MATHEMATICS OF COMPUTATION, VOLUME 33, NUMBER 146
APRIL 1979, PAGES 705–716

Quadrature Rule Methods for Volterra Integral Equations of the First Kind

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Abstract. A new class of quadrature rule methods for solving nonsingular Volterra integral equations of the first kind are introduced; these methods are based on an appropriate modification of the higher-order Newton-Gregory methods which are known to be divergent. Methods up to order six are constructed explicitly and illustrated with numerical examples.

1. Introduction. Consider the linear Volterra integral equation of the first kind,

$$\int_0^x K(x, t)y(t)\,dt = f(x), \tag{1.1}$$

where f and K (the kernel) are given functions which are continuously differentiable to sufficiently high order with respect to each of their arguments on

$$S_1 = \{x \mid 0 \leqslant x \leqslant a\} \qquad (a < \infty),$$

and on

$$S_2 = \{(x, t) \mid 0 \leqslant t \leqslant x \leqslant a\},$$

respectively. Furthermore, we assume that

(1) $f(0) = 0$, and

(2) $K(x, x) \neq 0$ for all $x \in S_1$.

Then it is well known, [11, p. 14], that a unique solution $y(x)$ exists and is continuously differentiable to sufficiently high order on S_1.

2. Method of Solution. We introduce the uniform mesh

$$I_N = \{x_n = nh \mid n = 0(1)N, Nh = a\}$$

on the basic interval S_1. The integral in (1.1) is then replaced by a quadrature rule of the form

$$\int_0^{x_n} g(x)\,dx = h \sum_{i=0}^{n} w_{ni} g(x_i) + E_n(g), \tag{2.1}$$

where w_{ni} are real-valued weights satisfying

Received March 17, 1975; revised December 29, 1975, August 23, 1976, April 14, 1977 and August 8, 1978.

AMS (MOS) subject classifications (1970). Primary 65R05; Secondary 45L10.

Key words and phrases. Quadrature rule methods, Volterra integral equations of the first kind.

0025-5718/79/0000-0065/$04.00

$$w_{ni} = \begin{cases} a_i^*, & i = 0(1)p, \\ 1, & i = p + 1(1)n - (p + 1), \\ a_{n-i}, & i = n - p(1)n. \end{cases}$$

Equation (2.1) will be denoted by $Q(A^*, A, m)$, where

(1) $A^* = (a_i^*) \in R^{p+1}$, $A = (a_i) \in R^{p+1}$.

(2) m is the degree of precision of the quadrature rule [8, p. 301], i.e. $E_n(g) = 0\ \forall g \in \pi_m$, space of real polynomials of degree m.

(3) (2.1) is "apparently" defined for $n \geqslant 2p + 1$. In Section 3, we shall extend the definition for $n \geqslant p + 1$.

This yields the numerical method for the solution of (1.1),

$$\sum_{i=0}^{n} w_{ni} K(x_n, x_i) y_i = f(x_n), \qquad n = p + 1(1)N, \tag{2.2}$$

where y_n denotes an approximation to the exact solution $y(x_n)$. If we have a set of starting values y_n, $n = 0(1)p$, then (2.2) may be used in a recursive fashion to generate approximations for y_n, $n = p + 1(1)N$.

In this paper we shall not deal with the problem of how to generate the necessary starting values; we mention, however, that some useful higher-order procedures have recently been derived in [1, p. 295] and [13, p. 45]. These methods could also be used to approximate y on the entire mesh I_N; however, they are r^2 times as expensive (in terms of the number of kernel evaluations per step) as the method defined in (2.2) (r is the order of the method, to be defined below).

Definition 2.1. Let $e_n = y_n - y(x_n)$, $n = 0(1)N$, denote the (global) discretization error of (2.2). Then (2.2) is said to be convergent of order r if r is the largest positive integer for which there exists a finite constant C such that, for all sufficiently smooth f and K,

$$|e_n| \leqslant Ch^r, \qquad n = 0(1)N.$$

The aim of this paper will be to give a characterization of the weights w_{ni} so that the method (2.2) is actually convergent of order r ($= m + 1$, as we shall see later). Furthermore, we shall construct explicitly families of such methods for $r \leqslant 6$.

This is significant in the light of the result in [5, p. 148], which essentially showed that methods of the form (2.2) based on a wide class of interpolatory quadrature rules yielded nonconvergent methods for $r > 2$.

