

An Inductive Algorithm to Construct Finite Lattices

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Abstract. G. Birkhoff [1] proposed the following problem: Enumerate all finite lattices which are uniquely determined (up to isomorphism) by their diagram, considered as a graph.

It is not known how many lattices of order n exist, except when the value of n is quite small. The aim of this note is to give an algorithm to construct inductively all finite lattices of order n . Using this algorithm, we have found that there exist 222 lattices for $n = 8$ and 1078 lattices for $n = 9$. All lattices of order $n \leq 8$ are shown at the end of this note.

1. A Graphical Representation. Since a lattice is an ordered set, it can be represented by a Hasse diagram. To represent a lattice here, we transform slightly its usual Hasse diagram, writing one of the longest chains straight up, and we "locate all elements upward closely." To "locate all elements upward closely" means that for any element a , except the maximal element, there exists at least one element which dominates a and is one level above a , where a level means the set of all points having equal Y -coordinates. Such a diagram is briefly called a "diagram closed upward."

Example. A lattice having the chains $f < e < d < c < b < a$, $f < h < d$, $f < h < g < b$, and $e < i < a$, is represented in the following diagram closed upward:

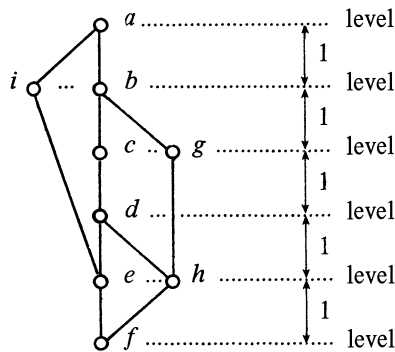


FIGURE 1

In Figure 1, a is the maximal element. For any element x except a we can find an element which dominates x with one level-difference.

The following theorem assures the possibility of representing a lattice by such a diagram closed upward.

THEOREM 1. *A lattice L can be represented by a diagram closed upward.*

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Proof. First make a Hasse diagram of a given lattice L . Let $E(L)$ be the element set of L . Denote the maximal element by m . Let the elements dominated by m be a_1, a_2, \dots, a_s . Let the elements dominated by a_i ($i = 1, 2, \dots, s$) be $b_{ij(i)}$ ($i = 1, 2, \dots, s; j(i) = 1, 2, \dots, t_i$).

We now transform the Hasse diagram into a diagram closed upward. Locate a_1, a_2, \dots, a_s at one level downward from m keeping invariant all connection relations. For an element $b_{ij(i)}$, if

$$\{x \in E(L) \mid b_{ij(i)} \preceq x\} \subseteq \{m, a_i; i = 1, 2, \dots, s\},$$

then we replace $b_{ij(i)}$ by $a_{ij(i)}$; otherwise we leave $b_{ij(i)}$ standing. Do this test for all $b_{ij(i)}$'s and locate all $a_{ij(i)}$'s at one level downward from the level of a_i 's, keeping invariant all connection relations.

The set of $a_{ij(i)}$'s is not empty. For if b_{11} is one of the elements left alone, let one chain which is not a subset of $\{m, a_i; i = 1, 2, \dots, s\}$ be the following:

$$b_{11} \preceq \dots \preceq b_{pq} \preceq a_r \preceq m.$$

If there is no chain upward from b_{pq} except $b_{pq} \preceq a_r \preceq m$, then we can replace b_{pq} by a_{pq} , and our statement will then be true. Hence, we assume that there exists the following chain upward from b_{pq} :

$$b_{pq} \preceq \dots \preceq b_{uv} \preceq a_w \preceq m.$$

If there is no chain from b_{uv} except $b_{uv} \preceq a_w \preceq m$, then our statement will be true. By continuing this argument, we obtain the following chain:

$$b_{11} \preceq b_{pq} \preceq b_{uv} \preceq \dots \preceq a_m \preceq m.$$

However, the finiteness of $b_{ij(i)}$ ($i = 1, 2, \dots, s; j(i) = 1, 2, \dots, t_i$) shows that the length of the above chain must be finite. Therefore, our statement is true.

Repeat the same procedures for the elements dominated by all $a_{ij(i)}$'s and so on. By the finiteness of the lattice we get to the minimal element. We draw one of the longest chains straight, and thus obtain the diagram. This completes the proof.

In the following we mean by 'lattice' the diagram closed upward.

2. To Construct the Lattice. Construction of all lattices of given order n will be accomplished by an inductive method. For this we prepare the following theorem:

THEOREM 2. *From an n -element lattice we can construct an $(n - 1)$ -element lattice by removing an element dominated by the maximal element and by rearranging the rest to make a lattice.*

Proof. Let m be the maximal element of a given n -element lattice. Let a_0 be an element dominated by m . Let a_i ($i = 1, 2, \dots, s$) be the elements dominated by a_0 .

For each $i \in \{1, 2, \dots, s\}$, if there exists at least another chain upward from a_i except $a_i \preceq a_0 \preceq m$, then take out the line segment connecting a_0 to a_i . If there exists no other chain upward except $a_i \preceq a_0 \preceq m$, then take out the line segment connecting a_0 to a_i and connect m with a_i by a new line segment. At last, take out the element a_0 and the line segment connecting m with a_0 . Then we can obtain a lattice of order $(n - 1)$.

