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Tables for the Gaussian Computation of $\int_0^\infty x^\alpha e^{-x} f(x)\,dx$ for Values of α Varying Continuously Between -1 and $+1$

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Abstract. Tables of coefficients of Chebyshev expansions for computing to 11 places the abscissas and weight factors of the 12-point generalized Gauss-Laguerre quadrature formula are presented.

1. Introduction. One of the most efficient methods of computing integrals of the form

$$I(\alpha) = \int_0^\infty x^\alpha e^{-x} f(x)\,dx, \tag{1}$$

when $f(x)$ is an arbitrary function easily represented by a polynomial of low degree, is certainly the Gaussian quadrature [1], [2]. In this method, the integral under consideration is approximated by the n-point expression

$$I(\alpha) \sim \sum_{i=1}^{n} w_i(\alpha) f(x_i(\alpha)), \tag{2}$$

where the functions $x_i(\alpha)$ are the zeros of the associated Laguerre polynomials, explicitly defined by [3]

$$L_n^\alpha(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}, \tag{3}$$

and where the weight factors $w_i(\alpha)$ are given by

$$w_i(\alpha) = \frac{\Gamma(n+\alpha+1)x_i(\alpha)}{n!\,[(n+1)L_{n+1}^\alpha(x_i(\alpha))]^2}. \tag{4}$$

In principle, the calculation of $x_i(\alpha)$ and $w_i(\alpha)$ is straightforward, since recurrence relations are known for $L_n^\alpha(x)$. Indeed, for some particular values of α, tables for $x_i(\alpha)$ and $w_i(\alpha)$ are available [4]–[13]. It should also be noted that Gaussian quadrature using zeros of Hermite polynomials is a special case of the Gauss-Laguerre quadrature for which $\alpha = -1/2$.

Nevertheless, some applications require the value of the integral equation (1) for arbitrary values of the parameter α. The complete calculation of the abscissas and

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weight factors at each calling of the integration subprogram would entail the loss of at least part of the efficiency inherent to the Gauss-Laguerre quadrature formula. On the other hand, not enough values of α have been considered up to now to allow precise interpolation between known results.

The purpose of this paper is to present tables permitting fast and accurate evaluation of the integral for any values of α lying within the interval $(-1, +1]$, by means of expansions of the functions $x_i(\alpha)$ and $w_i(\alpha)x_i(\alpha)$ in terms of Chebyshev polynomials of the variable α [14].

In Section 2, the coefficients of the Chebyshev expansions are obtained. The expected precision is then discussed in Section 3, where a typical example is treated and examined in order to illustrate the efficiency of the proposed algorithm.

2. Chebyshev Expansions of the Abscissas and Weight Factors. In this section, the proposed method for obtaining the abscissas and weight factors for Gauss-Laguerre integration of (1) is derived, for values of α ranging from -1 to $+1$. This method amounts to calculating $x_i(\alpha)$ and the product $w_i(\alpha)x_i(\alpha)$ for $i = 1, 2, \ldots, n$, in the following forms

$$x_i(\alpha) = \sum_{k=0}^{N} A_{i,k} T_k(\alpha), \tag{5}$$

$$w_i(\alpha)x_i(\alpha) = \sum_{k=0}^{N} B_{i,k} T_k(\alpha), \tag{6}$$

where $T_k(\alpha)$ is the Chebyshev polynomial of degree k and $A_{i,k}$, $B_{i,k}$ are coefficients to be determined numerically. The quantities $x_i(\alpha)$ and $w_i(\alpha)x_i(\alpha)$ are analytic in α in the neighborhood of $\alpha = -1$. In this connection, it can be readily shown that

$$L_n^\alpha(x) = \frac{\Gamma(n + \alpha + 1)}{n!}\left[\frac{1}{\Gamma(\alpha + 1)} + \sum_{k=1}^{n} \frac{(-n)_k x^k}{k!\Gamma(\alpha + 1 + k)}\right]. \tag{7}$$