It should be mentioned, however, that six interpolatory quadrature rules were displayed in [5, p. 148] which proved to yield convergent schemes up to order six. Unfortunately, these schemes give no indication as to whether methods of higher order exist.

3. Selection of the Weights. In order to choose the vectors A^* and A we consider the decomposition of (2.1) into simpler quadrature rules

$$\int_0^{x_n} g(x)\,dx = \int_0^{x_p} g(x)\,dx + \sum_{j=p+1}^{n} \int_{x_{j-1}}^{x_j} g(x)\,dx$$

(3.1)

$$= \left\{ h \sum_{i=0}^{p} c_i g(x_i) + \hat{E}_p(g) \right\} + \sum_{j=p+1}^{n} \left\{ h \sum_{i=0}^{p+1} b_i g(x_{j-i}) + \hat{E}_j(g) \right\},$$

where

$$E_n(g) = \hat{E}_p(g) + \sum_{j=p+1}^{n} \hat{E}_j(g).$$

Now we select:

(1) The c_i to be the weights of the closed Newton-Cotes rule with $p+1$ abscissae.

(2) The b_i to satisfy the system of linear equations

$$\sum_{i=0}^{p+1} i^j b_i = \frac{1}{j+1}, \qquad j = 0(1)r-1 \leqslant p+1 \quad \left(0^j = \begin{cases} 1, & j=0 \\ 0, & j>0 \end{cases}\right). \tag{3.2}$$

(In the following, conditions (3.2) will be referred to as the consistency conditions.)

With these choices, it follows that the degree of precision of (2.1) is $m = r-1$.

Furthermore, if $r-1 = p+1$, the system (3.2) yields a unique solution (since the coefficient matrix is essentially Vandermonde). In this case, the quadrature rule (2.1) is the Newton-Gregory rule to pth differences. This case is useless for $p > 0$; see [5, p. 148] or [6, p. 38]. Thus, in the remainder of the paper we shall assume $r-1 < p+1$ so that the weights of (2.1) will depend (linearly) on $p+2-r$ free parameters $b_0, b_1, b_2, \ldots, b_{p+1-r}$, say.

Finally, we associate with (2.1) the characteristic polynomial

$$\rho(z) = \sum_{i=0}^{p+1} b_i z^{p+1-i}. \tag{3.3}$$

4. Convergence of the Method. We need the following definition.

Definition 4.1 [10, p. 398]. Let $\rho(z) \in \pi_n$. Then

(1) $\rho(z)$ is said to be a *simple von Neumann* polynomial if

(a) $\rho(z) = 0 \Rightarrow |z| \leqslant 1$,

(b) $\rho(z) = 0$ and $|z| = 1 \Rightarrow \rho'(z) \neq 0$.

(2) $\rho(z)$ is a *Schur* polynomial if $\rho(z) = 0 \Rightarrow |z| < 1$.

We now give necessary conditions for (2.2) to define a convergent method of order 1.

THEOREM 4.1. *Assume* (2.2) *is convergent of order* 1. *Then the polynomial* $\rho(z)$, *defined in* (3.3), *satisfies*:

(1) $\rho(z)$ *is a simple von Neumann polynomial* (*stability condition*), *and*

(2) $\rho(1) = 1$ ($j = 0$ in (3.2)) (*consistency condition*).

Proof. For (1), we apply (2.2) to the test equation

$$\int_0^x y(t)\,dt = 0, \qquad y(x) \equiv 0,$$

to obtain

$$h \sum_{i=0}^{p} c_i y_i + \sum_{j=p+1}^{n} h \sum_{i=0}^{p+1} b_i y_{j-i} = 0.$$

Backward differencing of this relation at n and $n-1$ yields the difference equation

$$\sum_{i=0}^{p+1} b_i y_{n-i} = 0, \qquad n = p + 1(1)N.$$

If the method is convergent, we must have

$$\lim_{h \to 0} y_n = 0 = y(x_n) \qquad (n \longrightarrow \infty, nh = x_n \in S_1),$$

which entails that $\rho(z)$ must be a simple von Neumann polynomial. See [7, p. 218] for the details.

For (2), we apply the same procedure to the test equation

$$\int_0^x y(t)\, dt = x$$

(whose exact solution is $y(x) \equiv 1$) to obtain the difference equation

$$\sum_{i=0}^{p+1} b_i y_{n-i} = 1, \qquad n = p + 1(1)N.$$

Thus, the result follows.

