The Hankel Power Sum Matrix Inverse and the Bernoulli Continued Fraction

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Abstract. The $m \times m$ Hankel power sum matrix $W = VV^T$ (where V is the $m \times n$ Vandermonde matrix) has (i, j)-entry $S_{i+j-2}(n)$, where $S_p(n) = \Sigma_{k=1}^n k^p$. In solving a statistical problem on curve fitting it was required to determine $f(m)$ so that for $n > f(m)$ all eigenvalues of W^{-1} would be less than 1. It is proved, after calculating W^{-1} by first factoring W into easily invertible factors, that $f(m) = (13m^2 - 5)/8$ suffices. As by-products of the proof, close approximations are given for the Hilbert determinant, and a convergent continued fraction with mth partial denominator $m^{-1} + (m + 1)^{-1}$ is found for the divergent Bernoulli number series $\Sigma B_{2k}(2x)^{2k}$.

1. **Introduction.** Defined as the product $W = VV^T$ of the $m \times n$ Vandermonde matrix $V = (j^{i-1})$ with its transpose V^T, the $m \times m$ Hankel power sum matrix $W = W_m$ has (i, j)-entry $S_{i+j-2}(n)$, where

$$(1.1) \qquad S_p = S_p(n) = \sum_{k=1}^n k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \sum_{k=1}^{[p/2]} \frac{B_{2k}}{2k} \binom{p}{2k-1},$$

and where B_{2k} are the Bernoulli numbers [3], [6]:

$$(1.2) \quad B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \quad B_8 = -1/30, \quad B_{10} = 5/66, \ldots .$$

In solving a statistical problem involving the fitting of polynomial curves of degree m to $n > m$ points, for increasing m and n, it was required [4] to find a function $f(m)$, such that, whenever $n > f(m)$, the eigenvalues μ_k of $M = W^{-1}$ would all be less than 1. We evaluate $w_m = \det W_m$ in Section 2 as

$$(1.3) \qquad w_m = \det W_m = h_m \prod_{i,j=1}^m (n + i - j),$$

where h_m is the determinant of the Hilbert matrix H_m, and we obtain close estimates for h_m. In Section 3 we factor W_m as a product of easily invertible matrices of which only diagonal matrices involve n, and we also explicitly invert W_m and H_m. In Section 4 we estimate the trace of $M = W^{-1}$ and find that the function

$$(1.4) \qquad f(m) = (13m^2 - 5)/8$$

suffices for powers of M to converge when $n > f(m)$.

As a by-product of this investigation, we find in Section 5 that the divergent

asymptotic series $B(x)$ with Bernoulli number coefficients

$$(1.5) \qquad\qquad B(x) = \sum_{k=1}^{\infty} B_{2k}(2x)^{2k},$$

related to the Laplace transform of $x \coth x - 1$, has the convergent continued fraction expansion

$$(1.6) \qquad B(x) = \frac{x^2}{|1 + 1/2 +} \quad \frac{x^2}{|1/2 + 1/3 +} \quad \frac{x^2}{|1/3 + 1/4 +} \quad \cdots .$$

In fact, $B(1/12) = \pi^2 - 9.865$ is given with error $< 2 \times 10^{-12}$ by the sixth convergent of this continued fraction.

2. The Determinants. Since $S_0(n) = n$ and $n(n + 1)/2$ divides $S_k(n)$ for $k > 0$, it follows directly that $w_m = \det W_m$ has the algebraic factor $n^m(n + 1)^{m-1}$. For $r < m$, $(n - r)^{m-r}$ is also an algebraic factor of w_m, since the matrix $W_m(n)$ has rank r and nullity $m - r$, when $n = r$ for $r = 1, 2, \ldots, m - 1$. Since the polynomials $S_p(n)$ are generated by the function

$$(2.1) \qquad G(x, n) = (e^{xn} - 1)/(1 - e^{-x}) = \sum_{p=0}^{\infty} S_p(n)x^p/p!,$$

we see from the identity

$$G(x, n) + G(-x, -n - 1) + 1 = 0$$

that

$$(2.2) \qquad S_p(n) + (-1)^p S_p(-n - 1) + \delta_{p,0} = 0.$$

Hence, w_m has the algebraic factor $(n + 1 + r)^{m-1-r}$. We have found m^2 linear functions of n as factors of the polynomial $w_m(n)$ which is of degree m^2 in n. The remaining factor is the determinant of the leading coefficients $1/(i + j - 1)$, namely the determinant h_m of the ill-conditioned Hilbert matrix H_m of order m [5], [7]. This proves Eq. (1.3).

