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Calculation of the Regulator of $Q(\sqrt{D})$ by Use of the Nearest Integer Continued Fraction Algorithm

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Abstract. A computational method for determining the regulator of a real quadratic field $Q(\sqrt{D})$ is described. This method makes use of the properties of the nearest integer continued fraction of $\sqrt{D}$ and is about 25% faster than methods which use the ordinary (integer part) continued fraction of $\sqrt{D}$. Several tables are also presented.

1. **Introduction.** In Williams and Broere [5] it is shown how the regulator of a real quadratic field $Q(\sqrt{D})$ ($D > 0$ and D not a perfect square) can be determined by using the ordinary continued fraction (OCF) algorithm. If

$$\sqrt{D} = [q_0, q_1, q_2, q_3, \ldots, q_{n-1}, \theta_n]$$

is the OCF expansion of $\sqrt{D}$, then

$$\sqrt{D} = [q_0, \overline{q_1, q_2, \ldots, q_p}],$$

where p is the length of the shortest period in the OCF. Also, if

$$C_n = [q_0, q_1, q_2, \ldots, q_n],$$

then

$$C_n = A_n/B_n,$$

where

$$A_{-2} = 0, \quad A_{-1} = 1, \quad B_{-2} = 1, \quad B_{-1} = 0,$$

$$A_{k+1} = q_{k+1}A_k + A_{k-1} \qquad (k = -1, 0, 1, 2, \ldots),$$

$$B_{k+1} = q_{k+1}B_k + B_{k-1}.$$

We put $\theta_n = (P_n + \sqrt{D})/Q_n$, where P_n and Q_n are integers.

Let ϵ (> 0) be the fundamental unit of $Q(\sqrt{D})$. If $Q_r = 4$ and $r < [p/2]$,* ther

$$(1.1) \qquad \epsilon = (A_{r-1} + \sqrt{D}B_{r-1})/2 = 2\prod_{i=1}^{r} \theta_i.$$

If no such Q_r exists,

$$(1.2) \qquad \epsilon = A_{p-1} + \sqrt{D}B_{p-1} = (P_{(p+1)/2} + \sqrt{D}) \prod_{i=1}^{[p/2]} \theta_i^2 \qquad (p \text{ odd}),$$

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*We use the notation $[\alpha]$ for the largest integer $\leqslant \alpha$.

(1.3) $$\epsilon = A_{p-1} + \sqrt{D}B_{p-1} = Q_{p/2} \prod_{i=1}^{p/2} \theta_i^2 \qquad (p \text{ even}).$$

Since the regulator R of $Q(\sqrt{D})$ is $\log \epsilon$, these formulas can be used to calculate R by simply summing $\log \theta_i$. Further, p is odd if and only if $Q_k = Q_{k+1}$ for some pair of Q's and in this case $p = 2k + 1$. Also, p is even if and only if $P_k = P_{k+1}$ for some pair of P's, and then $p = 2k$.

After Minnegerode [2] and Hurwitz [1], we define the nearest integer continued fraction (NICF) of a real number θ in the following manner. We first put $\theta'_0 = \theta$, $q'_0 = N(\theta'_0)$, where by $N(\alpha)$ we mean that integer such that $-\frac{1}{2} \leqslant \alpha - N(\alpha) < \frac{1}{2}$.

We then define

$$\theta'_{k+1} = \frac{1}{q'_k - \theta'_k}, \quad q'_{k+1} = N(\theta'_{k+1});$$

thus,

$$\theta = q'_0 - \cfrac{1}{q'_1 - \cfrac{1}{q'_2 - \ddots \cfrac{1}{-\theta'_n}}}.$$

We denote the above expression by

$$\theta = (q'_0, q'_1, q'_2, \ldots, \theta'_n).$$

Several results concerning this type of continued fraction can be found in [1], [2] and Williams [6]. In [1] it is shown that the NICF expansion of $\sqrt{D}$ is periodic and how the fundamental solution of $x^2 - Dy^2 = \pm 1$ can be found by using only the first period of the NICF of $\sqrt{D}$.

Since the period length of the NICF of $\sqrt{D}$ is usually shorter and never longer than that of the OCF of $\sqrt{D}$, it would seem to give a faster algorithm for calculating R if formulas analogous to (1.1), (1.2) and (1.3) could be developed. In this paper we develop such formulas and describe some computer calculations of regulators performed by using both the OCF and NICF algorithms. We also present several tables which resulted from our computer runs.