We now proceed to show that stability and consistency are sufficient for convergence. Indeed, we shall show if we have the r conditions (3.2), we obtain convergence of order r.

Throughout the rest of the paper we assume that the following conditions hold:

A. f and K are sufficiently differentiable with respect to each of their arguments on S_1 and S_2, respectively.

B. $\rho(z)$, defined in (3.3), satisfies the stability condition and the r consistency conditions (3.2) (i.e. (3.2) with $j = 0, 1, \ldots, r-1$).

C. The starting errors, $\{e_n\colon n = 0(1)p\}$, are $O(h^r)$.

It is easily seen that the global discretization error, e_n, satisfies

$$h \sum_{i=0}^{p} c_i K(x_n, x_i) e_i + h \sum_{j=p+1}^{n} \sum_{i=0}^{p+1} b_i K(x_n, x_{j-i}) e_{j-i} = E_n \tag{4.1}$$

$$(n = p + 1(1)N),$$

where E_n is the quadrature rule error in (2.1) applied to the function $K(x_n, t) \cdot y(t)$. E_n can be estimated asymptotically ($n \longrightarrow \infty$, $h \longrightarrow 0$, $nh = x_n$ fixed), under assumptions A, B and C, by

$$E_n = h^r \phi_1(x_n) + h^{r+1} \phi_2(x_n) + h^{r+2} \phi_3(x_n) + O(h^{r+3}) \tag{4.2}$$

with (a) $\phi_i \in C^2[S_1]$, $i = 1(1)3$, (b) $\phi_1(0) = 0$.

We shall also need the following lemma.

LEMMA 4.1. *Let $P(z)$ be a real simple von Neumann polynomial (with $P(1) \neq 0$) of exact degree q and with no zeros at the origin. Furthermore, let the coefficients*

$\{\gamma_l: l = 0(1)\infty\}$ *be defined by*

$$\frac{1}{z^q P(1/z)} = \sum_{l=0}^{\infty} \gamma_l z^l. \tag{4.3}$$

Then, for any $\phi \in C^1[S_1]$, *we have*

$$\left| \sum_{l=0}^{n} \gamma_l \phi(x_l) \right| \leqslant M < \infty \quad \textit{for all } n = 0(1)N; x_l = lh.$$

Proof. Let $P(z) = \Sigma_{i=0}^{q} b_i z^{q-i}$. Then from (4.3) we obtain

$$\sum_{i=0}^{q} b_i \gamma_{l-i} = \begin{cases} 1, & l = 0, \\ 0, & l > 0, \end{cases} \tag{4.4}$$

with $\gamma_{l-i} = 0$ for $l < i$. For $l \geqslant q$, (4.4) is a linear, homogeneous difference equation with constant coefficients together with q starting values $\{\gamma_l: l = 0(1)q - 1\}$. Thus, using the hypotheses on $P(z)$, we may write

$$\gamma_l = \sum_{i=1}^{s} C_i z_i^l + D_1 \mu^l + \sum_{i=2}^{V} D_i l(l-1) \cdots (l-i+2)\mu^l, \qquad l \geqslant 0, \tag{4.5}$$

where (a) C_i, D_i are arbitrary constants, depending linearly on the starting values; (b) $P(z)$ is assumed (for ease of notation and without loss of generality) to have s simple zeros, z_i, on the unit circle and one zero, μ, of multiplicity v interior to the unit circle; here, $s + v = q$.

Thus, it suffices to see that

$$\left| \sum_{l=0}^{n} z^l \phi(lh) \right| \leqslant M^* < \infty \quad \text{for all } n = 0(1)N, |z| = 1, z \neq 1.$$

We have

$$\sum_{l=0}^{n} z_l \phi(x_l) = \sum_{l=0}^{n} (S_l - S_{l-1})\phi(x_l), \qquad x_l = lh,$$

where

$$S_n = \sum_{j=0}^{n} z^j = \frac{z^{n+1} - 1}{z - 1}, \quad S_{-1} = 0, \quad S_0 = 0,$$

or, using summation by parts,

$$\sum_{l=0}^{n} z^l \phi(x_l) = \sum_{l=0}^{n-1} S_l[\phi(x_l) - \phi(x_{l+1})] + S_n \cdot \phi(x_n).$$

The result now follows by applying the Mean Value Theorem of Differential Calculus. See [7, p. 269] for the details.