If we take $m = n$ in (1.3), the Vandermonde matrix V in $W = VV^T$ is square. Its determinant v_m is

$$(2.3) \qquad v_m = \det V_m = 1! \; 2! \; 3! \cdots (m - 1)! \equiv (m - 1)!!.$$

Hence by (1.3) and (2.3)

$$(2.4) \qquad h_m = \det V_m^2 \Big/ \prod_{i,j=1}^{m} (m + i - j) = v_m^4/v_{2m}.$$

The ratio of successive h_m's is

$$(2.5) \qquad h_m/h_{m+1} = (2m + 1)!(2m)!/(m!)^4 = (2m + 1)\binom{2m}{m}^2 .$$

The following theorem gives a close approximation for h_m.

THEOREM 2.1. *The determinant h_m of the Hilbert matrix H_m of order m is*

given to 10 *significant figures for* $m > 4$ *by*

(2.6) $$h_m = 4^{-m(m-1)}(\pi/2)^{m-1}m^{-1/4}\exp R_m,$$

where the remainder function R_m *is defined by*

(2.7) $$R_m = \int_0^\infty (e^{-2t} - e^{-2mt})\tanh^2 (t/2)(4t)^{-1}\, dt$$

and is approximated to 9 *decimals for* $m \geqslant 5$ *by*

(2.8) $$\begin{aligned} R_m &= 0.013081539 - 2^{-6}m^{-2} + 2^{-8}m^{-4} - 2^{-8.5}m^{-6} \\ &\quad + 2^{-8}m^{-8} - 2^{-7}m^{-10}. \end{aligned}$$

   Proof. The ratio h_m/h_{m+1} in (2.5) is related to the Wallis approximation π_m for π by

(2.9) $$\frac{\pi_m}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\cdots\frac{2m}{2m-1}\frac{2m}{2m+1} = 2^{4m}h_{m+1}/h_m.$$

If we express $\ln n$ in the form

(2.10) $$\ln n = \int_1^n \frac{ds}{s} = \int_0^\infty \int_1^n e^{-st}\, ds\, dt = \int_0^\infty (e^{-t} - e^{-nt})t^{-1}\, dt,$$

then $\ln(\pi_m/2)$ and its limit $\ln(\pi/2)$ are expressible as

(2.11) $$\begin{aligned} \ln(\pi_m/2) &= \int_0^\infty (e^{-t} - 2e^{-2t} + 2e^{-3t} - 2e^{-2mt} + e^{-(2m+1)t})t^{-1}\, dt \\ &= \int_0^\infty (e^{-t} - e^{-(2m+1)t})(1 - e^{-t})(1 + e^{-t})^{-1}t^{-1}\, dt, \end{aligned}$$

(2.12) $$\ln(\pi/2) = \int_0^\infty e^{-t}\tanh (t/2)t^{-1}\, dt,$$

(2.13) $$\ln(\pi/\pi_k) = \int_0^\infty e^{-(2k+1)t}\tanh (t/2)t^{-1}\, dt,$$

(2.14) $$(\pi/2)^{m-1}\prod_{k=1}^{m-1}(\pi_k/\pi) = 2^{2m(m-1)}h_m/h_1.$$

Summing in (2.13) from $k = 1$ to $m - 1$ yields

(2.15) $$\begin{aligned} \ln[(\pi/2)^{m-1}2^{-2m(m-1)}/h_m] &= \int_0^\infty (e^{-2t} - e^{-2mt})(e^{t/2} + e^{-t/2})^{-2}t^{-1}\, dt \\ &= (1/4)\ln(2m/2) - R_m \end{aligned}$$

by (2.10), where R_m is defined by (2.7). Equation (2.6) follows from (2.15). To obtain (2.8) we evaluate $R_4 = .012119610988$ from (2.6) setting $h_4 = 1/6048000$ in (2.6). Then we compute $R_\infty - R_m$ from (2.7) by replacing $\tanh(t/2)$ by the first five terms of its series, and set $m = 4$ to get R_∞ in (2.8). We check the tenth decimal by working from h_5 instead. This gives $\exp R_m$ and h_m accurate to 10 significant figures.

