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On a Relationship Between the Convergents of the Nearest Integer and Regular Continued Fractions

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Abstract. In this paper we derive a relation concerning the speed of convergence of the nearest integer and regular continued fractions. If A_n/B_n, p_k/q_k denote the convergents of the nearest integer and regular continued fractions of an irrational number α, then for all n there is a $k(n)$ such that $A_n/B_n = p_{k(n)}/q_{k(n)}$. It is shown that

$$\lim_{n\to\infty} \frac{n}{k(n)} = \frac{\log\left(\frac{1+\sqrt{5}}{2}\right)}{\log 2}$$

for almost all α. This problem is reduced to a special case of a general result concerning the frequency of partial quotients in the regular continued fraction (Theorem 2).

1. Introduction. Williams and Buhr [7] compared the lengths of the algorithms for computing the fundamental unit of real quadratic fields using the regular continued fraction* (RCF) and the nearest integer continued fraction (NICF) (see Section 4 for the definition of the NICF). It is known (see [5], say, for RCF and [6], [7], [8] for NICF) that, for square-free positive integers D, if $a + b\sqrt{D}$ is a fundamental unit of $Q(\sqrt{D})$ (a, b positive integers or halves of positive integers if $D \equiv 1 \pmod 4$), then a/b is a convergent of both the RCF and NICF. Let us suppose that the fundamental unit is given by the $p(D)$th convergent of the RCF and the $\pi(D)$th convergent of the NICF. It is observed in [7] that the ratio $\pi(D)/p(D)$ is near $\log 2 = .693147181\ldots$. Indeed, in an example of D. Shanks [4] of a prime $D = 26437680473689$ with an unusually long period for its RCF, it is observed that $\pi(D)/p(D) = .6942215829\ldots$. Moreover, it is noted that

$$\frac{\displaystyle\sum_{\substack{d\leqslant D\\ d \text{ squarefree}}} \pi(d)}{\displaystyle\sum_{\substack{d\leqslant D\\ d \text{ squarefree}}} p(d)} = .7017174\ldots,$$

for $D = 100000$.

The purpose of this paper is to derive what the asymptotic value of these ratios should be on the assumption that, within the period of $\sqrt{D}$, the RCF behaves like

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*The RCF is called the ordinary continued fraction (OCF) in [7].

almost all numbers with respect to the frequency of certain partial quotients. It will be shown that, with this assumption, the above ratios will be, as $D \longrightarrow \infty$,

$$\frac{\log\left(\frac{1+\sqrt{5}}{2}\right)}{\log 2} = .6942419 \ldots .$$

This agrees with Shanks' example to four places. As Shanks pointed out, this exceptional accuracy was really to be expected since the frequencies of the small partial quotients of the RCF, as reported in [4], agree with the Gauss-Kuzmin law to that same accuracy.

Now let α be any irrational number. Denote by p_k/q_k, the convergents of the RCF ($p_k = p_k(\alpha)$), and by A_n/B_n, the convergents of the NICF. So p_k, q_k are relatively prime positive integers with $q_0 < q_1 < q_2 < \cdots$, and A_n, B_n are relatively prime integers such that $|B_0| < |B_1| < |B_2| < \cdots$. As we shall see below, each A_n/B_n is one of the p_k/q_k. So there is a strictly increasing function $k(n)$ such that for all $n \geqslant 0$

$$A_n/B_n = p_{k(n)}/q_{k(n)}.$$

We will show the following theorem.

THEOREM 1. *For almost all real numbers* α

$$\lim_{n\to\infty} \frac{n}{k(n)} = \frac{\log\left(\frac{1+\sqrt{5}}{2}\right)}{\log 2}.$$

This result will be deduced from the following general metrical theorem concerning the RCF. For irrational α with $0 < \alpha < 1$ let

$$\alpha = [0; a_1(\alpha), a_2(\alpha), \ldots]$$

denote its RCF. Set $\alpha_j = [a_j(\alpha); a_{j+1}(\alpha), \ldots]$.

THEOREM 2. *Let* α, β *be irrational numbers with* $0 < \alpha, \beta < 1$.