From Theorem 2, we can see how to construct the set of all lattices of order n . It will be derived from the set of all lattices of order $(n - 1)$ by applying the inverses of all possible procedures to make an $(n - 1)$ -element lattice from an n -element lattice, as stated in the proof of Theorem 2.

In the following, we show the detail of the procedure to construct the set of all lattices of order n .

Algorithm to produce from an $(n - 1)$ -element lattice L a set of n -element lattices:

Let $E(L)$ be the element set of L .

Let m be the maximal element of L .

Let the elements dominated by m be $a_1 \cdots a_k$.

Let the elements of $E(L) - \{m, a_1, \dots, a_k\}$ be $e_1 \cdots e_l$.

We shall produce a list of n -element lattices, LIST(n).

1. Let LIST(n) = \emptyset .

2. In L , add an element a_0 which dominates m . Add this lattice to LIST(n).

3. If $k > 1$ do:

For each q -subset $s_1 \cdots s_q$ of $a_1 \cdots a_k$, in turn for $1 \leq q < k$, do:

for all s_i , let m no longer dominate s_i .

for all s_i , let a_0 dominate s_i .

Let m dominate a_0 .

For each result obtained above, check for all a_j of $\{a_1, \dots, a_k\} - \{s_1, \dots, s_q\}$ whether all $a_0 \wedge a_j$ are unique, where $a_0 \wedge a_j$ denotes the g.l.b. of a_0 and a_j . Append any result for which all $a_0 \wedge a_j$ are unique to LIST(n).

4. For each e_i ($1 \leq i \leq l$) do:

Let m dominate a_0 .

Let a_0 dominate e_i .

Append the result to LIST(n).

5. For each lattice M except the one obtained in 2 in the list LIST(n) created thus far, do:

Let $a_0, x_1 \cdots x_m$ be the elements dominated by m in M .

Let $L_{a_0} = \{a_i \in E(M) \mid a_i \not\leq a_0\}$.

Let $P = E(M) - L_{a_0} - \{m, a_0, x_1, \dots, x_m\}$.

Let $p_1 \cdots p_t$ be the elements of P .

For each p_i ($1 \leq i \leq t$) do:

If $a_0 \wedge p_i$ is not dominated by a_0 , and moreover if for any $a_j \in L_{a_0}$, $a_j \vee p_i = m$ or $a_j \vee p_i = p_i$, then let a_0 dominate p_i and append this lattice to LIST(n).

6. For each lattice M newly constructed in 5 in the list LIST(n), do: the same procedures stated in 5.

7. For each lattice constructed in 6 in the list LIST(n), do: the same procedures stated in 5.

8. Repeat the same procedures stated as in 7 for each lattice M newly constructed in the previous step until the set P becomes empty.

3. A Matrix Representation. To find all isomorphic lattices we consider a matrix representation. For a given diagram closed upward, denote the maximal element by 1 and number the levels 1, 2, 3, . . . , from the top to bottom. Denote the elements on the same level by the same numbers and distinguish different elements by the suffixes a, b, c, \dots

Examples.

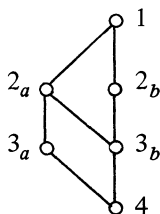


FIGURE A

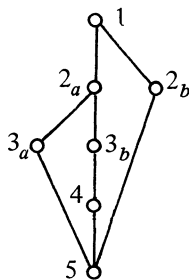


FIGURE B

Make a matrix whose columns represent all chains from the maximal element to the minimal element. Put 0's between two nonadjacent integers to make the lengths of all columns the same.

Examples. Matrices corresponding to Figures A and B.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2_a & 2_a & 2_b \\ 3_a & 3_b & 3_b \\ 4 & 4 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 2_a & 2_a & 2_b \\ 3_a & 3_b & 0 \\ 0 & 4 & 0 \\ 5 & 5 & 5 \end{pmatrix}.$$

We may define the equality of two such $l \times m$ -matrices P, Q as follows:

If some permutation of all columns of P and some replacements of all suffixes on each row of P transform P to Q , then we say that P and Q are equal.

By this definition, for a lattice we have a unique matrix representation. Conversely, if a matrix is given, the suffixes of each row determine the elements which should be located on the corresponding level, and each column shows which element should be connected with which element. Thus a matrix determines a unique lattice.