For $m = 20$ we find $R_{20} = .0130425009$ and

$$(2.16) \qquad\qquad h_{20} = 4.206178954 \times 10^{-226}.$$

The matrices H_m and $W_m(n)$ are ill conditioned. In fact, $W_3(3)$ has the eigenvalues $\lambda_1 = 113.4132$, $\lambda_2 = 1.564253$, $\lambda_3 = .02254695$ and the conditioning ratio $\lambda_1/\lambda_3 = 5030$. So the usual computer methods for inverting $W_m(n)$ are unreliable [5], [7].

   3. Inversion by Factoring. To invert the ill-conditioned $m \times m$ matrix $W = W_m(n)$ with (i, j)-entry $S_{i+j-2}(n)$, we first factor it into easily invertible factors, restricting the variable n to diagonal matrix factors $EP = (\text{diag } e_i p_i)$ and $Q = (\text{diag } q_j)$, where

$$(3.1) \qquad e_i = (-1)^{i-1}, \quad p_i = \binom{n+i-1}{n-m}, \quad q_j = \binom{n-j}{n-m}.$$

We denote by $T = (t_{ij})$ the lower Pascal triangle matrix with

$$(3.2) \qquad t_{ij} = \binom{i-1}{j-1} = \binom{i-1}{i-j} = (-1)^{i+j}\binom{-j}{i-j}.$$

We note that ETE has entries $\binom{-j}{i-j}$, so

$$(3.3) \qquad (TETE)_{ij} = \sum_{k=j}^{i} \binom{i-1}{i-k}\binom{-j}{k-j} = \binom{i-j-1}{i-j} = \delta_{ij}$$

and $T^{-1} = ETE$. Next, we define an $m \times m$ lower triangular row stochastic matrix $A = (a_{ij})$ that converts the integral powers in V into binomial coefficients by the formula

$$(3.4) \qquad (AV)_{ik} = \sum_{r=1}^{i} a_{ir} k^{r-1} = \binom{k+i-2}{i-1}, \qquad k = 1, 2, \ldots, n.$$

The a_{ir} are related to Stirling numbers of the first kind, and

$$(3.5) \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1/2! & 1/2! & 0 & 0 & \cdots \\ 0 & 2/3! & 3/3! & 1/3! & 0 & \cdots \\ 0 & 6/4! & 11/4! & 6/4! & 1/4! & \cdots \end{bmatrix} = (a_{ij}).$$

From (3.5) and (3.4) we obtain

$$(3.6) \qquad (AV)_{ik} = \binom{i+k-2}{i-1} = \sum_{j=1}^{i}\binom{i-1}{i-j}\binom{k-1}{j-1} = \sum_{j=1}^{k} t_{ij}\binom{k-1}{j-1}.$$

   To factor W we now write

$$(3.7) \quad (AV(T^{-1}AV)^T)_{ij} = \sum_{k=j}^{n} \binom{k+i-2}{i-1}\binom{k-1}{j-1}$$

$$= \binom{i+j-2}{i-1} \sum_{k=j}^{n} \binom{k+i-2}{j+i-2}.$$

Summing over k yields

$$(3.8) \quad (AW(T^{-1}A)^T)_{ij} = \binom{i+j-2}{i-1}\binom{n+i-1}{j+i-1}$$

$$= p_i \binom{i+j-2}{i-1}\binom{m+i-1}{j+i-1} \Big/ q_j.$$

THEOREM 3.1. *The inverse matrix $M = W^{-1}$ has the factorization*

$$(3.9) \quad M = W^{-1} = A^T EBEA, \quad B = T^T QTT^T TP^{-1},$$

where E, P, Q, T, A are defined in (3.1), (3.2) *and* (3.5).