If $x_1(\alpha)$ is the smallest zero, then

$$x_1(\alpha) \sim \frac{\alpha + 1}{n} \tag{8}$$

for α near -1, and

$$\lim_{\alpha \to -1} w_1(\alpha)\, x_1(\alpha) = \frac{1}{n}. \tag{9}$$

The coefficients of the expansions (5) and (6) are given by [14], [15]

$$A_{i,k} = \delta_k (N + 1)^{-1} \sum_{l=0}^{N} x_i(\alpha_l) T_k(\alpha_l), \tag{10}$$

$$B_{i,k} = \delta_k (N + 1)^{-1} \sum_{l=0}^{N} w_i(\alpha_l) x_i(\alpha_l) T_k(\alpha_l), \tag{11}$$

where δ_k is equal to 1 if $k = 0$ and equal to 2 otherwise. The α_l ($l = 0, 1, \ldots, N$) are the zeros of the Chebyshev polynomials of degree $N + 1$, i.e.

$$\alpha_l = \cos\left(\frac{2l + 1}{N + 1}\,\frac{\pi}{2}\right). \tag{12}$$

The factors $x_i(\alpha_l)$ appearing in expressions (10) and (11) have been computed by use of the Bairstow iteration method [16], using the explicit expression (3) of the associated Laguerre polynomials. The iterations were carried out in order to obtain the zeros $x_i(\alpha_l)$ to at least 12 places.

The computation of the coefficients $A_{i,k}$ and $B_{i,k}$ defined in the expansions (5) and (6) is straightforward if one uses the explicit expressions (10) and (11). The following table (Table 1) displays the values of $A_{i,k}$ and $B_{i,k}$ to 12 places for $N = 20$ and $n = 12$.

3. Discussion. To check the accuracy of expansions (10) and (11), one may use the coefficients given in the following table to evaluate the integral

$$I(\alpha) = \int_0^\infty x^\alpha e^{-x} x^{23}\, dx \tag{13}$$

by Gauss-Laguerre quadrature, provided that the 12-point method should give the exact result

$$I(\alpha) = \Gamma(\alpha + 24). \tag{14}$$

Most of the difference between Gauss-Laguerre and exact results arises in this case from the use of the limited expansions (5) and (6). The relative error produced by the above technique is plotted in Figure 1 as a function of α. As expected, the relative discrepancy between exact and Gauss-Laguerre values of (13) never exceeds $3 \cdot 10^{-12}$ in the whole range of variation of α. This provides an estimate of the accuracy obtained in computing $x_i(\alpha)$ and $w_i(\alpha)$ by the Chebyshev expansions obtained in the present paper. It indicates that errors in the computed abscissas and weight factors are expected only in the 12th place.

To illustrate the application of this method, one can consider the following example

$$I(\alpha) = \int_0^\infty x^\alpha e^{-x}(e^{-x} - 1)^2\, dx, \tag{15}$$

which can be exactly expressed as

$$I(\alpha) = \Gamma(\alpha + 1)[1 - 2^{-\alpha} + 3^{-\alpha-1}]. \tag{16}$$

The relative error produced in this case is plotted on Figure 2 as a function of the parameter α. Here, the relative precision obtained is reduced to 10^{-6} in contrast to the case of the integral (13). The oscillations specific to the interpolation scheme have here completely disappeared: the discrepancy is now mainly due to the error inherent to the use of a Gauss-associated Laguerre quadrature formula.

TABLE 1

Coefficients $A_{i,k}$ and $B_{i,k}$ ($i = 1, 2, \ldots, 12$; $k = 1, 2, \ldots, 20$) of the Chebyshev expansion of $x_i(\alpha)$ and $x_i(\alpha)w_i(\alpha)$, respectively.