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LATTICES

$n = 1$



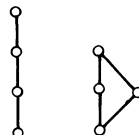
$n = 2$



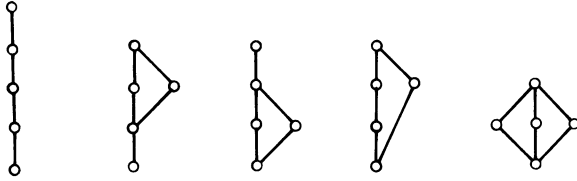
$n = 3$



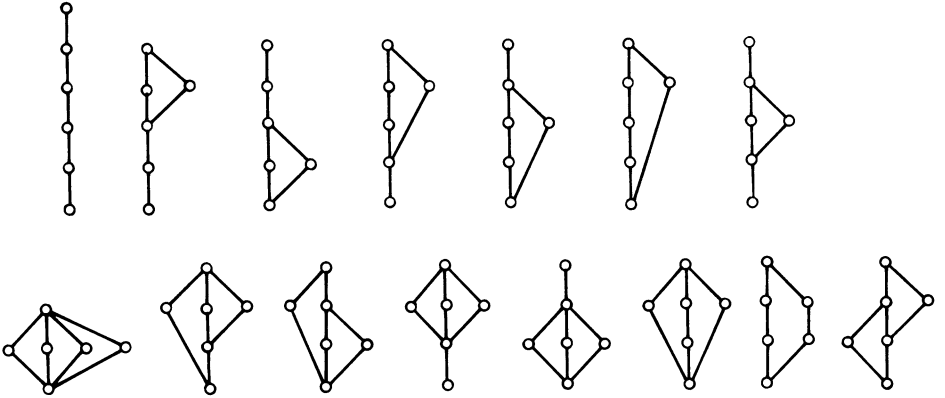
$n = 4$



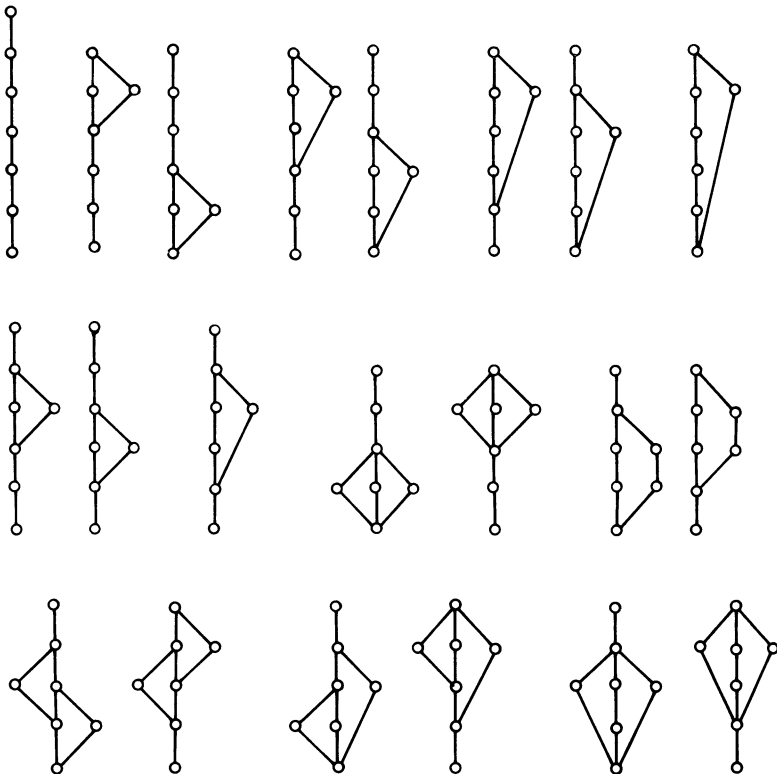
$n = 5$

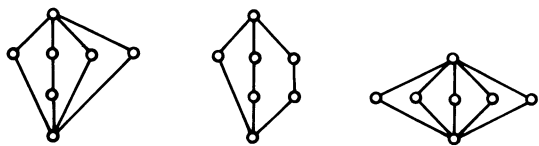
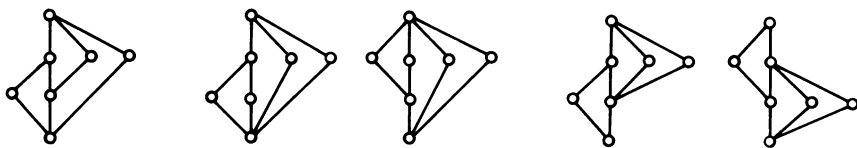
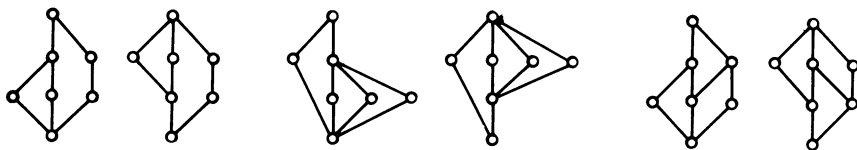
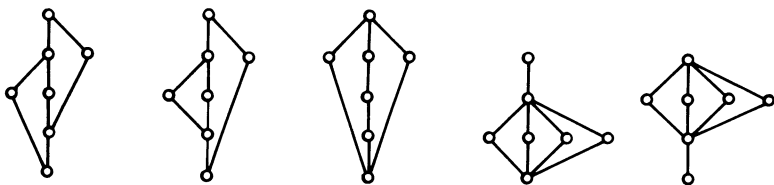
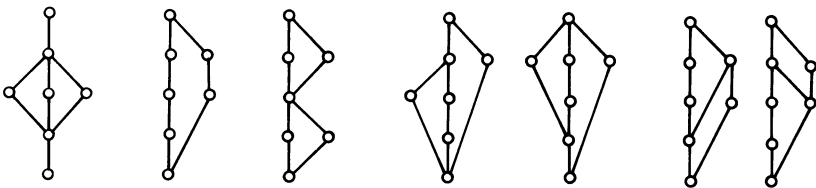
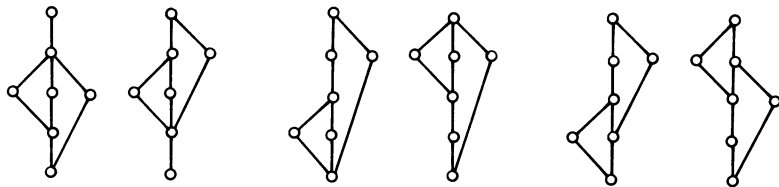