   Proof. Entries of TT^T and $TT^T T$ are

$$(3.10) \quad (TT^T)_{ik} = \sum_{r=1}^{i} \binom{i-1}{i-r}\binom{k-1}{r-1} = \binom{i+k-2}{i-1},$$

$$(3.11) \quad (TT^T T)_{ij} = \sum_{k=j}^{m} \binom{i+k-2}{k-1}\binom{k-1}{j-1} = \binom{i+j-2}{i-1} \sum_{k=j}^{m} \binom{k+i-2}{j+i-2}.$$

Summing over k yields

$$(3.12) \quad (TT^T T)_{ij} = \binom{i+j-2}{i-1}\binom{m+i-1}{j+i-1}.$$

   Combining (3.8) and (3.12), we have

$$(3.13) \quad AWA^T = PTT^T TQ^{-1} T^T.$$

Since the diagonal sign matrix E commutes with P and Q but transforms T and T^T into their inverses,

$$(3.14) \quad (AWA^T)^{-1} = ET^T QTT^T TP^{-1}E = EBE$$

and (3.9) is proved.

   Equation (3.12) provides a simple method for inverting the Hilbert matrix H.

   THEOREM 3.2. *The inverse of the $m \times m$ Hilbert matrix $H = (h_{ij})$ with (i, j)-entry $h_{ij} = 1/(i+j-1)$ is given by*

$$(3.15a) \quad (H^{-1})_{ij} = d_i d_i' h_{ij} d_j d_j',$$

$$(3.15b) \quad d_i' = \binom{-m-1}{i-1}, \quad d_j = m\binom{m-1}{j-1}.$$

Proof. Factoring (3.12) yields

$$(3.16) \quad (ETT^TT)_{ij} = (-1)^{i-1} \binom{m+i-1}{i-1} \frac{m}{i+j-1} \binom{m-1}{j-1} = d_i' h_{ij} d_j.$$

Since ETT^TT is involutory, $D'HD = (D'HD)^{-1}$. Also,

$$(3.17) \qquad\qquad h_m = \det H_m = \pm 1 \bigg/ \prod_{i=1}^{m} d_i d_i'.$$

Although the matrix B in (3.9) is symmetric, its symmetry is not obvious from formula (3.9).

THEOREM 3.3. *The symmetric matrix $B = (b_{ij})$ in (3.9) has entries expressible in terms of descending factorials $(x)_r = x(x-1) \cdots (x-r+1)$ as follows:*

$$(3.18) \qquad b_{ij} = \sum_{r \geqslant i+j-1} \frac{(m+i-1)_r (m+j-1)_r}{(n-m+r)_r r!} \binom{r-1}{i-1,\,j-1},$$

where $\binom{r-1}{i-1,\,j-1}$ is the trinomial coefficient

$$\binom{r-1}{i+j-2} \binom{i+j-2}{i-1}.$$

Proof. To transform B in (3.9) we evaluate

$$(T^TQTT^T)_{is} = \sum_{r=i}^{m} \binom{r-1}{i-1}\binom{n-r}{n-m}\binom{r+s-2}{r-1}$$

$$(3.19) \qquad\qquad = \binom{i+s-2}{i-1} \sum_{r=i}^{m} \binom{r+s-2}{i+s-2}\binom{n-r}{n-m}$$

$$= \binom{i+s-2}{i-1}\binom{n+s-1}{m-i},$$

$$(BP)_{ij} = \sum_{s=j}^{m} \binom{i+s-2}{i-1}\binom{n+s-1}{m-i}\binom{s-1}{j-1}$$

$$(3.20) \qquad = \binom{i+j-2}{i-1} \sum_{s=j}^{m} \binom{i+s-2}{s-j} \sum_{r \geqslant i+j-1} \binom{s-j}{r-i-j+1}\binom{n+j-1}{m+j-1-r}$$

$$= \sum_{r \geqslant i+j-1} \sum_{s=j}^{m} \binom{i+s-2}{r-1}\binom{r-1}{i-1,\,j-1}\binom{n+j-1}{n-m+r}.$$

Summing over s and dividing by p_j, we have

$$(3.21) \quad b_{ij} = \sum_{r \geqslant i+j-1} \binom{m+i-1}{r}\binom{r-1}{i-1,\,j-1}\binom{m+j-1}{r} \bigg/ \binom{n-m+r}{r}.$$

Writing $(x)_r = r!\binom{x}{r}$, Eq. (3.21) becomes (3.18).