$A_{1,K}$	$B_{1,K}$	$A_{2,K}$	$B_{2,K}$
(-1) 1.2850 46876 87	(-2) 4.1966 70945 98	(-1) 6.2061 05395 60	(-1) 2.8280 27455 60
(-1) 1.4208 73601 54	(-2)-2.3016 64516 05	(-1) 3.2305 50217 75	(-2)-2.9346 33000 48
(-2) 1.2851 30857 66	(-2) 1.2565 37415 03	(-3) 8.8361 09371 35	(-2) 5.4751 68961 49
(-4)-6.4617 11907 35	(-3)-3.8538 76260 81	(-4)-1.6243 15724 38	(-3)-8.2847 42445 57
(-5) 7.0857 35220 93	(-3) 1.3481 57882 00	(-5)-1.8520 29055 45	(-3) 3.7987 86270 24
(-5)-1.1530 25291 02	(-4)-4.0850 30279 64	(-6) 6.6666 42576 73	(-4)-8.7479 32609 48
(-6) 2.2146 07821 56	(-4) 1.2376 97531 11	(-6)-1.6607 55665 80	(-4) 2.6545 03470 07
(-7)-4.6074 86955 99	(-5)-3.5976 78092 96	(-7) 3.9031 07398 36	(-5)-6.9270 99951 98
(-7) 1.0041 30130 29	(-5) 1.0302 85187 15	(-8)-9.0856 17938 03	(-5) 1.8928 87726 36
(-8)-2.2561 11737 06	(-6)-2.9016 38419 93	(-8) 2.1209 31472 21	(-6)-5.0352 52111 86
(-9) 5.1800 27336 97	(-7) 8.0859 12799 12	(-9)-4.9832 07323 12	(-6) 1.3453 41676 08
(-9)-1.2087 82174 17	(-7)-2.2340 93039 56	(-9) 1.1795 13118 86	(-7)-3.5795 26977 34
(-10) 2.8567 24119 92	(-8) 6.1341 60507 83	(-10)-2.8124 82983 13	(-8) 9.5252 66292 69
(-11)-6.8208 02129 46	(-8)-1.6762 15052 13	(-11) 6.7531 99131 59	(-8)-2.5318 83611 88
(-11) 1.6424 54326 23	(-9) 4.5638 29170 48	(-11)-1.6320 60638 89	(-9) 6.7426 53047 00
(-12)-3.9835 58807 72	(-9)-1.2391 44484 42	(-12) 3.9666 33605 94	(-9)-1.7949 32807 32
(-13) 9.7232 86548 66	(-10) 3.3572 55452 93	(-13)-9.7046 23935 73	(-10) 4.7804 11149 21
(-13)-2.3851 91583 28	(-11)-9.0807 51621 32	(-13) 2.3783 95834 18	(-10)-1.2788 86212 56
(-14) 5.8910 01386 10	(-11) 2.4529 32918 51	(-14)-5.8369 97551 96	(-11) 3.3957 84579 45
(-14)-1.4564 64519 34	(-12)-6.6193 59445 85	(-14) 1.4888 29635 70	(-12)-9.0582 10971 42

$A_{3,K}$	$B_{3,K}$	$A_{4,K}$	$B_{4,K}$
1.5186 36586 70	(-1) 4.3710 34620 57	2.8373 73140 11	(-1) 3.2103 55629 27
(-1) 4.9192 41979 10	(-1) 1.6798 72087 34	(-1) 6.5139 31003 45	(-1) 2.2274 80093 08
(-3) 6.0175 79369 34	(-2) 7.0282 27735 28	(-3) 3.6159 40443 63	(-2) 6.7129 66878 51
(-5)-7.8918 51678 42	(-3) 7.6007 74958 04	(-5)-2.2945 82041 45	(-2) 1.3222 16160 05
(-6)-9.0598 68859 95	(-3) 2.4368 29055 91	(-6)-5.9669 57068 51	(-3) 2.4583 37337 06
(-6) 2.0390 02508 31	(-4) 1.1111 23582 46	(-7) 8.9721 32211 00	(-4) 3.4723 54028 38
(-7)-3.2974 23598 73	(-5) 5.8374 43174 62	(-7)-1.0636 16664 86	(-5) 4.9277 24041 00
(-8) 5.0322 97687 09	(-6)-1.6342 67005 05	(-8) 1.2013 29952 38	(-6) 5.4311 70565 69
(-9)-7.6053 65756 92	(-6) 1.3532 95675 91	(-9)-1.3460 92307 57	(-7) 6.5671 21222 58
(-9) 1.1525 41336 03	(-7)-1.4638 52028 99	(-10) 1.5133 45333 82	(-8) 5.6801 76412 66
(-10)-1.7573 49348 34	(-8) 3.6721 65261 58	(-11)-1.7158 43927 84	(-9) 6.5666 09933 79
(-11) 2.6983 09785 48	(-9)-5.8194 00999 15	(-12) 1.9301 92631 24	(-10) 3.9811 91991 30
(-12)-4.1708 99222 70	(-9) 1.1215 16164 91	(-13)-2.2235 71121 46	(-11) 5.6513 52739 68
(-13) 6.4699 06914 74	(-10)-1.9567 27526 92	(-15) 7.7304 41801 09	(-12) 1.0831 95261 88
(-14)-9.9511 96245 62	(-11) 3.5349 18037 29	(-14)-2.6809 83007 61	(-13) 4.4395 04319 71
(-14) 2.6978 41949 83	(-12)-6.1753 42130 65	(-14)-3.5132 39082 36	(-13)-1.1217 98127 64
(-15) 4.1643 64326 62	(-12) 1.1673 77922 72	(-14)-2.9770 42480 84	(-14)-4.0977 50945 14
(-15) 3.4560 83156 28	(-13)-1.6002 42572 19	(-15) 5.5264 43500 35	(-14) 1.5645 92077 29
(-15) 2.4280 98874 22	(-14) 5.4585 96537 74	(-14) 1.0625 24553 93	(-14) 1.9466 93834 91
(-15) 2.0097 09271 42	(-15) 6.7466 60843 16	(-15)-1.6015 99511 45	(-14) 2.8112 28616 15