$n = 6$ (15 pieces)

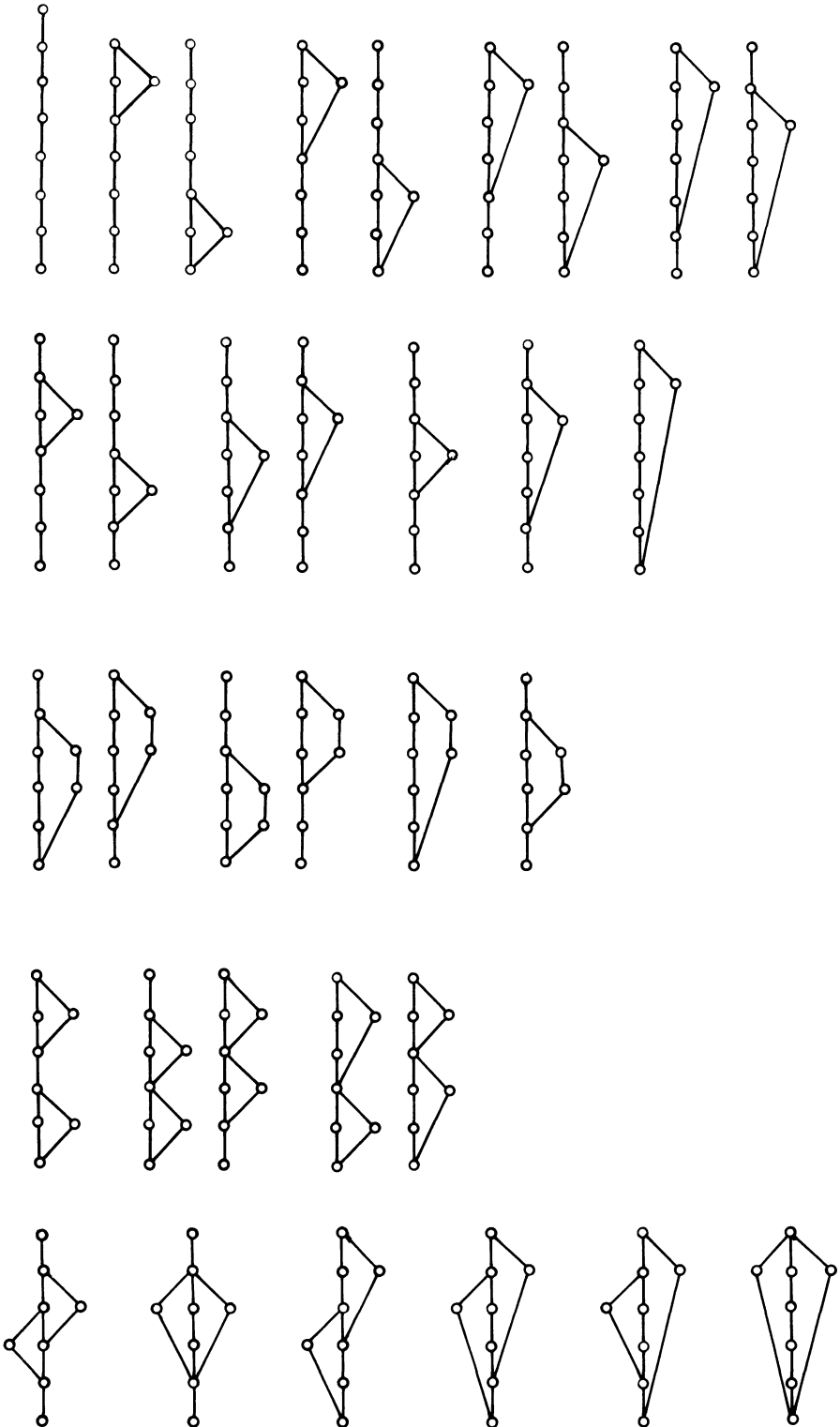


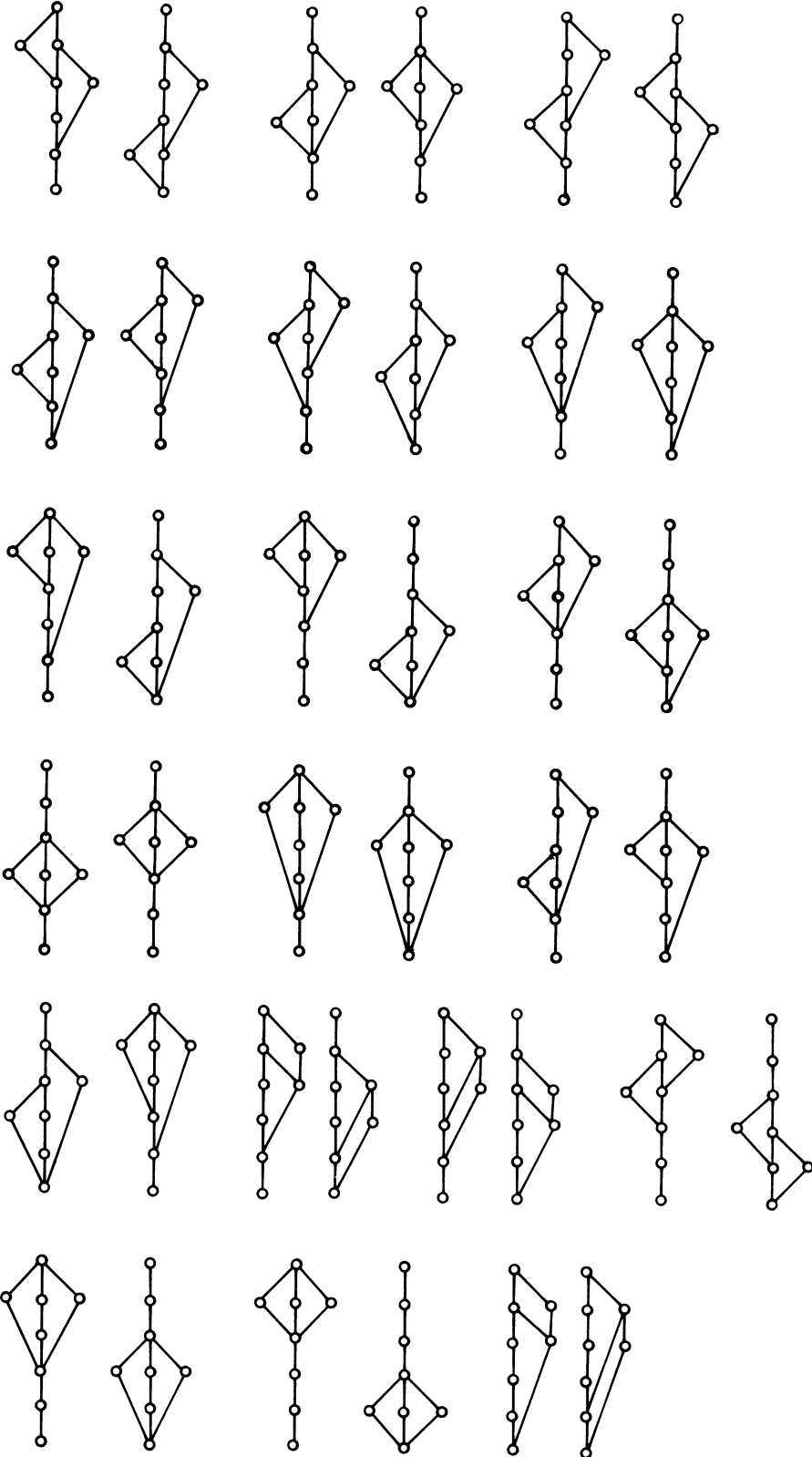
$n = 7$ (53 pieces)