We are now prepared to show that stability and consistency are sufficient for convergence.

THEOREM 4.2. *Assume that* e_n *satisfies* (4.1) *and that the conditions* A, B *and* C *are fulfilled. Then the method defined by* (2.2) *is convergent of order* r; *i.e.*

$$e_n = O(h^r), \qquad n = 0(1)N, \textit{ as } h \longrightarrow 0, Nh = a.$$

Proof. By using backward differencing in (4.1), and by observing Taylor's formula and assumption A, together with

(a) $K_{n\cdot i} = K(x_n, x_i)$, $i = 0(1)n$,

(b) $K_{n\cdot n-i} = K_{n\cdot n} - ih\theta_{n\cdot i}$, $|\theta_{n\cdot i}| \leqslant \theta = \max_{S_2}\{|K_t|\}$,

(c) $K_{n-1\cdot i} = K_{n\cdot i} - h\chi_{n\cdot i} + \theta^*_{n\cdot i}h^2$,

$$|\chi_{n\cdot i}| \leqslant \chi = \max_{S_2}\{|K_x|\}, \qquad |\theta^*_{n\cdot i}| \leqslant \theta^* = \max_{S_2}\{|K_{xx}|\},$$

(d) $|K_{nn}| \geqslant \lambda > 0$ for all $n = 0(1)N$,

we obtain

$$\sum_{i=0}^{p+1} b_i e_{n-i} = K^{-1}_{n\cdot n}\left\{h \sum_{i=0}^{n-1} d_{n\cdot i}e_i + h^{-1}(E_n - E_{n-1})\right\}, \qquad n = p+1(1)N, \tag{4.6}$$

where

$$d_{n\cdot i} = \begin{cases} (-\chi_{n\cdot i} + \theta^*_{n\cdot i}h)s_i, & i = 0(1)p, \\ (-\chi_{n\cdot i} + \theta^*_{n\cdot i}h), & i = p+1(1)n-(p+2), \\ (-\chi_{n\cdot i} + \theta^*_{n\cdot i})t_i + u_i\theta_{n\ n-i}, & i = n-(p+1)(1)n-1, \end{cases}$$

and where s_i, t_i, u_i depend (linearly) only on c_i and b_i.

Here, the functions $d_{n\cdot i} = d_i(x_n)$ are sufficiently smooth in view of assumption A. Multiply (4.6) by γ_{N-n} for each $n = p+1(1)N$ and form the sum. We first note that (from (4.4))

$$\sum_{n=p+1}^{N} \gamma_{N-n} \sum_{i=0}^{p+1} b_i e_{n-i} = e_N + \sum_{k=1}^{p+1}\sum_{i=k}^{p+1} b_k\gamma_{N-(p+k)}e_{p+k-i}. \tag{4.7a}$$

Secondly, we have

$$\sum_{n=p+1}^{N} \gamma_{N-n}K^{-1}_{nn} \sum_{i=0}^{n-1} d_{n\cdot i}e_i$$

$$= \sum_{i=0}^{p}\left\{\sum_{n=p+1}^{N} \gamma_{N-n}K^{-1}_{nn}d_{n\cdot i}\right\}e_i + \sum_{i=p+1}^{N-1} \gamma^*_{N\cdot i}e_i, \tag{4.7b}$$

where $\gamma^*_{N\cdot i} = \Sigma_{n=i+1}^{N}\ \gamma_{N-n}K^{-1}_{nn}d_{n\cdot i}$, $i = p+1(1)N-1$. We note

(a) $|\Sigma_{n=p+1}^{N}\ \gamma_{N-n}K^{-1}_{nn}d_{n\cdot i}| \leqslant \Gamma_1 < \infty$, and

(b) $|\gamma^*_{N\cdot i}| \leqslant \Gamma_2 < \infty$,

for all i and N, in view of Lemma 4.1.