(a) *Define, for all integers* $\nu \geqslant 0$, $\psi_\nu(n, \alpha, \beta)$ *to be the number of integers* j *satisfying* $0 \leqslant j \leqslant n-1$

$$(1) \quad a_{j+1}(\alpha) = a_1(\beta), \quad a_{j+2}(\alpha) = a_2(\beta), \ldots, a_{j+\nu}(\alpha) = a_\nu(\beta), \quad \alpha_{\nu+j+1} > a_{\nu+1}(\beta).$$

Then, for almost all α,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{\infty} (-1)^\nu \psi_\nu(n, \alpha, \beta) = \frac{\log(1+\beta)}{\log 2}.$$

(b) *Define, for all integers* $\nu \geqslant 0$, $\varphi_\nu(n, \alpha, \beta)$ *to be the number of integers* j *satisfying* $0 \leqslant j \leqslant n-\nu$ *and* (1). *Then for almost all* α

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{\infty} (-1)^\nu \varphi_\nu(n, \alpha, \beta) = \frac{\log(1+\beta)}{\log 2}.$$

We note that if

$$\beta = \frac{\sqrt{5} - 1}{2} = [0; 1, 1, \ldots],$$

then the last condition in (1) is automatic. Thus ψ_ν and φ_ν, in this case, are simply counting the number of sequences of ν 1's in a row.

The relevance of the frequency of occurrence of the partial quotients is seen in

THEOREM 3. *For a given integer* $n \geqslant -1$, *suppose*

$$\frac{A_n}{B_n} = \frac{p_k}{q_k}.$$

Then

$$\frac{A_{n+1}}{B_{n+1}} = \frac{p_{k+1}}{q_{k+2}} \quad or \quad \frac{p_{k+2}}{q_{k+2}},$$

where the latter occurs if and only if $a_{k+2} = 1$ $(a_{k+2} = a_{k+2}(\alpha))$. *Moreover,*

$$\frac{A_{-1}}{B_{-1}} = \frac{p_{-1}}{q_{-1}}.$$

Actually, the relevant frequencies for Theorem 1 are the frequencies of the sequences 1; 1, 1; 1, 1, 1; . . . . This accounts for the appearance of the number $(1 + \sqrt{5})/2$ in Theorem 1.

Now suppose that $\alpha = \sqrt{D}$, and suppose that A_n/B_n gives the fundamental unit of $Q(\sqrt{D})$. Then, of course, $p_{k(n)}/q_{k(n)}$ also gives the fundamental unit, and so $\pi(D) = n$ and $p(D) = k(n)$. Thus, on the assumption that as D gets large, the first $p(D)$ partial quotients behave like almost all numbers with respect to the sequences of digits 1; 1, 1; 1, 1, 1; etc., I conjecture that

$$\lim_{\substack{D \to \infty \\ D \text{ squarefree}}} \frac{\pi(D)}{p(D)} = \frac{\log\left(\frac{1 + \sqrt{5}}{2}\right)}{\log 2}.$$

I am indebted to H. C. Williams for allowing me the use of his manuscripts [7], [8] in advance of publication. The present manuscript was prepared before [8] was available to me, but it should be pointed out that Theorem 3 was already contained in [8].

I would also like to take this opportunity to thank D. Shanks for his encouragement during the course of the preparation of this paper.

Finally, as this paper was being typed, the author discovered that G. J. Rieger [9] has proved a result which yields an analogous result to Theorem 1 for the average length of RCF and NICF for rational numbers.

2. Proof of Theorem 2(a). The purpose of this section is to prove Theorem 2(a). We will follow the Ergodic Theorem proof for results of this nature (see [1, pp. 40–50]). For any $\nu \geqslant 0$ define a function $f_\nu(u_1, \ldots, u_\nu, x)$, for positive integers $u_1, \ldots, u_\nu$ and real numbers $x > 1$, to be 1 provided $u_j = a_j(\beta)$ $(1 \leqslant j \leqslant \nu)$ and $x > a_{\nu+1}(\beta)$ and to be zero otherwise. Define the transformation T of the unit interval $0 < \alpha < 1$ by

$$T(\alpha) = \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right],$$

where $[y]$ denotes the largest integer $\leqslant y$. If T^j denotes T composed with itself j times, we have

$$\psi_\nu(n, \alpha, \beta) = \sum_{j=0}^{n-1} f_\nu(a_1(T^j\alpha), \ldots, a_\nu(T^j\alpha), T^j\alpha_{\nu+1}).$$

Set

$$f(\alpha) = \sum_{\nu=0}^{\infty} (-1)^\nu f_\nu(a_1(\alpha), \ldots, a_\nu(\alpha), \alpha_{\nu+1}).$$

We note that if $T^j\alpha \neq \beta$ for all $j \geqslant 0$ (in particular if α is not equivalent to β), then this sum for $f(T^j\alpha)$ is in fact finite $(j \geqslant 0)$. There are only a countable number of such α and so $f(T^j\alpha)$ is defined for almost all α in $(0, 1)$, for all $j \geqslant 0$. Then, for α not equivalent to β,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{\infty} (-1)^\nu \psi_\nu(n, \alpha, \beta) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j\alpha) \tag{2}$$

(the interchange of summations in (2) is justified since the sums are actually finite).