To conserve space in displaying the symmetric matrices $M_m(n) = W_m^{-1}(n)$ we show the upper half of M_3 and the lower half of M_4.

(3.22)

$$M_4 \qquad\qquad\qquad\qquad M_3$$

$$\begin{bmatrix} \dfrac{16n^3 + 24n^2 + 56n + 24}{(n)_4} & & \dfrac{9(n^2 + n) + 6}{(n)_3} & \dfrac{-18(2n + 1)}{(n)_3} & \dfrac{30}{(n)_3} \\[3mm] \dfrac{-120(n^2 + n) - 100}{(n)_4} & \dfrac{1200(n^4 + 4.5n^3 + 7n^2 + 5n + 11/6)}{(n + 3)_7} & \dfrac{(24n + 12)(8n + 1)}{(n + 2)_5} & \dfrac{-180(n + 1)}{(n + 2)_5} \\[3mm] \dfrac{120(2n + 1)}{(n)_4} & \dfrac{-300(n + 1)(3n + 2)(3n + 5)}{(n + 3)_7} & \dfrac{360(2n + 1)(9n + 13)}{(n + 2)_5} & \dfrac{180}{(n + 2)_5} \\[3mm] \dfrac{-140}{(n)_4} & \dfrac{280(6n^2 + 15n + 11)}{(n + 3)_7} & \dfrac{-4200(n + 1)}{(n + 3)_7} & \dfrac{2800}{(n + 3)_7} \end{bmatrix}.$$

4. **Estimation of** $\operatorname{tr} M$. Since W and M are positive definite for $n > m$, all eigenvalues μ_k of M will satisfy $\mu_k < 1$ if $\operatorname{tr} M \leqslant 1$, $m > 1$. For A in (3.5) and $e_i = (-1)^{i-1}$, the first and last diagonal entries of $M = M_m$ are b_{11} and $b_{mm}/((m - 1)!)^2$. Numerical computation shows that the maximum n for which $\det(W_m(n) - I) = 0$ are given for $m = 1, 2, 3, 4$ by

(4.1) $\qquad (m, n) = (1, 1), (2, 5.82090), (3, 13.3776), (4, 24.24453).$

The parabola through the first three points is

(4.2) $\qquad\qquad n = g(m) = 1.5679m^2 - .0828m - .4851,$

and we find $g(4) = 24.270 > 24.24453$. A slightly higher value than (4.2) will be required for $\operatorname{tr} M \leqslant 1$. We first estimate the dominant diagonal entry b_{11} of M.

(4.3)
$$\binom{n}{m} b_{11} = \sum_{r=1}^{m} \binom{m}{r}^2 \binom{n}{m} \Big/ \binom{n - m + r}{r} = \sum_{r=1}^{m} \binom{m}{r} \binom{n}{m - r}$$

$$= \binom{n + m}{m} - \binom{n}{m}.$$

(4.4) $\qquad 1 + b_{11} = \binom{n + m}{m} \Big/ \binom{n}{m} = \prod_{k=1}^{m} \frac{2n + 1 + (2k - 1)}{2n + 1 - (2k - 1)},$

(4.5a) $\quad \ln(1 + b_{11}) = \sum_{k=1}^{m} \ln \frac{1 + (2k - 1)/(2n + 1)}{1 - (2k - 1)/(2n + 1)} = \sum_{r=1}^{\infty} \frac{\theta(m, r)}{(2r - 1)(2n + 1)^{2r-1}},$

where

(4.5b) $\qquad \theta(m, r) = \sum_{k=1}^{m} 2(2k - 1)^{2r-1} < \int_0^{2m} x^{2r-1}\, dx = (2m)^{2r}/2r.$

We now assume the inequalities $n > f(m)$ in (1.4).