$A_{5,K}$	$B_{5,K}$	$A_{6,K}$	$B_{6,K}$
4.6007 53364 70	(-1) 1.2542 21518 46	6.8441 98533 83	(-2) 2.6848 14882 21
(-1) 8.0320 04176 28	(-1) 1.1197 49176 18	(-1) 9.4889 46551 16	(-2) 2.7577 40274 18
(-3) 1.5208 82523 74	(-2) 3.4387 84839 69	(-4)-3.3142 46291 65	(-3) 9.0365 39931 01
(-5) 2.0784 23235 37	(-3) 7.7543 02441 09	(-5) 5.6395 60111 64	(-3) 2.1950 79535 27
(-6)-4.8881 17573 01	(-3) 1.4528 02125 42	(-6)-4.5291 68191 91	(-4) 4.3285 50927 76
(-7) 5.0102 28662 56	(-4) 2.2953 08879 21	(-7) 3.3199 48085 49	(-5) 7.2043 14243 84
(-8)-4.5712 23253 42	(-5) 3.2128 12651 30	(-8)-2.3948 92738 44	(-5) 1.0487 18203 10
(-9) 4.0640 59015 24	(-6) 3.9916 40997 33	(-9) 1.7324 75743 11	(-6) 1.3596 60040 36
(-10)-3.6031 68956 61	(-7) 4.5409 84364 60	(-10)-1.2778 09792 13	(-7) 1.5960 11558 18
(-11) 3.2425 63419 41	(-8) 4.7026 13763 53	(-12) 7.9515 98227 59	(-8) 1.7146 10083 30
(-12)-2.7130 72564 96	(-9) 4.5570 51620 12	(-12)-1.5186 20620 20	(-9) 1.7028 63841 25
(-13) 3.9499 26805 38	(-10) 4.0720 00338 51	(-13)-3.7438 36516 52	(-10) 1.5736 94238 83
(-15) 1.5460 88360 21	(-11) 3.4780 52169 82	(-16)-4.6053 69583 63	(-11) 1.3613 63934 39
(-13) 1.5707 59982 98	(-12) 2.8321 64672 53	(-15)-7.4837 25573 39	(-12) 1.0890 08769 99
(-13) 1.0230 49957 50	(-13) 2.8032 45675 71	(-16) 8.0593 96771 35	(-14) 6.4081 13494 57
(-13) 1.3459 19260 81	(-14) 6.0466 03547 07	(-13)-3.0046 74698 34	(-14)-2.7071 06658 18
(-13) 1.4964 16159 70	(-14) 1.6715 47416 94	(-13)-6.0170 79838 49	(-14)-3.4727 11605 16
(-14)-6.7304 18691 50	(-14)-6.0531 31247 26	(-13) 2.6612 45709 39	(-14)-1.2286 23343 85
(-14)-7.2468 77994 81	(-14)-6.5998 58262 78	(-14) 3.6678 47918 39	(-15)-8.9293 00361 10
(-14) 8.3159 81648 15	(-14)-2.6908 64506 52	(-13)-5.9805 65836 79	(-15)-4.1121 37906 42

TABLE 1 (*continued*)

