Eqs. (5) and proposed here.
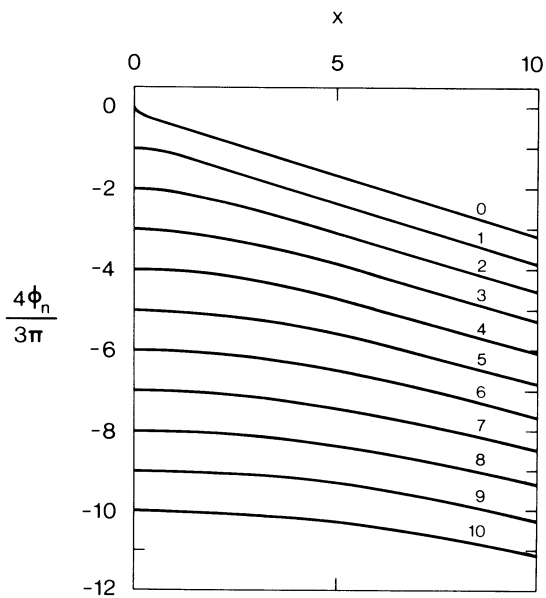*Numerals on the curves denote values of n*

FIGURE 4

The set of $\phi_n(x)$ satisfying Eqs. (5) and proposed here.
Numerals on the curves denote values of n

The Young-Kirk tables may be corrected so that they conform with (5) and yield well-ordered functions. The required changes are set out in Table 1.

4. Amendments to Other Tables. Various other tables of $\phi_1(x)$ require amendment. McLachlan and Meyers [7] present as $\phi_1(x)$ what, from the present viewpoint, is $\phi_1(x) + 360°$. McLachlan [8], on the other hand, gives a table of $\phi_1(x)$ satisfying (5). Fletcher et al. [4] incorrectly list the two tables as identical. Jahnke-Emde-Lösch [3, pp. 241–247] describes the graphed and tabulated quantity η_1 as $-\phi_1 + 90°$; it is, according to us, $-\phi_1 - 270°$. The graphed and tabulated quantity $\eta_0 - \eta_1$ is described as $-180° + \phi_1 - \phi_0$; it is, according to us, $\phi_1 - \phi_0 + 180°$. Fletcher et al. [4, p. 432] offer, as 'help in identification', the value $\phi_1(1) = 198°.10$. We consider that $\phi_1(1) = -161°.90$ and that their identified function is $\phi_1(x) + 360°$. The tabulated function in [9] which Fletcher et al. describe as $\phi_1(x) + 90°$ is $\phi_1(x) + 270°$. Apart from [8], the only known table of $\phi_1(x)$ which conforms to (5) is that of Olver [5].

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