Collecting the above results, we find

$$|e_N| \leqslant hS \sum_{i=0}^{N-1} |e_i| + R, \qquad N \geqslant 0, \tag{4.8}$$

where $S = \max\{\Gamma_1, \Gamma_2\}$,

$$R = \sum_{k=1}^{p+1}\sum_{i=k}^{p+1} |b_k||\gamma_{N-(p+k)}||e_{p+k-i}| + \left|\sum_{n=p+1}^{N} \gamma_{N-n}K^{-1}_{nn}h^{-1}(E_n - E_{n-1})\right|.$$

The double sum is finite and $O(h^r)$, since it only depends on the starting errors which, due to assumption C, satisfy $e_n = O(h^r)$ $(n = 0(1)p)$. The second sum is $O(h^r)$,

too, which may be easily verified by inserting (4.2) (using appropriate backward differencing), applying the Mean Value Theorem, and observing Lemma 4.1.

Thus the solution of the recursive inequality (4.8) is [7, p. 244],

$$|e_N| \leqslant R(1 + Sh)^N \leqslant R \cdot \exp(NhS), \qquad Nh = a,$$

so that $|e_N| = O(h^r)$ for $n = 0(1)N$. Thus the proof is complete.

5. Construction of Methods. We first consider the choice of the $p + 2 - r$ free parameters so that $\rho(z)$ is a Schur polynomial. An equivalent (and more convenient) approach is to ensure that the polynomial

$$\tau(w) = (w-1)^{p+1}\rho\left(\frac{w+1}{w-1}\right) = \sum_{j=0}^{p+1} \tau_j w^j \tag{5.1}$$

is a Hurwitz polynomial [3, p. 206]. (A Hurwitz polynomial is a polynomial such that $\tau(w) = 0 \Rightarrow \operatorname{Re} w < 0$.)

It is easy to see [3, p. 204] that

$$\tau_j = \sum_{i=0}^{p+1} b_{p+1-i}\gamma_{ij}, \qquad j = 0(1)p+1,$$

where $\gamma_{ij} = \gamma_{i\cdot j-1} - \gamma_{i-1\cdot j-1} - \gamma_{i-1\cdot j}$ $(i, j = 1(1)p+1)$, with

$$\gamma_{i\cdot 0} = (-1)^{p+1-i}, \qquad \gamma_{0\cdot j} = \binom{p+1}{j}(-1)^{p+1-j},$$

$$\gamma_{i\cdot p+1} = 1, \qquad \gamma_{p+1\cdot j} = \binom{p+1}{j},$$

where we have used (3.3).

Remark. It can be easily shown that, for $\rho(z)$ satisfying (3.2) and (3.3)

(i) $\partial\tau_i/\partial b_0 = 0$, $i = 1(1)p+1$, $r = p+1$,

(ii) $\partial\tau_i/\partial b_0 = \partial\tau_i/\partial b_1 = 0$, $i = 2(1)p+1$, $r = p$.

This permits the use of Theorem 5.1 which is a simpler form of the usual Hurwitz criterion. It should be noted, however, that Theorem 5.1 is valid for arbitrary real polynomials. In order to introduce a Hurwitz criterion, we define the sequence of determinants

$$\delta_k = \begin{vmatrix} \tau_1 & \tau_3 & \cdots & \tau_{2k-1} \\ \tau_0 & \tau_2 & \cdots & \tau_{2k-2} \\ 0 & \tau_1 & \cdots & \tau_{2k-3} \\ 0 & \tau_0 & \cdots & \tau_{2k-4} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \tau_k \end{vmatrix}, \qquad k = 1(1)p+1 \ (\tau_j = 0 \text{ if } j > p+1), \tag{5.2}$$

where τ_j are as in (5.1).

The reader should also note that $\tau(w)$ is a Hurwitz polynomial if and only if

$$\tau^*(w) = w^{p+1}\,\tau\left(\frac{1}{w}\right) = \sum_{j=0}^{p+1} \tau_j w^{p+1-j}$$

is a Hurwitz polynomial. We then have the following result.

THEOREM 5.1. *The following conditions are necessary and sufficient for* $\tau^*(w)$ *to be a Hurwitz polynomial*:

(i) $\tau_{p+1} > 0, \tau_{p-1} > 0, \tau_{p-3} > 0, \ldots ; \delta_p > 0, \delta_{p-2} > 0, \ldots ,$ *or*

(ii) $\tau_{p+1} > 0, \tau_p > 0, \tau_{p-2} > 0, \ldots ; \delta_p > 0, \delta_{p-2} > 0, \ldots .$

Proof [2, p. 158]. To construct a simple von Neumann polynomial we chose the free parameters so that

(a) $\tau(0) = 0$,

(b) $\tau(w)/w$ is a Hurwitz polynomial.