2. Results on Nearest Integer Continued Fractions. We define the nth convergent C'_n of the NICF of θ to be

$$C'_n = (q'_0, q'_1, q'_2, \ldots, q'_n).$$

If

$$A'_{-2} = 0, \quad A'_{-1} = 1, \quad B'_{-2} = -1, \quad B'_{-1} = 0,$$

and

$$A'_{k+1} = q'_{k+1}A'_k - A'_{k-1} \qquad (k = -1, 0, 1, 2, \ldots),$$

$$B'_{k+1} = q'_{k+1}B'_k - B'_{k-1},$$

then $C'_n = A'_n/B'_n$ and $A'_{n-1}B'_n - B'_{n-1}A'_n = 1$.

In [1] Hurwitz showed that if

(2.1) $$\theta = (r_0, r_1, r_2, \ldots, r_n, \phi_{n+1}),$$

where $r_0, r_1, r_2, \ldots, r_n$ are integers and ϕ_{n+1} is real then (2.1) represents the NICF of θ when the following conditions hold.

(1) $|r_i| \geqslant 2$ $(i = 1, 2, 3, \ldots, n)$,

(2) if any $r_i = 2$ (-2), then r_{i+1} is negative (positive),

(3) $\phi_{n+1} \geqslant 2$ or $\phi_{n+1} < -2$ and $|r_n - 1/\phi_{n+1}| > 2$.

It is also shown that if $\gamma = (3 - \sqrt{5})/2 \approx .382$, and $T_n = (q'_n, q'_{n-1}, \ldots, q'_1)^{-1}$, where the q'_i are integers such that $|q'_i| \geqslant 2$, we have

$$-\gamma < T_n < 1 - \gamma \quad \text{when } q'_n \neq -2,$$

and

$$-1 + \gamma < T_n < \gamma \quad \text{when } q'_n \neq 2.$$

Also, $T_n = B'_{n-1}/B'_n$.

We use these results to prove the following

THEOREM. *If $\beta = 3 - \gamma$ and a and b are coprime integers such that for some irrational θ*

$$|a/b - \theta| < 1/\beta b^2,$$

then a/b is a convergent in the NICF expansion of θ.

Proof. Put $\eta = a/b - \theta$ and expand a/b into a NICF. Then

$$a/b = (q'_0, q'_1, q'_2, \ldots, q'_n) = A'_n/B'_n.$$

Now if $q_n = 2$ and $\eta > 0$, replace q'_{n-1} by $q'_{n-1} - 1$ and q'_n by -2. If $q'_n = -2$ and $\eta < 0$, replace q'_{n-1} by $q'_{n-1} + 1$ and q'_n by 2. Thus, if $|q_n| = 2$, $\eta q_n < 0$.

Define θ'_{n+1} by

$$\theta = (q'_0, q'_1, q'_2, \ldots, q'_n, \theta'_{n+1});$$

then

$$\theta = \frac{A'_n\theta'_{n+1} - A'_{n-1}}{B'_n\theta'_{n+1} - B'_{n-1}}$$

and

$$\theta'_{n+1} = \frac{\theta B'_{n-1} - A'_{n-1}}{\theta B'_n - A'_n} = \frac{1 + \eta B'_n B'_{n-1}}{\eta B'^2_n}.$$

Since

$$|\eta B'_n B'_{n-1}| = \left|\frac{B'_{n-1}}{B'_n}\right| |\eta B'^2_n| < \frac{1}{\beta} < 1,$$

we see that η and θ'_{n+1} must have the same sign.

If $\theta'_{n+1} > 0$, then $\eta > 0$ and $-1 + \gamma < B'_{n-1}/B'_n < \gamma$; hence,

$$\theta'_{n+1} = \frac{1}{\eta B'^2_n} + \frac{B'_{n-1}}{B'_n} > \beta - 1 + \gamma = 2.$$

If $\theta'_{n+1} < 0$, then $\eta < 0$ and $-\gamma < B'_{n-1}/B'_n < 1 - \gamma$; hence

$$\theta'_{n+1} < -\beta + \frac{B'_{n-1}}{B'_n} < -\beta + 1 - \gamma = -2.$$

Thus, $|\theta'_{n+1}| > 2$. Also,

$$\left|q'_n - \frac{1}{\theta'_{n+1}}\right| > |q'_n| - \frac{1}{|\theta'_{n+1}|} > 2,$$

when $|q'_n| > 2$. If $|q'_n| = 2$,

$$\left|q'_n - \frac{1}{\theta'_{n+1}}\right| = |q'_n| + \frac{1}{|\theta'_{n+1}|} > 2$$

by selection of q'_n.

The theorem now follows from the result of Hurwitz above.

With this result it is easy to show that if x, y are two integers such that $(x, y) = 1$ and $x^2 - Dy^2 = N$, where $|N| < 2\sqrt{D}/\beta \approx .764\sqrt{D}$, then x/y must be a convergent in the NICF expansion of $\sqrt{D}$.