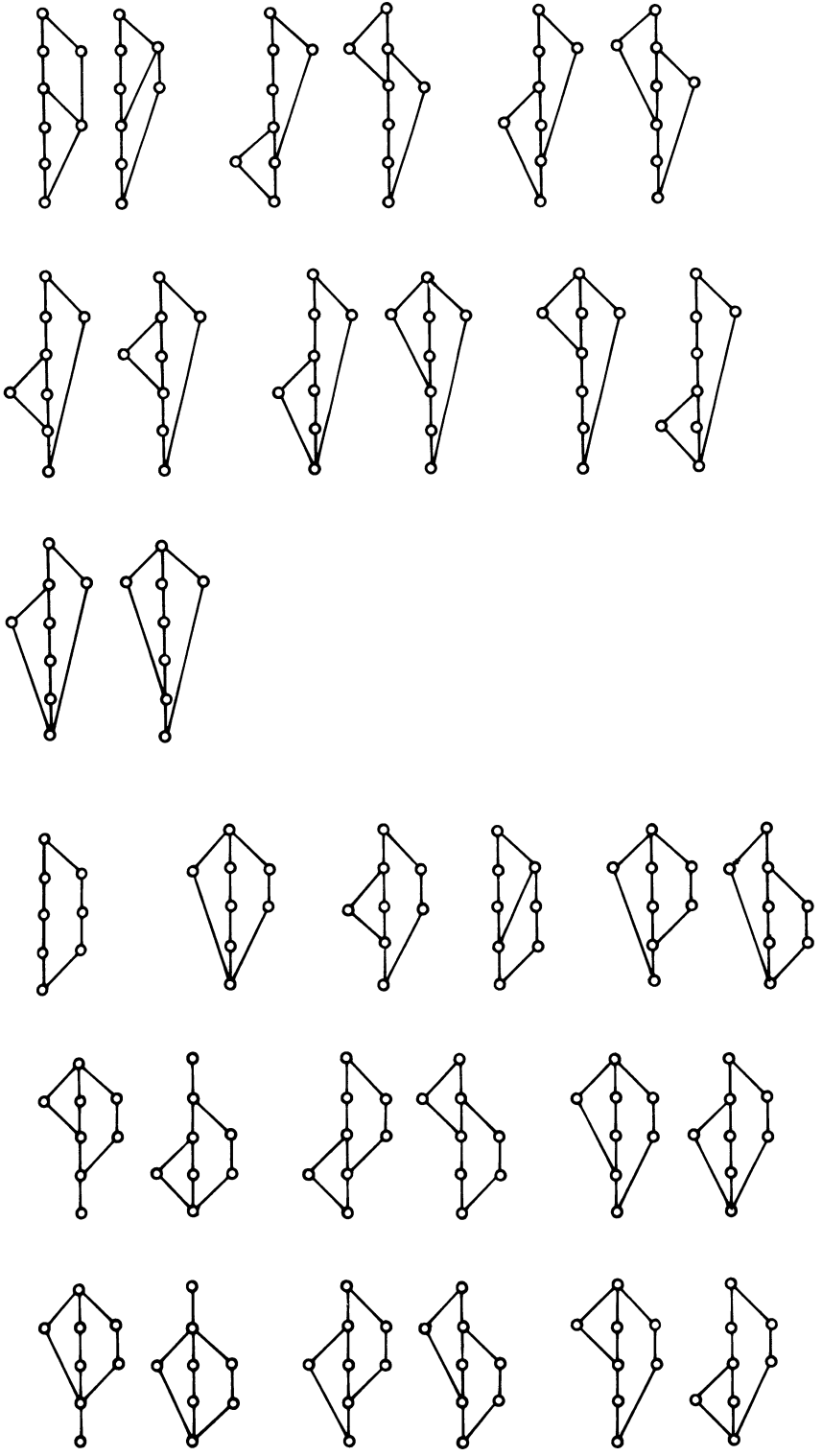


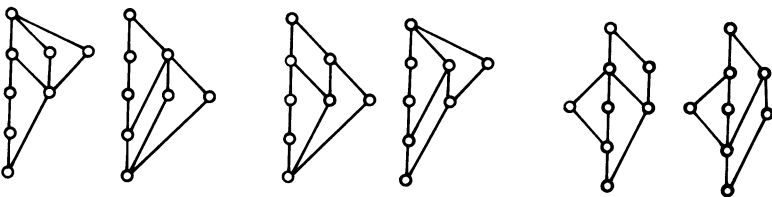
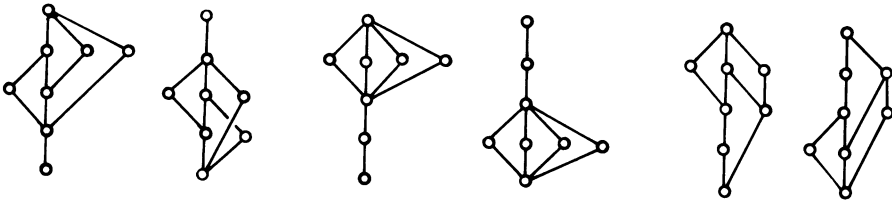
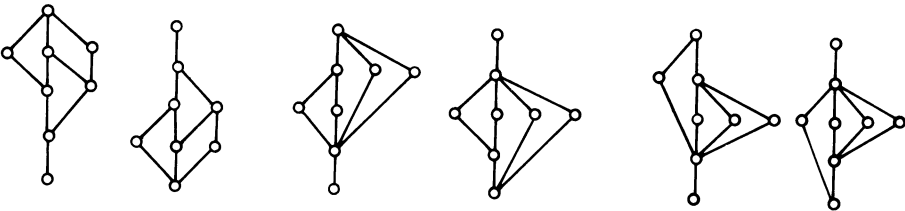