TABLE I

	r	p	DOMAIN OF STABILITY
$\rho(z)$: SCHUR	1	0	$b_0 > 1/2$
	2	1	$b_0 > 1/2$
	3	2	$11/24 < b_0 < 15/24$
	4	4	$b_0 = 0,\ 45/24 < b_1 < 77/24$
	5	5	$b_0 = 0$ $3251/1440 < b_1 < (-14518/5 + 7(289845)^{1/2})/288$
	6	6	$b_0 = 0$ $941/360 < b_1 < 133/60 + (1005)^{1/2}/40$
	r	p	DOMAIN OF STABILITY
$\rho(z)$: SIMPLE VON NEUMANN	1	1	$b_0 > 1/4\ ,\ b_1 = 1/2$
	2	2	$b_0 > 3/8\ ,\ b_0 + b_1 = 1$
	3	3	$5/12 < b_0 < 11/12\ ,\ 16b_0 + 8b_1 = 35/3$
	4	4	$5/12 < b_0 < 25/48\ ,\ 3b_0 + b_1 = 15/8$
	5	5	$1151/2880 < b_0 < -1199/480 + (5/8)\,(1307/60)^{1/2}$ $128b_0 + 32b_1 = 3251/45$
	6	6	$b_0 = 0$ $b_1 = 941/360$

For $p = 3$ and $r = 4$; $\tau_1 = 0$ in the polynomial $\tau(w)$. Thus, $\rho(z)$ cannot be a Schur polynomial for any b_0 [3, p. 198].

Conditions (a) and (b) are obviously sufficient for $\rho(z)$ to have a single root at $z = -1$ and p roots in the interior of $|z| = 1$.

In the following table we list a number of polynomials ρ satisfying the conditions mentioned above; the derivation of these results is then illustrated by means of two examples.

Examples. 1. $\rho(z)$ is a Schur polynomial, with $r = 3, p = 2$. Here,

$$\rho(z) = b_0 z^3 + \left(\frac{23}{12} - 3b_0\right) z^2 + \left(3b_0 - \frac{16}{12}\right) z + \left(\frac{5}{12} - b_0\right)$$

and

$$\tau(w) = \left(8b_0 - \frac{44}{12}\right) + \frac{2}{3} w + 2w^2 + w^3.$$

For Theorem 5.1(ii) we use

$$\tau_3 = 1, \quad \tau_2 = 2, \quad \tau_1 = \frac{2}{3}, \quad \tau_0 = 8b_0 - \frac{44}{12},$$

and obtain

$$\delta_2 = \begin{vmatrix} \frac{2}{3} & 1 \\ 8b_0 - \frac{44}{12} & 2 \end{vmatrix} = \frac{60}{12} - 8b_0,$$

which then yields the condition stated in Table I.

2. $\rho(z)$ is a simple von Neumann polynomial, with $r = 2, p = 2$. Here,

$$\rho(z) = b_0 z^3 + b_1 z^2 + \left(\frac{5}{2} - 3b_0 - 2b_1\right) z + \left(2b_0 + b_1 - \frac{3}{2}\right),$$

and

$$\tau(w) = (4 - 4b_0 - 4b_1) + (12b_0 + 4b_1 - 7)w + 2w^2 + w^3.$$

Thus, condition (a) yields

$$4 - 4b_0 - 4b_1 = 0,$$

while (b) implies

$$12b_0 + 4b_1 - 7 > 0,$$

which may be written as

$$12b_0 + 4(1 - b_0) - 7 > 0.$$

6. Examples and Conclusions. It can be shown that the global discretization error, e_n, contains terms of the form $z_k^n d_k(x_n)$, where z_k are the roots of $\rho(z)$ and $d_k(x)$ are solutions of the initial value problems

$$K(x, x)d_k'(x) - z_k K_t(x, x)d_k(x) = 0, \qquad d_k(x_0) = \delta_k. \tag{6.1}$$

(See [6] for details; compare also Kobayasi [9] where a corresponding analysis is given for the trapezoidal method.) Thus, in the case of quadrature rule methods based on simple von Neumann polynomials, we get "marginally stable" approximations depending on whether $|d_k(x)|$ is monotone increasing or decreasing (with increasing x).