The singular continued fraction (SCF) was also discussed in [1]. As we shall require some of its properties, we discuss some simple results concerning this continued fraction here. If $M(\alpha)$ is that integer such that

$$-\gamma < \alpha - M(\alpha) < 1 - \gamma \quad \text{when } \alpha > 0,$$

$$-1 + \gamma < \alpha - M(\alpha) < \gamma \quad \text{when } \alpha < 0,$$

then the SCF expansion of θ is given by

$$\theta = (q''_0, q''_1, q''_2, \ldots, \theta''_n),$$

where

$$\theta''_0 = \theta, \quad q''_0 = M(\theta''_0), \quad \theta''_{k+1} = \frac{1}{q''_k - \theta''_k}, \quad q''_{k+1} = M(\theta''_{k+1}).$$

If

$$C''_n = (q''_0, q''_1, q''_2, \ldots, q''_n),$$

then

$$C''_n = A''_n/B''_n, \quad \text{where } A''_{-2} = 0, \quad A''_{-1} = 1, \quad B''_{-2} = -1, \quad B''_{-1} = 0,$$

and

$$A''_{k+1} = q''_{k+1}A''_k - A''_{k-1},$$

$$B''_{k+1} = q''_{k+1}B''_k - B''_{k-1} \qquad (k = -1, 0, 1, 2, \ldots).$$

3. Nearest Integer Continued Fraction of $\sqrt{D}$. The NICF expansion of $\sqrt{D}$ has the form [1]

$$\sqrt{D} = (q'_0, \overline{q'_1, q'_2, \ldots, q'_s, q'_0 + b}),$$

where $b = M(\sqrt{D})$ and the periodic part of the expansion has a bar drawn over it. If

(3.1) $$x^2 - Dy^2 = -1$$

is solvable in integers x, y, then $s = 2m - 1$ and the NICF of $\sqrt{D}$ is

$$\sqrt{D} = (q'_0, \overline{q'_1, q'_2, \ldots, q'_m, -q'_1, -q'_2, \ldots, -q'_m}),$$

where $-q'_m = q'_0 + b$. Also, (A'_{m-1}, B'_{m-1}) is the fundamental solution of (3.1) and (A'_{s-1}, B'_{s-1}) is the fundamental solution of

(3.2) $$x^2 - Dy^2 = 1.$$

If we define $P'_0 = P''_0 = 0$, $Q'_0 = Q''_0 = 1$,

$$P'_{k+1} = q'_k Q'_k - P'_k, \qquad P''_{k+1} = q''_k Q''_k - P''_k,$$
$$Q'_{k+1} Q'_k = P'^2_{k+1} - D, \quad Q''_{k+1} Q''_k = P''^2_{k+1} - D \qquad (k = 0, 1, 2, \dots),$$

then $\theta'_n = (P'_n + \sqrt{D})/Q'_n$, $\theta''_n = (P''_n + \sqrt{D})/Q''_n$, and

$$A'^2_n - DB'^2_n = Q'_{n+1}.$$

It can also be shown from results of [1] that $P'_n > 0$ for $n > 0$.

If (3.1) is solvable in integers, put $\pi = m$; otherwise put $\pi = s + 1$. In [6] it is shown that

$$Q''_k = (-1)^p Q'_{\pi-k} \quad \text{and} \quad P''_{k+1} = P'_{\pi-k} \qquad (k = 0, 1, 2, \dots, \pi).$$

If

$$x^2 - Dy^2 = \pm 4 \tag{3.3}$$

has no solutions in odd integers x, y, then

$$\epsilon = |A'_{\pi-1} + \sqrt{D} B'_{\pi-1}|.$$

We now deal with the case when (3.3) is solvable in odd integers.

THEOREM. *If $D \neq 5,13$ and (3.3) is solvable in odd integers, there must be some $\kappa \leqslant \rho = [\pi/2]$ such that $|Q'_\kappa| = 4$; further $\epsilon = |A'_{\kappa-1} + \sqrt{D} B'_{\kappa-1}|$.*

Proof. If (3.3) is solvable in integers, there must be some Q_j in the OCF expansion of $\sqrt{D}$ such that $Q_j = 4$ and $j < [p/2]$. Also, $\epsilon = A_{j-1} + \sqrt{D} B_{j-1}$. Now if $D > 28$, C_{j-1} must be a convergent $C'_{\kappa-1}$ in the NICF expansion of $\sqrt{D}$. Since $j - 1 < [p/2] - 1$, it follows by Theorem 5 of [6] that $\kappa \leqslant \rho$. The theorem can be verified directly for $D = 21$.