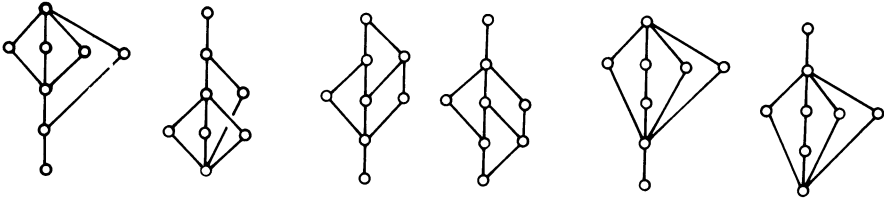
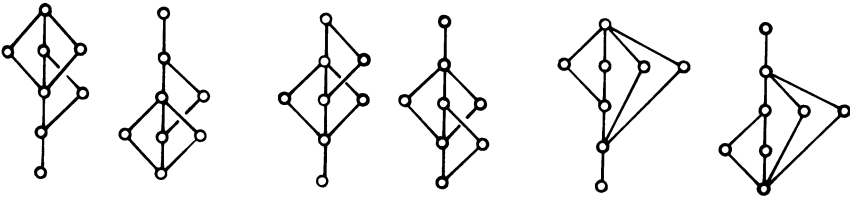
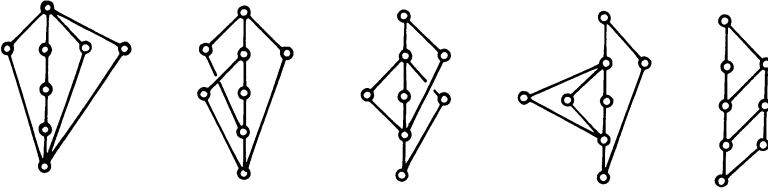
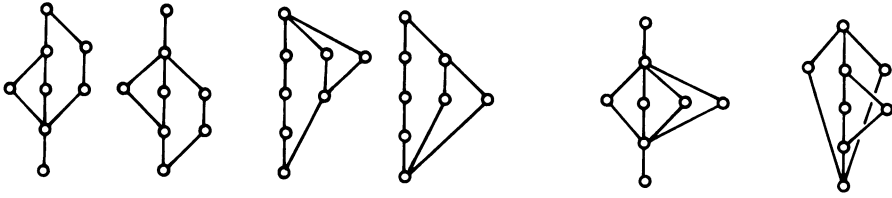