THEOREM 4.1. *The matrix* $M = W_m^{-1}(n)$ *has trace* < 1 *if*

(4.6) $\qquad\qquad n > 1.625m^2 - .625 \quad and \quad m \geqslant 5.$

Proof. If (4.6) is satisfied for $m = 5$, then $n \geqslant 40$, and

(4.7a) $$b_{11} \leqslant (45)_5/(40)_5 - 1 = 62639/73112 = .856754.$$

If (4.6) is satisfied for $m \geqslant 6$, then $(2n + 1)/2m^2 > 1.6215$ and

(4.8)
$$\ln(1 + b_{11}) < \frac{2m^2}{2n+1} \sum_{r=1}^{\infty} \left(\frac{2m}{2n+1}\right)^{2r-2} \Big/ r(2r-1)$$

$$< \frac{1}{1.6215} \sum_{r=1}^{\prime \infty} \left(\frac{1}{9.729}\right)^{2r-2} \Big/ r(2r-1) < .62777,$$

(4.7b) $$b_{11} < .8734 \quad \text{for } m \geqslant 6, n > (13m^2 - 5)/8.$$

The rest of $\operatorname{tr} M$ is given by

(4.9) $$\operatorname{tr} M - b_{11} = \sum_{k=2}^{m} \sum_{i,j=k}^{m} a_{ik}(-1)^i b_{ij}(-1)^j a_{jk}.$$

We replace i, j, r by $i + 1, j + 1, r + 2$ and write

(4.10) $$b_{i+1,j+1} = \sum_{k=1}^{m-1} y_{ij}^{(r)}, \quad y_{ij}^{(r)} = \frac{(m+i)_{r+2}(m+j)_{r+2}}{(n-m+r+2)_{r+2}(r+2)!} \binom{r+1}{i,j} = y_{ji}^{(r)}.$$

Then

(4.11) $$\operatorname{tr} M - b_{11} = y_{11}^{(1)} \sum_{r=1}^{2m-1} \varphi_{mn}(r), \quad \varphi_{mn}(r) = \sum_{i+j=2}^{r+1} c_{ij} y_{ij}^{(r)}/y_{11}^{(1)},$$

where the entries of the $(m - 1) \times (m - 1)$ matrix $C = (c_{ij})$ are

(4.12a) $$c_{ij} = (-1)^{i+j} \sum_{k=1}^{m-1} a_{i+1,k} a_{j+1,k} = c_{ji},$$

(4.12b) $$C = \frac{1}{720} \begin{bmatrix} 720 & -360 & 240 & -180 & 144 & \cdots \\ -360 & 360 & -300 & 255 & -222 & \cdots \\ 240 & -300 & 280 & -255 & 233 & \cdots \\ -180 & 255 & -255 & 242.5 & -228.5 & \cdots \\ 144 & -222 & 233 & -228.5 & 220.1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

The dominant term $y_{11}^{(1)}$ satisfies

(4.13) $$y_{11}^{(1)} < \frac{(m+1)_3(m+1)_3/3}{(13m^2/8 - m + 19/8)_3} \leqslant \frac{6_3 6_3/3}{(38)_3} = \frac{200}{2109} < .094832,$$

since the rational function decreases for $5 \leqslant m$. The function $\varphi_{mn}(1)$ is 1, but for $r > 1$, then $\varphi_{mn}(r)$ in (4.11) are bounded by rational functions which increase for

$m \geqslant 5$, and which we replace by their limits as $m \longrightarrow \infty$.

$$\varphi_{mn}(2) = (y_{11}^{(2)} - y_{12}^{(2)})/y_{11}^{(1)} = 3(m-2)(m-6)/(13m^2 - 8m + 27)$$

(4.14a)

$$< 3/13 = .23077,$$

$$\varphi_{mn}(3) = (y_{11}^{(3)} - y_{12}^{(3)} + 2y_{13}^{(3)}/3 + y_{22}^{(3)}/2)/y_{11}^{(1)}$$