TABLE II(a)

$$\left\{ \begin{matrix} |e_n| \\ |e_{n+1}| \end{matrix} \right\} \text{ at } x_n = 2$$

METHOD	h	PROBLEM I(a=1)	I(a=-1)	II(a=1)	II(a=-1)
A	.1	-3.0×10^{-3} $+3.0 \times 10^{-4}$	$+1.4 \times 10^{-3}$ -2.1×10^{-3}	-4.6×10^{-2} $+2.7 \times 10^{-2}$	-2.7×10^{-2} -3.4×10^{-2}
	.05	-1.9×10^{-4} $+2.1 \times 10^{-4}$	$+7.0 \times 10^{-5}$ -3.2×10^{-4}	-5.1×10^{-3} $+4.7 \times 10^{-3}$	-4.1×10^{-3} -4.7×10^{-3}
B	.1	-5.7×10^{-5} $+7.0 \times 10^{-4}$	-2.2×10^{-4} $+6.9 \times 10^{-5}$	$+5.4 \times 10^{-3}$ -2.0×10^{-3}	-1.4×10^{-3} -2.3×10^{-3}
	.05	$+1.6 \times 10^{-5}$ $+5.5 \times 10^{-5}$	-1.3×10^{-5} $+1.1 \times 10^{-5}$	$+3.6 \times 10^{-4}$ -3.2×10^{-4}	-1.3×10^{-4} -1.7×10^{-4}
C	.1	$+1.7 \times 10^{-5}$ $+3.3 \times 10^{-5}$	-1.9×10^{-4} -2.3×10^{-4}	-2.2×10^{-4} $+7.6 \times 10^{-4}$	-6.3×10^{-3} -7.0×10^{-3}
	.05	$+1.7 \times 10^{-5}$ $+1.6 \times 10^{-5}$	-2.8×10^{-5} -2.8×10^{-5}	$+1.9 \times 10^{-5}$ $+1.7 \times 10^{-5}$	-8.5×10^{-4} -9.0×10^{-4}
D	.1	-8.2×10^{-4} -9.0×10^{-4}	$+2.6 \times 10^{-6}$ $+4.7 \times 10^{-6}$	$+1.1 \times 10^{-4}$ $+1.1 \times 10^{-4}$	$+3.3 \times 10^{-3}$ $+3.7 \times 10^{-3}$
	.05	-6.1×10^{-5} -6.4×10^{-5}	-6.0×10^{-7} -5.0×10^{-7}	$+6.2 \times 10^{-6}$ $+5.9 \times 10^{-6}$	$+2.4 \times 10^{-4}$ $+2.6 \times 10^{-4}$
E	.1	$+2.2 \times 10^{-6}$ $+2.5 \times 10^{-6}$	$+4.4 \times 10^{-8}$ $+4.0 \times 10^{-8}$	-1.8×10^{-7} -1.7×10^{-7}	-8.5×10^{-6} -9.4×10^{-6}
	.05	$+1.4 \times 10^{-7}$ $+1.5 \times 10^{-7}$	$+3.0 \times 10^{-9}$ $+3.0 \times 10^{-9}$	-1.1×10^{-8} -1.0×10^{-8}	-5.5×10^{-7} -5.8×10^{-7}
$y(x_n)$		7.3890561	.1353353	.1353353	7.3890561

We shall illustrate this phenomenon with the following examples.

I. $\int_0^x (a^2 + 1) \cos(x - t) y(t)\, dt = ae^{ax} + \sin x - a \cos x$, $y(x) = e^{ax}$, $d_k(x) = \delta_k$.

II. $a \int_0^x e^{a(x-t)} y(t)\, dt = \sinh ax$, $y(x) = e^{-ax}$, $d_k(x) = \delta_k e^{ax}$, $z_k = -1$ (for our methods).

Thus, we expect serious stability problems in example II with $a > 0$.

The following methods will be used for the solution.

A: $\rho(z)$ (simple von Neumann polynomial): $r = p = 3$, $b_0 = \frac{3}{4}$, $b_1 = \frac{-1}{24}$;

B: $\rho(z)$ (simple von Neumann polynomial): $r = p = 4$, $b_0 = \frac{1}{2}$, $b_1 = \frac{3}{8}$;

C: $\rho(z)$ (Schur polynomial): $r = 3$, $p = 2$, $b_0 = \frac{1}{2}$;

D: $\rho(z)$ (Schur polynomial): $r = p = 4$, $b_0 = 0$, $b_1 = \frac{54}{24}$;

E: block method: $u_i = \frac{i}{4}$, $i = 1(1)4$ (see Weiss [13, p. 51]).