A 7,K	B 7,K	A 8,K	B 8,K
9.6193 33049 45	(-3) 3.1110 11288 32	(1) 1.3002 57172 35	(-4) 1.8588 81555 01
1.0899 62074 76	(-3) 3.4924 93818 15	1.2279 61643 54	(-4) 2.2186 80981 88
(-3)-1.9882 32378 68	(-3) 1.2187 97121 10	(-3)-3.4877 54317 61	(-5) 8.1719 94320 05
(-5) 8.6015 42729 86	(-4) 3.1365 11595 45	(-4) 1.1105 02386 82	(-5) 2.2075 34021 41
(-6)-4.4539 08457 98	(-5) 6.4820 08489 41	(-6)-4.4945 78454 96	(-6) 4.7606 14285 46
(-7) 2.4937 80406 37	(-5) 1.1281 24419 92	(-7) 2.0487 35093 67	(-7) 8.6081 13067 98
(-8)-1.4586 61377 69	(-6) 1.7061 54890 78	(-8)-1.0009 72779 56	(-7) 1.3475 14440 47
(-10) 8.7843 13030 06	(-7) 2.2911 10176 45	(-10) 5.1156 32526 64	(-8) 1.8673 58074 95
(-11)-5.0900 05783 11	(-8) 2.7758 99130 81	(-11)-3.1568 65715 33	(-9) 2.3287 98042 08
(-12) 7.1104 60368 64	(-9) 3.0715 96991 91	(-12)-4.4238 52231 04	(-10) 2.6467 15050 32
(-12) 2.2239 65867 43	(-10) 3.1342 89332 38	(-12)-4.9586 83669 43	(-11) 2.7686 13963 77
(-13) 7.2069 09965 92	(-11) 2.9724 86591 43	(-13)-3.6074 84681 34	(-12) 2.6867 98042 39
(-13)-8.2973 95692 33	(-12) 2.6332 70072 86	(-12) 3.3988 77886 95	(-13) 2.4311 45326 72
(-12)-1.9143 37001 67	(-13) 2.1243 43934 75	(-12) 6.4197 70065 34	(-14) 2.0215 17738 92
(-12)-1.6330 31158 85	(-14) 1.0163 33927 38	(-12) 5.8019 10392 45	(-15) 1.1972 80192 58
(-14) 4.0017 37213 20	(-15)-2.4700 93638 37	(-12) 1.5392 62544 58	(-16)-3.1165 06518 35
(-12) 1.1955 70391 65	(-15)-2.3717 68548 02	(-13)-5.9542 49439 17	(-16)-4.5324 19341 62
(-13)-6.5215 32285 39	(-15)-1.5495 53791 60	(-12) 1.2582 69208 99	(-16)-3.4652 23708 42
(-13) 9.7651 92769 63	(-15) 2.8981 92873 96	(-12)-3.2682 16972 77	(-16)-2.6465 63384 59
(-12) 1.9246 49739 99	(-15) 3.6035 82838 50	(-12)-3.1327 36868 27	(-16)-2.3024 18603 86