For computing the NICF expansion of $\sqrt{D}$ when IBM/360 assembler language is being used, the following algorithm is quite efficient. We put $d = [\sqrt{D}]$, $Q'_{-1} = -D$, and

$$F = \begin{cases} 1 & \text{when } \sqrt{D} - d > ½, \\ 0 & \text{when } \sqrt{D} - d < ½. \end{cases}$$

Define

$$\langle \alpha \rangle = \begin{cases} [\alpha] & \text{for } \alpha > 0, \\ [\alpha] + 1 & \text{for } \alpha < 0, \end{cases} \qquad T_k = d + \left[\frac{|Q'_k| + F}{2} \right];$$

then

$$P'_{k+1} = T_k - R_k$$
$$Q'_{k+1} = Q'_{k-1} + q'_k (P'_{k+1} - P'_k)$$
$$q'_{k+1} = \langle (P'_{k+1} + T_{k+1})/Q'_{k+1} \rangle$$
$$R'_{k+1} = \text{remainder on dividing } P'_{k+1} + T_{k+1} \text{ by } Q'_{k+1}.$$

This algorithm is easily derived from the fact that

$$N(\alpha) = \begin{cases} \langle \alpha + ½ \rangle & \text{for } \alpha > 0, \\ \langle \alpha - ½ \rangle & \text{for } \alpha < 0; \end{cases}$$

thus,

$$\begin{aligned} q'_k &= N((\sqrt{D} + P'_k)/Q'_k) \\ &= \langle (\sqrt{D} + P'_k + ½|Q'_k|)/Q'_k \rangle \\ &= \langle (P'_k + T_k)/Q'_k \rangle . \end{aligned}$$

4. Determination of R. To calculate R by the method indicated in the introduction we make use of the fact that there are two very simple criteria for determining the middle of the period of the OCF expansion of $\sqrt{D}$. Unfortunately, there are 6 possible mid-period criteria for the NICF expansion of $\sqrt{D}$. These are given in [6] as

Condition 1 $P'_\rho = P'_{\rho+1}$ $(\pi = 2\rho, p = 2r)$,

Condition 2 $|P'_{\rho+1} - P'_\rho| = |Q'_\rho|$ $(\pi = 2\rho, p = 2r)$,

Condition 3 $Q'_\rho = Q'_{\rho+1}$ $(\pi = 2\rho + 1, p = 2r)$,

Condition 4 $Q'_\rho = -Q'_{\rho+1}$ $(\pi = 2\rho + 1, p = 2r + 1)$,

Condition 5 $P'_{\rho+1} = |Q'_\rho + Q'_{\rho+1}/2|$, $Q'_\rho Q'_{\rho+1} > 0$ $(\pi = 2\rho + 1, p = 2r + 1)$,

Condition 6 $P'_\rho = |Q'_\rho + Q'_{\rho-1}/2|$, $Q'_{\rho-1} Q_\rho > 0$ $(\pi = 2\rho, p = 2r + 1)$.

In Table 1 we give the frequency of occurrence of each of these criteria for the NICF expansion of $\sqrt{D}$ for each nonsquare $D < M$.

TABLE 1

Condition	M = 10000	M = 100000	M = 1000000
1	7370	75666	768987
2	880	10245	109976
3	324	2282	17371
4	785	6819	60702
5	153	1295	11633
6	382	3370	30325

In the following theorem we give the formulas for the regulator when the NICF expansion of $\sqrt{D}$ is being used. We assume that $D \neq 5,13$.

THEOREM. *Let* $\rho = [\pi/2]$ *and put* $\Theta = \sum_{i=1}^{\rho} \log|\theta_i|$. *If there exists a* $\kappa \leqslant \rho$ *such that* $|Q_\kappa| = 4$, *then* $R = \log 2 + \sum_{i=1}^{\kappa} \log|\theta_i|$. *If no such* κ *exists, the value of* R *is given in Table* 2 *according to the mid-period criterion for the NICF expansion of* $\sqrt{D}$.

TABLE 2

Condition	Value of R
1,2	$\log Q'_\rho + \Theta$
3,4	$\log(P'_{\rho+1} + \sqrt{D}) + \Theta$
5	$\log(\sqrt{D} + \lvert Q'_{\rho+1}/2\rvert) + \Theta$
6	$\log(\sqrt{D} + \lvert Q'_{\rho-1}/2\rvert) + \Theta$