Exact starting values were used, when needed. All computations were performed on the CDC 6400 at Dalhousie University (in double precision 120 bit word).

TABLE II(b)

$$\left\{ \begin{matrix} |e_n| \\ |e_{n+1}| \end{matrix} \right\} \text{ at } x_n = 4$$

		PROBLEM			
METHOD	h	I(a=1)	I(a = -1)	II(a=1)	II(a = -1)
A	.1	-1.7×10^{-2} -1.6×10^{-2}	$+2.8 \times 10^{-4}$ -1.9×10^{-3}	-2.3×10^{0} $+2.1 \times 10^{0}$	-2.2×10^{-1} -2.5×10^{-1}
	.05	-1.1×10^{-3} -8.6×10^{-4}	$+2.8 \times 10^{-5}$ -3.1×10^{-4}	-3.5×10^{-2} $+3.6 \times 10^{-2}$	-3.2×10^{-2} -3.3×10^{-2}
B	.1	$+3.2 \times 10^{-3}$ $+4.1 \times 10^{-3}$	-1.3×10^{-4} $+7.2 \times 10^{-5}$	$+2.8 \times 10^{-2}$ -2.6×10^{-2}	-1.4×10^{-2} -1.6×10^{-2}
	.05	$+2.5 \times 10^{-4}$ $+3.0 \times 10^{-4}$	-1.2×10^{-5} $+1.1 \times 10^{-5}$	$+2.7 \times 10^{-3}$ -2.9×10^{-3}	-1.1×10^{-3} -1.1×10^{-3}
C	.1	-1.6×10^{-3} -1.8×10^{-3}	-2.0×10^{-4} -2.0×10^{-4}	$+6.1 \times 10^{-5}$ -2.5×10^{-5}	-4.7×10^{-2} -5.2×10^{-2}
	.05	-9.3×10^{-5} -1.0×10^{-4}	-2.8×10^{-5} -2.8×10^{-5}	$+2.5 \times 10^{-6}$ $+2.4 \times 10^{-6}$	-6.3×10^{-3} -6.6×10^{-3}
D	.1	-6.0×10^{-3} -6.6×10^{-3}	$+2.7 \times 10^{-5}$ $+2.7 \times 10^{-5}$	$+1.6 \times 10^{-5}$ $+1.5 \times 10^{-5}$	$+2.5 \times 10^{-2}$ $+2.7 \times 10^{-2}$
	.05	-4.5×10^{-4} -4.7×10^{-5}	-7.0×10^{-7} -7.0×10^{-7}	$+8.0 \times 10^{-7}$ $+8.0 \times 10^{-7}$	$+1.8 \times 10^{-3}$ $+1.9 \times 10^{-3}$
E	.1	$+1.6 \times 10^{-5}$ $+1.8 \times 10^{-5}$	$+6.0 \times 10^{-9}$ $+5.0 \times 10^{-9}$	-2.5×10^{-8} -2.3×10^{-8}	-6.3×10^{-5} -6.9×10^{-5}
	.05	$+1.0 \times 10^{-6}$ $+1.1 \times 10^{-6}$	$0^{(1)}$ $0^{(1)}$	-1.0×10^{-9} -1.0×10^{-9}	-4.1×10^{-6} -4.3×10^{-6}
$y(x_n)$		54.5981500	.0183156	.0183156	54.5981500

Remarks. (1) The errors for II($a = 1$) are diverging from the exact solution for methods A and B (simple von Neumann polynomial), although not as markedly for $h = 0.05$.

(2) Methods A and C are order 3 while B and D are order 4.

(3) Method E requires r^2 as many kernel evaluations as the others for a given stepsize, h.

The results in this paper are a summary of the author's Ph.D. thesis written at Dalhousie University. Families of quadrature rule methods for the numerical solution of Volterra integral equations of the first kind have been constructed.

The question of existence, for arbitrary r, hinges on the possibility of whether one can construct polynomials $\rho(z)$ (which are Schur or simple von Neumann polynomials) by choosing $p + 2 - r$ (the number of free parameters) sufficiently large.

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