A 9,K	B 9,K	A 10,K	B 10,K
(1) 1.7111 99640 50	(-6) 5.2092 51402 59	(1) 2.2144 94722 77	(-8) 5.7619 08505 13
1.3647 35455 20	(-6) 6.5033 29944 36	1.5028 37558 60	(-8) 7.4472 86088 81
(-3)-4.8633 55880 74	(-6) 2.5067 13708 19	(-3)-6.1478 28320 37	(-8) 2.9888 98928 05
(-4) 1.3253 89903 73	(-7) 7.0601 22548 95	(-4) 1.5131 99590 55	(-9) 8.7175 16367 38
(-6)-4.5810 14235 19	(-7) 1.5808 92074 98	(-6)-4.6827 63329 75	(-9) 2.0188 22055 00
(-7) 1.7904 66782 73	(-8) 2.9577 03984 20	(-7) 1.6310 76263 68	(-10) 3.8954 77317 59
(-9)-7.5436 31415 00	(-9) 4.7762 27419 47	(-9)-6.1125 76519 02	(-11) 6.4720 02085 19
(-10) 3.3332 20777 57	(-10) 6.8106 43022 24	(-10) 2.4224 40190 82	(-12) 9.4747 64485 55
(-11)-1.0379 51617 65	(-11) 8.7209 44884 63	(-11)-1.3870 05736 60	(-12) 1.2482 85267 00
(-12) 6.6269 78801 10	(-11) 1.0157 88401 81	(-12)-3.4671 85386 53	(-13) 1.4816 37120 71
(-12) 6.6759 93091 67	(-12) 1.0872 18228 98	(-12)-5.9693 48472 31	(-14) 1.6202 74845 11
(-13)-1.1977 25046 71	(-13) 1.0782 26813 99	(-14)-8.3422 98045 77	(-15) 1.6395 11812 69
(-12)-6.2423 64649 88	(-14) 1.0010 13137 87	(-12) 6.1089 89857 18	(-16) 1.5402 99395 27
(-11)-1.0074 60781 46	(-16) 9.2336 90299 45	(-12) 8.7593 47157 07	(-17) 1.2784 63462 20
(-12)-9.6459 63040 66	(-16) 1.2637 31550 73	(-12) 8.5265 45724 61	(-19) 2.5443 76927 89
(-12)-3.4354 57679 65	(-17) 3.8363 70546 61	(-12) 3.3920 52071 32	(-19)-5.9521 60690 08
(-12)-1.7934 13154 60	(-18) 9.3164 68329 46	(-12) 3.4950 80768 01	(-19)-4.8269 20249 21
(-12)-1.9748 31820 70	(-18)-4.9344 16904 00	(-12) 2.2288 17953 24	(-19)-2.8382 01897 82
(-12) 4.2205 41611 67	(-17)-1.2649 02338 49	(-12)-1.8848 79084 08	(-20)-9.8707 40888 21
(-12) 2.3792 49061 14	(-17)-1.1445 62387 34	(-13)-1.1204 20628 70	(-20)-6.5228 80157 57

A 11,K	B 11,K	A 12,K	B 12,K
(1) 2.8480 59055 68	(-10) 1.7896 82231 18	(1) 3.7090 48418 47	(-14) 6.5697 96941 15
1.6466 86008 50	(-10) 2.3793 43670 46	1.8072 62506 45	(-14) 8.9527 53240 26
(-3)-7.3814 76406 38	(-11) 9.8626 55183 78	(-3)-8.6417 48351 77	(-14) 3.8305 03110 21
(-4) 1.6817 69970 30	(-11) 2.9781 70177 71	(-4) 1.8418 56544 99	(-14) 1.1938 72014 14
(-6)-4.7874 52260 59	(-12) 7.1133 62726 20	(-6)-4.8932 33235 88	(-15) 2.9386 95398 97
(-7) 1.5266 20794 55	(-12) 1.4125 30450 48	(-7) 1.4530 90319 01	(-16) 6.0028 10833 98
(-9)-5.2181 23217 47	(-13) 2.4101 95823 34	(-9)-4.6163 53258 50	(-16) 1.0517 60493 18
(-10) 1.8589 24536 93	(-14) 3.6171 88209 38	(-10) 1.5382 25020 60	(-17) 1.6182 16926 57
(-12)-4.8216 90372 07	(-15) 4.8580 91580 01	(-12)-5.7664 49046 65	(-18) 2.2248 19876 28
(-12) 1.7695 14576 01	(-16) 5.9171 24330 10	(-14)-4.0000 92438 35	(-19) 2.7703 27620 98
(-12) 3.0213 85610 83	(-17) 6.6050 01686 74	(-13)-6.6554 16958 28	(-20) 3.1577 17603 09
(-13) 3.5842 93355 94	(-18) 6.8151 55313 26	(-13)-1.2658 18725 55	(-21) 3.3282 99998 27
(-12)-3.1016 50623 58	(-19) 6.5572 20852 55	(-13) 6.5633 09566 61	(-22) 3.2516 28194 59
(-12)-3.9714 07564 94	(-20) 6.0652 54479 41	(-13) 7.8396 54851 21	(-23) 2.9687 98278 47
(-12)-3.8935 11027 98	(-21) 6.8041 64947 34	(-13) 6.6317 32200 42	(-24) 2.4156 37070 54
(-12)-1.4871 39629 54	(-21) 1.8922 48771 49	(-13) 2.6632 19439 21	(-26) 7.5966 74639 68
(-12)-2.4700 57080 68	(-21) 1.2381 68925 99	(-13) 6.2869 87391 59	(-25)-1.0712 39034 88
(-12)-1.3954 43431 58	(-22) 8.7855 48622 92	(-13) 3.6527 15989 75	(-25)-1.0530 49801 51
(-13)-2.8666 78088 32	(-22) 5.7480 70686 81	(-13) 3.5079 75802 84	(-26)-8.8400 53094 35
(-13)-8.2311 11265 82	(-22) 4.7830 21851 91	(-13) 3.5099 49532 66	(-26)-7.4329 00176 65