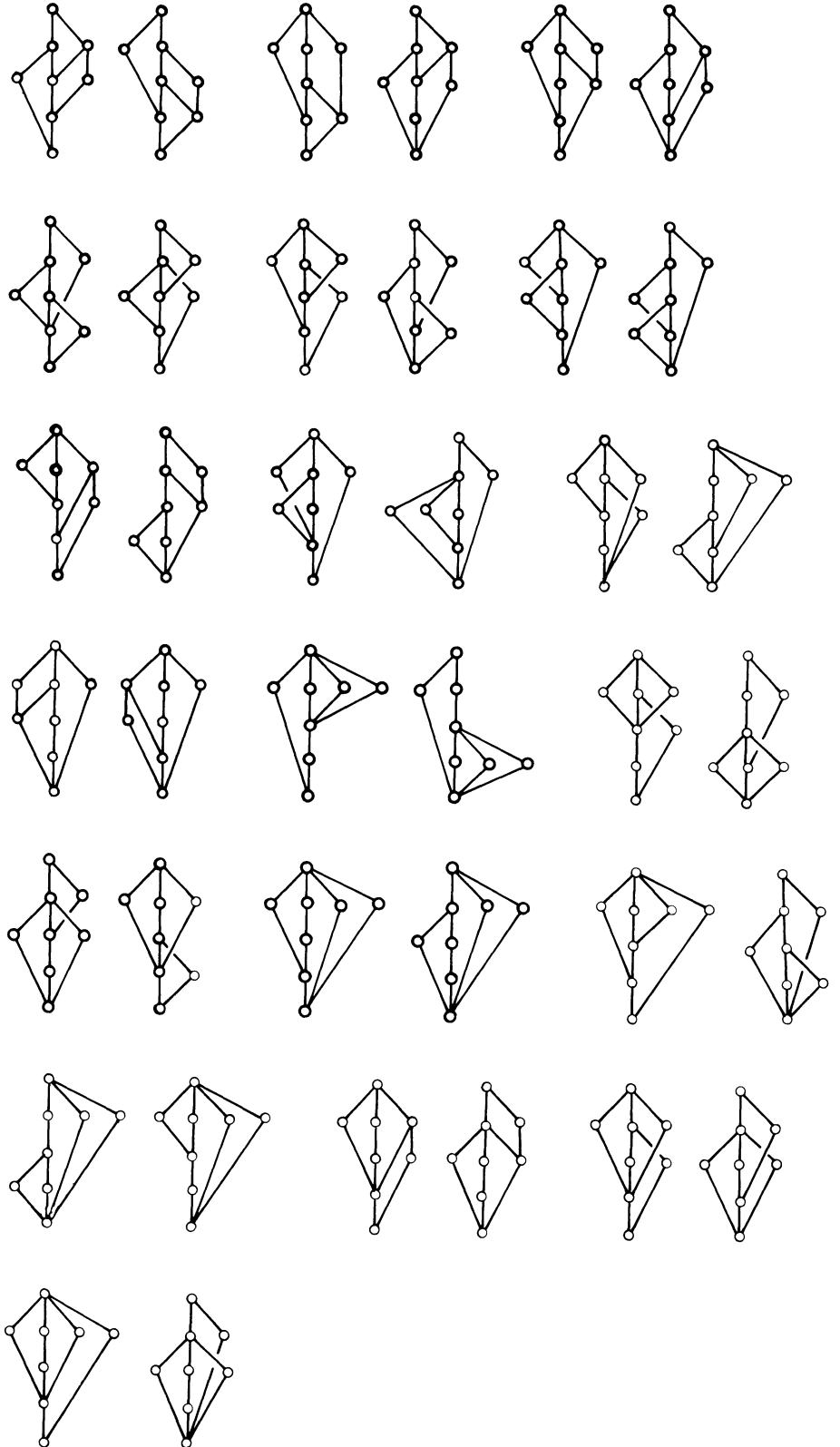
$n = 8$ (222 pieces)