(4.14b) $$< 17(m^2 - 6m + 32)(m-2)(m-2.4)/(120)(13m^2/8 - m + 35/8)_2$$

$$< (17/120)(8/13)^2 = .05365.$$

Similar calculations yield

(4.14c) $$\varphi_{mn}(4) < (1/32)(8/13)^3 = .00728.$$

Since the coefficients of $(8/13)^{r-1}$ in $\varphi_{mn}(r)$ decrease as r increases, the remaining sum of $\varphi_{mn}(r)$ is $< 2.6\varphi_{mn}(4)$. Hence, (4.11) implies

$$\operatorname{tr} M < .8734 + .095(1.23077 + .05365 + 3.6(.00728))$$

(4.15)

$$< .8734 + .095(1.3107) < .998 < 1.$$

This proves Theorem 4.1. We check directly for $m = 2, 3, 4$ that

$$\operatorname{tr} M_2(6) = 97/105, \quad \operatorname{tr} M_3(14) = .95 + 1/7280,$$

(4.16)

$$\operatorname{tr} M_4(25) = .87755 + .09359 + .0073 + .0000005 < .9719.$$

This proves the parabolic bound $n > f(m) = (13m^2 - 5)/8$ to be sufficient for $\operatorname{tr} M < 1$. Although some bound between this and $n > g(m)$ in (4.2) might also suffice for all n, the tight inequality (4.15) indicates that it would be difficult to prove.

   5. The Bernoulli Continued Fraction. The entries $S_{i+j-2}(n)/n$ of the matrix $W_m(n)/n$ have as constant terms the Bernoulli numbers B_{i+j-2} given in (1.2). The limit as $n \longrightarrow 0$ of the leading principal minor of $W_m(n)/n$ is the determinant b_{m-1}^* of order $m-1$ expressible as

(5.1) $$b_{m-1}^* = \det(B_{i+j}) = \lim_{n \to 0} (nb_{11})(n^{-m}w_m(n)).$$

Recalling b_{11} from (4.3), $w_m(n)$ from (1.3), v_m from (2.3) and h_m from (2.4), we have

(5.2) $$\lim_{n=0} nb_{11} = \binom{m}{m}m \bigg/ \binom{-1}{m-1} = (-1)^{m-1}m,$$

(5.3) $$\lim_{n=0} n^{-m}w_m(n) = h_m v_m^2 (-1)^{m(m-1)/2},$$

(5.4) $$b^*_{m-1} = (-1)^{(m-1)(m-2)/2} m v^6_m / v_{2m},$$

(5.5) $$b^*_m / b^*_{m-1} = (-1)^{m-1} (m-1)! (m!)^4 (m+1)! / (2m)! (2m+1)!.$$

Since $B_{i+j} = 0$ for odd $i + j$, we can rearrange rows and columns of the matrix (B_{i+j}) so the odd numbered ones precede the even numbered ones, and thus factor b^*_{m-1} as the product $d_{m-1} d_{m-2}$ of two determinants, where

(5.6)
$$d_{2k-1} = \begin{vmatrix} B_2 & B_4 & \cdots & B_{2k} \\ B_4 & B_6 & \cdots & B_{2k+2} \\ \cdot & \cdot & \cdot & \cdot \\ B_{2k} & B_{2k+2} & \cdots & B_{4k-2} \end{vmatrix},$$

$$d_{2k} = \begin{vmatrix} B_4 & B_6 & \cdots & B_{2k+2} \\ B_6 & B_8 & \cdots & B_{2k+4} \\ \cdot & \cdot & \cdot & \cdot \\ B_{2k+2} & B_{2k+4} & \cdots & B_{4k} \end{vmatrix},$$

(5.7) $$d_m / d_{m-2} = b^*_m / b^*_{m-1},$$

(5.8)
$$-d_{m-3} d_m / d_{m-1} d_{m-2} = (m-1) m^4 (m+1) / (2m-1)(2m)^2 (2m+1)$$
$$= (1/4)((m-1)m/(2m-1))(m(m+1)/(2m+1)).$$

THEOREM 5.1. *The divergent asymptotic alternating series*

(5.9) $$B(x) = \sum_{k=1}^{\infty} B_{2k}(2x)^{2k} = 4x^2/6 - 16x^4/30 + 64x^6/42 \cdots$$

has the convergent continued fraction expansion (1.6).