Proof. Put $\eta'_n = \theta' B'_n - A'_n$. Then, $\eta'_{-2} = -\theta'_0$ and $\theta'_{n+1} = \eta'_{n-1}/\eta'_n$; consequently,

$$\theta'_0\theta'_1\theta'_2 \cdots \theta'_n = \eta'_{-2}/\eta'_{n-1}$$

or

$$\theta'_1\theta'_2\theta'_3 \cdots \theta'_n = -1/\eta'_{n-1}.$$

If we put $\eta''_n = \theta'' B''_n - A''_n$ and $\eta_n = \theta B_n - A_n$ we get

$$\theta''_1\theta''_2\theta''_3 \cdots \theta''_n = -1/\eta''_{n-1}$$

and

$$\theta_1\theta_2\theta_3 \cdots \theta_n = \frac{(-1)^{n-1}}{\eta_{n-1}}.$$

If $|Q'_\kappa| = 4$ ($\kappa \leqslant \rho$), then $2|\eta_{k-1}|\,|\epsilon| = 4$ and

$$R = \log 2 + \sum_{i=1}^{k} \log|\theta_i|.$$

We suppose now that $|Q'_\kappa| \neq 4$ for any $\kappa \leqslant \rho$. We have

$$\epsilon = |A'_{\pi-1} + \theta B'_{\pi-1}| \quad \text{and} \quad |\epsilon|\,|\eta_{\pi-1}| = 1;$$

thus, $\epsilon = |\theta'_1\theta'_2\theta'_3 \cdots \theta'_\pi|$.

For Conditions 1 and 2 we have π even and

$$\epsilon = |\theta'_1\theta'_2 \cdots \theta'_\rho|\,|\theta'_{\rho+1}\theta'_{\rho+2} \cdots \theta'_\pi|.$$

Since

$$\theta'_{\pi-j} = \frac{P'_{\pi-j} + \sqrt{D}}{Q'_{\pi-j}} = \sigma\frac{P''_{j+1} + \sqrt{D}}{Q''_j} = \sigma\frac{Q''_{j+1}}{Q''_j}\,\theta''_{j+1} \qquad (|\sigma| = 1),$$

we get

$$|\theta'_{\rho+1}\theta'_{\rho+2} \cdots \theta'_\pi| = \frac{1}{|\eta''_{\rho-1}|},$$

$|Q''_\rho| = |Q'_\rho|$ and $C''_{\rho-1} = C'_{\rho-1}$ [6]; hence, $|\eta''_{\rho-1}| = |\eta'_{\rho-1}|$ and $\epsilon = |Q'_\rho|(\theta'_1\theta'_2 \cdots \theta'_\rho)^2$.

For Conditions 3 and 4 we have $\pi = 2\rho + 1$ and

$$|\theta'_{\rho+2}\theta'_{\rho+3} \cdots \theta'_{2\rho+1}| = |Q''_\rho|\,\frac{1}{|\eta''_{\rho-1}|},$$

$|Q''_\rho| = |Q'_{\rho+1}|$ and $C'_{\rho-1} = C''_{\rho-1}$ [6]; hence,

$$\epsilon = (P'_{\rho+1} + \sqrt{D})(\theta_1\theta_2\theta_3 \cdots \theta_\rho)^2.$$

For Condition 5, we have $\pi = 2\rho + 1$, $\rho = 2r + 1$, $2P_{r+1} = Q_{r+2}$, $C''_{\rho-1} = C_{r-2}$ and $C'_{\rho-1} = C_{r-1}$ [6]; thus

$$|\eta''_{\rho-1}| = |\eta_{r-2}| \quad \text{and} \quad |\eta'_{\rho-1}| = |\eta_{r-1}|.$$

Since $1/|\eta_{r-1}| = \theta_r/|\eta_{r-2}|$, we have

$$\frac{1}{|\eta'_{\rho-1}|} = \frac{1}{|\eta''_{\rho-1}|}\theta_r.$$

It follows that

$$\epsilon = \frac{|Q''_\rho|\,|\theta'_{\rho+1}|}{\theta_r}\,\frac{1}{|\eta'_{\rho-1}|^2}.$$

Since $P'_{\rho+1} = Q_r + P_{r+1}$ and $|Q'_{\rho+1}| = Q_{r+2} = Q_{r-1} = |Q''_\rho|$, we get

$$\begin{aligned}\frac{|Q''_\rho|\,|\theta'_{\rho+1}|}{\theta_r} &= Q_r\,\frac{Q_r + P_{r+1} + \sqrt{D}}{P_r + \sqrt{D}}\\ &= (\sqrt{D} - P_r)(Q_r + P_{r+1} + \sqrt{D})/Q_{r-1}\\ &= P_{r+1} + \sqrt{D} = \sqrt{D} + |Q'_{\rho+1}/2|.\end{aligned}$$

Hence

$$\epsilon = (\sqrt{D} + |Q'_{\rho+1}/2|)(\theta_1\theta_2\theta_3 \cdots \theta_\rho)^2.$$

The result for Condition 6 is proved in a similar way.