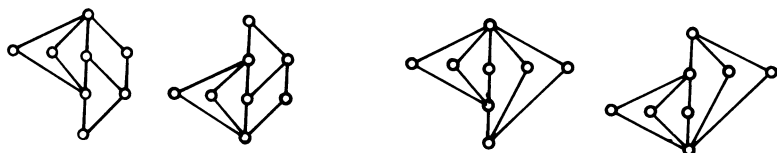
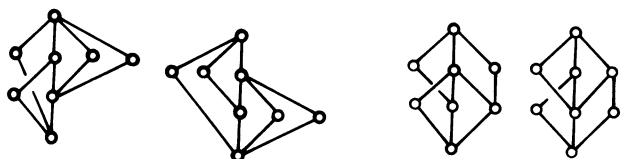
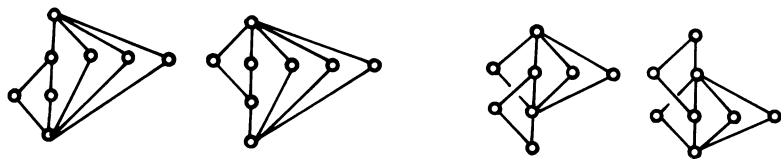
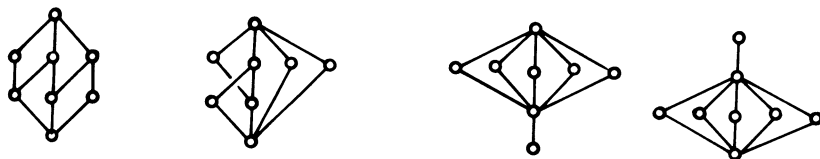
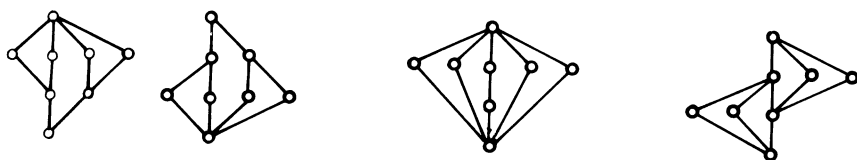
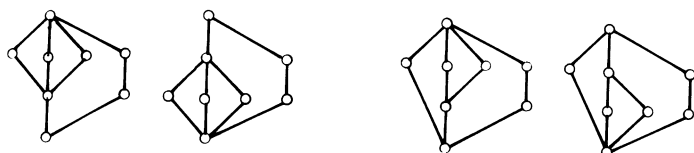
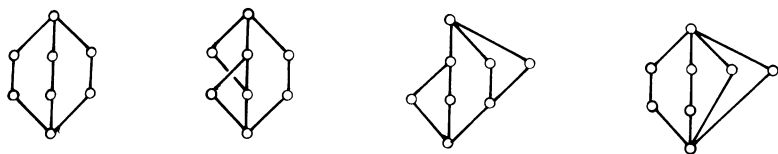
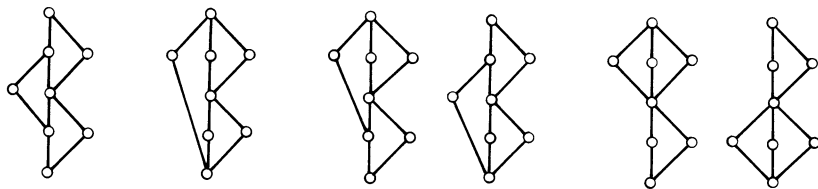


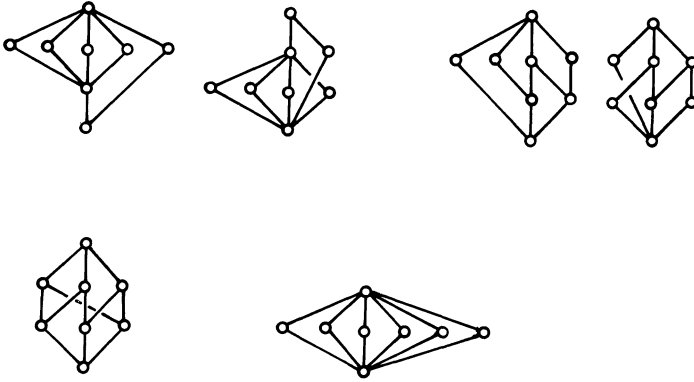












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1. G. BIRKHOFF, *Lattice Theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1967.