Proof. By the general theory of continued fractions [2], [9], if a formal power series (5.9) with arbitrary coefficients B_{2k} is expanded into continued fractions of the form

(5.10) $$\frac{a_1(2x)^2|}{|\ 1\ +} \quad \frac{a_2(2x)^2|}{|\ 1\ +} \quad \cdots = \frac{x^2/c_0|}{|c_1\ +} \quad \frac{x^2\ |}{|c_2\ +} \quad \frac{x^2\ |}{|c_3\ +} \cdots$$

and if the d_k's are defined by (5.6), then

(5.11) $a_m = 1/4 c_{m-1} c_m = -d_{m-3} d_m / d_{m-2} d_{m-1}, \quad m \geqslant 1.$

For the Bernoulli series Eqs. (5.5) and (5.11) imply

(5.12) $c_m = (m(m+1)/(2m+1))^{-1} = 1/m + 1/(m+1), \quad m \geqslant 1,$

while the condition $1/c_0 c_1 = 4B_1 = 2/3$ implies $c_0 = 1$. Since Σc_m is divergent, the continued fraction (1.6) converges, and Theorem 5.1 is proved.

We can apply this continued fraction to approximate π^2. It would require about a billion terms of the series $\Sigma_1^\infty (1/k^2)$ to approximate $\pi^2/6$ to nine decimals. But the Euler-Maclaurin summation formula gives the remainder after 5 terms by the expression

(5.13) $\displaystyle\int_6^\infty x^{-2} \, dx + 1/2 \cdot 6^2 + \sum_{k=1}^\infty B_{2k}(1/6)^{2k+1}.$

This alternating series diverges, with minimum remainder of about 10^{-15} after the 19th term. Using the convergent continued fraction instead, we have

$$\pi^2 = 6(1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/6 + 1/72) + B(1/12)$$

(5.14)
$$= 9.865 + \frac{12^{-2}}{|1 + 1/2 +} \frac{12^{-2}}{|1/2 + 1/3 +} \frac{12^{-2}}{|1/3 + 1/4 +},$$

(5.15)
$$\begin{aligned}\pi^2 &= 9.865 \\ &+ \frac{1/12}{|12 + 6 +} \frac{1}{|6 + 4 +} \frac{1}{|4 + 3 +} \frac{1}{|3 + 2.4 +} \frac{1}{|2.4 + 2 +} \frac{1}{|2 + r},\end{aligned}$$

where the sixth convergent with $r = 12/7$ has an error about 10^{-12}, and the tenth convergent (which changes this r to 1.9976) has an error less than 10^{-15}, giving $\pi^2 = 9.869604401089359$.

The function $s^{-1} B(s^{-1})$ is the Laplace transform of $x \coth x - 1$.

Continued fractions for the Laplace transforms of $\tanh x$, $\operatorname{sech} x$ and $x \operatorname{csch} x$ can also be obtained by similar methods, but have already been derived by Stieltjes [8] and others, and are listed by Wall [9, p. 369]. The author has not found the continued fraction (1.6) in the literature, nor the determinantal formula (5.4) which evaluates the first principal $m \times m$ minor $b_m^* = |B_{i+j}|, i, j = 1, \dots,$ (omitting B_0 and B_1) of the determinant $|B_{i+j-2}|$ of order $m+1$ called $\Delta_m(B)$ by Al-Salam and Carlitz [1, p. 93, (3.1)] which in the notation of (2.3) becomes

(5.16) $\Delta_m(B) = (-1)^{m(m+1)/2} (m!!)^6 / (2m+1)!!.$

Comparing (5.16) with (5.4) for order m, we have

(5.17) $|B_{i+j}|_m = (-1)^m (m+1) |B_{i+j-2}|_{m+1}.$

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