5. Computations. If, by $\pi(D)$ we denote the value of π for the NICF expansion of $\sqrt{D}$ and by $p(D)$ we denote the value of p for the OCF expansion of $\sqrt{D}$, we get

$$r(D) = \sum_{d=2}^{D}{}' \pi(d) \Big/ \sum_{d=2}^{D}{}' p(d) = .7017174$$

for $D = 10^5$. Here we denote by the symbol

$$\sum_{d=2}^{D}{}'$$

the sum taken over all the nonsquare $d \leqslant D$. Usually, for any D with a long OCF period length we find that the value of $\pi(D)/p(D)$ is near log 2. For example with

$$D = 26437680473689 \qquad \text{(see Shanks [3])}$$

we get $\pi = 12726393$, $p = 18331889$ [4]; hence, $\pi(D)/p(D) = .6942215829$. It seems possible that $\lim_{D\to\infty} r(D) = \log 2$; and we tentatively conjecture that this is true. However, a heuristic computation that leads to this proposed limit would certainly make the conjecture more plausible.

Since, when $p(D)$ is large, $\pi(D)$ is about $.7\,p(D)$, it seemed reasonable to use the NICF method to calculate the regulator of $Q(\sqrt{D})$. To determine the increase in speed in using the NICF over the OCF algorithm two programs were written in assembler language and run on an IBM/370-168 computer. These programs calculated the regulators for each $Q(\sqrt{D})$, $D < 10^6$. The program which used the OCF algorithm completed this task in about 75 minutes of CPU time; the program which used the NICF algorithm finished in 56 minutes for an increase in speed of about 25%. The reason the increase is not higher is due to the larger amount of time the NICF program has to use in order to determine the middle of the period.

Two other programs were also written to find $p(D)$ and $\pi(D)$ for all $D < 5 \times 10^6$. The programs ran at about the same speed. This occurred because the amount of time the NICF program spends searching for the middle of the period offsets the amount of time saved by having a shorter period.

Some results of these computations are presented in the tables below.

In Table 3 we give those values of D and $R = R(D)$ where $R(D)$ attains a new maximum:

$$R(D) > R(d) \quad \text{for all } 2 \leqslant d < D.$$

This table extends Table 2 of [5] and also corrects some of the errors in the early entries of that table. That is, the entry for $D = 109$ has been deleted and the entries for $D = 127$ and $D = 454$ have been added.

TABLE 3

D	R	D	R
2	0.8813735870	1291	69.4273184672
3	1.3169578969	1366	77.5074598259
6	2.2924316696	1699	77.6876332419
7	2.7686593833	1726	91.4834493664
11	2.9932228461	1999	91.8871616482
14	3.4000844141	2011	100.5330045343
19	5.8289369670	2311	110.3031685568
22	5.9763444674	2326	111.2288678348
31	8.0196126862	2566	114.0560290161
43	8.8485092803	2671	119.5949359041
46	10.7928181024	3019	127.4968135077
67	11.4894930579	3259	132.1264896763
94	15.2710021030	3691	137.3982413669
127	16.0627148562	3931	147.2572667307
139	18.8597514711	4174	153.0173430331
151	21.9634633555	4846	162.4648752348
199	24.2055021388	4951	166.6589816408
211	27.0453080448	5119	172.5083888229
214	27.9608415496	6211	174.4907308575
331	36.2563832043	6379	175.3152117927
379	37.7923393812	6406	188.3791730941
454	38.0601858647	6451	196.1309977901
526	46.5711631870	7606	215.6813117646
571	47.3388626944	8254	221.1925364815
631	52.9384699466	8779	231.7579182597
739	53.6325614062	9619	239.9527441496
751	57.9421480629	10399	255.8485157573
886	58.0020463701	10651	270.8720689096
919	64.3629254888	12919	283.2448208522
991	68.8018425044	13126	298.6426033189

TABLE 3 (Continued)