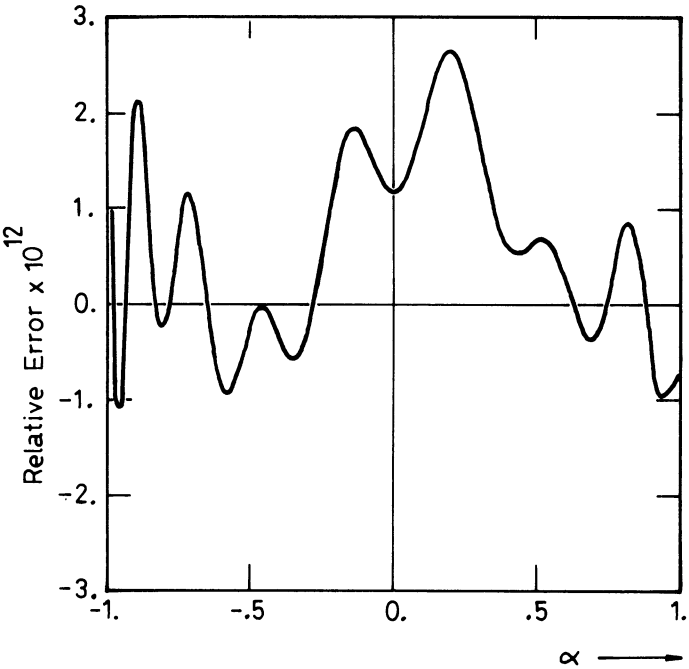

FIGURE 1

Relative error produced by the computing technique presented in the present paper on the evaluation of the 23rd moment of the weight function of the generalized Gauss-Laguerre quadrature formula. This error is plotted against the parameter α.

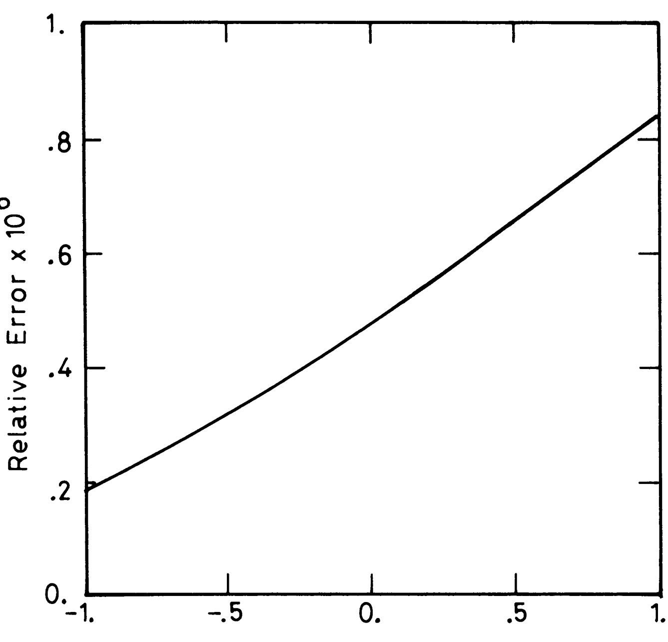

FIGURE 2

Relative error produced when $f(x)$ is the function $f(x) = (e^{-x} - 1)^2$. The parameter α varies from -1 to $+1$.

4. Conclusion. The method presented here has been developed to allow fast computation of zeros and weight factors for the generalized Gauss-Laguerre quadrature formula, for arbitrary values of the parameter appearing in the weight function. The precision obtained is better than $3 \cdot 10^{-12}$.

The Gauss-associated Laguerre quadrature method is a special case of a Gaussian formula involving a weight function depending on a parameter α. The method used here, i.e., an α-wise expansion in orthogonal polynomials of the abscissas and weight factors could easily be applied to other types of weight functions in Gaussian formulas. The summation of the expansions can then be achieved by use of the recurrence relations verified by the orthogonal polynomials, the advantage being then that the computing time of zeros and weight factors is not any more prohibitive and, at the same time, the efficiency of the Gaussian formula is preserved.

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