D	R
15031	303.7361309346
16699	306.3140636634
16879	318.4517115485
17494	335.6569396021
17614	336.7982397982
18379	367.1977320417
21319	392.0102622714
23566	397.8561015547
23599	400.3884707647
25939	415.0636719610
26959	423.4032864806
27934	433.0545764572
28414	447.0803714909
31606	456.8954759278
32839	458.8319317155
32971	502.2498400138
34654	508.5862719586
38119	525.2411587024
42046	532.1198798454
42571	538.6911444813
43726	550.8678249363
46006	585.6837169002
48799	619.5703884996
53299	645.0226905353
55819	646.5879132757
56611	647.6311525131
58774	649.8430400137
60811	653.0194860976
61051	700.8274150643
67846	725.3253021387
72934	737.9722442638
76651	754.2932571294
78094	795.2507812599
78439	813.5634679127
82471	817.8618423939
84991	822.1613668169
85999	826.1184149736
87151	841.0124809508
90931	867.1952157000
98011	867.7624599955
100291	879.4415113297
102859	894.9227568186
104311	907.8337387654
106591	922.6947724216
111094	971.0454916162
122719	982.4675389675
132694	986.1702524442
133519	1029.9098380696
139591	1063.7848268402
145006	1076.1524960560
152959	1081.9106380987
155299	1099.6787898320
156979	1112.1565681808
162094	1173.1754996429
162451	1180.6485586331
166846	1188.8221393564
181399	1205.4474050689
187366	1275.2294528905
189814	1283.9621566595
196771	1291.3224081185
217519	1339.0474507336
221659	1359.5350576911
230239	1422.7564155302
241894	1466.0032736143
253639	1466.9987850554
256699	1474.4287253323
257371	1529.8495498315
285574	1570.9730666900
288766	1593.5654321329
289111	1598.5735104690
290614	1622.2713107321
294694	1668.5275429675
313699	1684.8638950626
338251	1686.2367870567
339091	1710.4050010237
351751	1736.9355786734
358471	1778.9022034134
363379	1795.5601386665
369979	1802.6572076590
375631	1832.9022408924
395014	1835.7837079228
399499	1890.8635586712
404251	1932.2658189175
424414	1975.7658936684
438166	2014.9843659014
462214	2042.7710971743
462331	2075.1170289558
493399	2143.4726643288
505411	2145.4728706727
528334	2220.2767581032
553414	2231.7286391589
580486	2239.5186437879
580759	2251.5301065154
583339	2326.2371872327
584791	2376.5183924506
630919	2428.0031512070
636574	2463.2742248858
647854	2472.7815418044
677791	2537.4875384785
710854	2542.3172869764
723046	2542.6899463012
727939	2561.7142512252
749974	2700.2195358195
784939	2730.5569309958
820759	2739.4966172282
861799	2835.2857171115
909451	2948.6956215459
955894	2958.1996638340
966211	2969.1193861384
969406	3064.7655131335
978091	3092.4558654611

In Table 4 we give those values of D and $p(D)$ where $p(D)$ attains a new maximum and in Table 5 we give those values of D and $\pi(D)$ where $\pi(D)$ attains a new maximum.

TABLE 4

D	P	D	P	D	P
2	1	10774	238	155299	958
3	2	12541	239	162094	1016
7	4	12919	248	166846	1028
13	5	13126	262	173671	1040
19	6	15031	268	187366	1106
31	8	16669	280	189814	1110
43	10	17341	281	196771	1122
46	12	17494	290	217519	1152
94	16	17614	300	221659	1174
139	18	18379	322	230239	1224
151	20	19231	332	241894	1262
166	22	21319	348	253621	1272
211	26	23599	352	256699	1274
331	34	25939	374	257371	1318
421	37	27589	380	285574	1358
526	40	28414	388	289111	1400
571	42	31606	394	294694	1438
604	44	32839	400	313699	1442
631	48	32971	438	333334	1454
751	52	34654	440	339091	1458
886	54	38119	444	351751	1484
919	60	39439	448	358471	1524
1291	62	39901	449	363379	1538
1324	64	40429	451	375631	1548
1366	70	40639	456	383839	1560
1516	76	42046	464	395014	1578
1621	79	42571	474	395509	1585
1726	88	43261	489	399499	1654
2011	94	46006	506	424414	1716
2311	96	48799	544	438166	1738
2566	102	53299	566	462214	1742
2671	104	60811	574	462331	1798
3004	108	61051	614	493399	1836
3019	114	67846	618	505411	1866
3334	118	72934	642	528334	1900
3691	122	73516	644	553414	1918
3931	130	76651	654	576046	1920
4174	136	78094	692	580759	1956
4846	152	78439	696	583339	1986
5119	156	82471	716	584791	2016
6211	158	85999	720	609046	2026
6451	170	90931	734	630919	2052
6679	172	95131	750	636574	2104
6694	174	100291	754	647854	2120
7606	194	102859	790	675151	2124
8254	196	106591	808	677791	2156
8779	202	111094	834	698086	2166
8941	207	127219	838	710854	2186
9739	210	131884	864	749974	2270
9949	217	133519	876	784039	2280
10399	228	139591	904	784939	2338
10651	234	145006	944	799621	2383

TABLE 4 (Continued)

D	P	D	P	D	P
861799	2432	1627861	3395	3082699	4938
909451	2522	1660411	3526	3318214	5030
966211	2526	1702639	3548	3462229	5237
969406	2664	1834309	3609	3573574	5250
1013879	2672	1890079	3732	3574411	5326
1059829	2750	1957099	3898	3874126	5520
1089334	2778	2139349	3900	4103719	5652
1138999	2872	2151451	3934	4144534	5730
1174429	2882	2185726	4012	4355311	5916
1202191	2908	2237134	4212	4760926	5968
1214326	2922	2494606	4236	4836679	5992
1234531	3030	2519911	4364	4840159	6000
1336141	3117	2631511	4480	4865974	6018
1365079	3196	2765239	4504	4938124	6036
1427911	3308	2797414	4522	4960069	6054
1526086	3310	2810014	4600	4966966	6182
1526431	3336	2847079	4784		

TABLE 5

D	P	D	P	D	P
2	1	6406	116	72934	442
6	2	6451	120	76651	452
13	3	7606	132	78094	476
19	4	8254	136	78439	488
31	6	8779	142	82471	490
46	8	8941	143	84991	494
94	10	9619	144	87151	506
109	11	9949	149	90931	522
139	12	10399	154	100291	528
151	14	10651	166	102859	532
181	15	12541	167	104311	544
199	16	12919	172	106591	552
211	18	13126	184	111094	582
301	20	16669	194	122719	588
331	24	17341	195	131884	606
421	25	17494	202	133519	614
526	30	17614	204	139591	636
631	34	18379	224	145006	644
751	36	21319	238	152959	648
919	40	23566	242	155299	654
991	44	25939	250	156979	662
1324	46	26959	260	162094	702
1366	48	27589	264	166846	712
1549	51	28414	270	181399	716
1621	56	31606	274	187366	762
2011	62	32341	277	189814	772
2311	68	32971	302	216661	773
2326	70	34654	306	217519	802
2671	74	37591	308	221659	814
3019	78	38119	318	230239	850
3259	80	42046	320	241894	876
3469	81	42571	324	256699	880
3691	84	42604	330	257371	914
3931	92	43261	333	285574	936
4741	94	46006	352	288766	954
4846	100	48799	374	290614	968
4951	102	53299	390	294694	994
5119	104	56611	392	313699	1008
5149	106	61051	422	339091	1022
6211	108	67846	436	351751	1032

TABLE 5 (Continued)

D	P	D	P	D	P
358471	1056	784939	1626	2139349	2706
363379	1068	799621	1639	2185726	2804
369979	1070	861799	1684	2237134	2898
375631	1100	909451	1754	2471869	2922
395509	1115	955894	1756	2494606	2926
399499	1128	966211	1762	2519911	3010
404251	1150	969406	1826	2631511	3120
424414	1174	978091	1838	2765239	3142
438166	1200	1018879	1868	2797414	3158
455701	1207	1059829	1914	2810014	3170
462214	1218	1089334	1918	2847079	3328
462331	1240	1138999	1998	3082699	3408
493399	1278	1214326	2018	3318214	3474
528334	1324	1234531	2096	3360109	3486
553414	1328	1261654	2098	3462229	3605
580486	1330	1336141	2174	3573574	3648
580759	1334	1365079	2228	3574411	3676
583339	1386	1427911	2276	3740851	3696
584791	1416	1526086	2284	3799819	3708
630919	1442	1526431	2304	3874126	3850
636574	1466	1627861	2363	4103719	3900
647854	1470	1660411	2442	4144534	3954
677791	1508	1717951	2466	4355311	4118
710854	1510	1834309	2526	4760926	4166
712891	1514	1890079	2602	4865974	4176
727939	1526	1957099	2674	4938124	4198
749974	1608	2039566	2676	4966966	4306

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1. A. HURWITZ, "Über eine besondere Art die Kettenbruchentwicklung reeller Grössen," *Acta Math.*, v. 12, 1889, pp. 367–405.

2. B. MINNEGERODE, "Über eine neue Methode, die Pellshe Gleichung aufzulösen," *Gött. Nachr.*, 1873, pp. 619–653.

3. D. SHANKS, "The infrastructure of a real quadratic field and its applications," *Proceedings of the* 1972 *Number Theory Conference*, Boulder, Colorado, 1972, pp. 217–224.

4. D. SHANKS, "Review of UMT File: Two related quadratic surds having continued fractions with exceptionally long periods," *Math. Comp.*, v. 28, 1974, pp. 333–334.

5. H. C. WILLIAMS & J. BROERE, "A computational technique for evaluating $L(1, \chi)$ and the class number of a real quadratic field," *Math. Comp.*, v. 30, 1976, pp. 887–893.

6. H. C. WILLIAMS, "Some results concerning the nearest integer continued fraction expansion of $\sqrt{D}$," *J. Reine Angew. Math.* (To appear.)