Now define the Gauss measure μ on $(0, 1)$ by

$$\mu(E) = \frac{1}{\log 2} \int_E \frac{dx}{1+x},$$

for Lebesgue measurable sets E in $(0, 1)$. It is well known, [1, p. 44] that T preserves μ and is ergodic with respect to μ. Then, if we knew that $f \in L_1(\mu)$, we would have from the Ergodic Theorem [1, p. 13], and (2) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{\infty} (-1)^\nu \psi_\nu(n, \alpha, \beta) = \int_0^1 f(x)\, d\mu(x) \tag{3}$$

for almost all x in $(0, 1)$. This will now be verified and at the same time the integral will be evaluated.

Set

$$\varphi^*(\nu) = \int_0^1 f_\nu(a_1(x), \ldots, a_\nu(x), x_{\nu+1})\, d\mu(x).$$

It is not hard to show (see [2, Chapter III]) that $f_\nu(a_1(x), \ldots, a_\nu(x), x_{\nu+1})$ is the characteristic function of the set of x in interval with endpoints

$$\frac{p_\nu(\beta)}{q_\nu(\beta)} = \frac{p_\nu}{q_\nu}, \quad \frac{p_{\nu+1}(\beta)}{q_{\nu+1}(\beta)} = \frac{p_{\nu+1}}{q_{\nu+1}}.$$

Hence,

$$\begin{aligned} \varphi^*(\nu) &= \frac{1}{\log 2}\left|\log\left(1+\frac{p_\nu}{q_\nu}\right) - \log\left(1+\frac{p_{\nu+1}}{q_{\nu+1}}\right)\right| \\ &\leqslant \frac{1}{\log 2}\left|\frac{p_\nu}{q_\nu} - \frac{p_{\nu+1}}{q_{\nu+1}}\right| = \frac{1}{\log 2}\frac{1}{q_\nu q_{\nu+1}}. \end{aligned} \tag{4}$$

Thus,

$$\sum_{\nu=0}^{\infty} \varphi^*(\nu) < \infty$$

(it is well known that $q_\nu \geqslant 2^{(\nu-1)/2}$). Hence by Fubini's theorem, $f \in L_1(\mu)$ and, moreover, (using Fubini's theorem again and (4))

$$\begin{aligned} \int_0^1 f(x)\,d\mu(x) &= \sum_{\nu=0}^{\infty} (-1)^\nu \varphi^*(\nu) \\ &= -\frac{1}{\log 2}\sum_{\nu=0}^{\infty}\left(\log\left(1+\frac{p_\nu}{q_\nu}\right) - \log\left(1+\frac{p_{\nu+1}}{q_{\nu+1}}\right)\right) \\ &= \frac{\log(1+\beta)}{\log 2} \end{aligned}$$

(since $p_0/q_0 = 0$ and $p_\nu/q_\nu \longrightarrow \beta$ $(\nu \longrightarrow \infty)$ and $p_\nu q_{\nu+1} - p_{\nu+1}q_\nu = (-1)^{\nu+1}$). This completes the proof of Theorem 2(a).

Using the techniques of [3], it should be possible to derive an explicit error term in Theorem 2(a) and, hence, using the result of the next section, an error term in Theorem 2(b), also.

3. Proof of Theorem 2(b). Define

$$\rho_\nu(n, \alpha, \beta) = \psi_\nu(n, \alpha, \beta) - \varphi_\nu(n, \alpha, \beta).$$

It suffices to show that, except on a set of measure zero in α, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{\nu=0}^{\infty} \rho_\nu(n, \alpha, \beta) = 0.$$

Now $\rho_\nu(n, \alpha, \beta)$ is the number of integers j, $n - \nu + 1 \leqslant j \leqslant n - 1$ satisfying (1). Denote by $E_{\nu,n,j}$ the set of α satisfying (1). We first estimate the measure $|E_{\nu,n,j}|$ of $E_{\nu,n,j}$.

We follow the argument given in [2, pp. 57–60]. If $\alpha \in E_{\nu,n,j}$, then for some positive integers $b_1, \ldots, b_j$,

$$\alpha = [0; b_1, b_2, \ldots, b_j, a_1(\beta), \ldots, a_\nu(\beta), \alpha_{\nu+j+1}]$$

with $\alpha_{\nu+j+1} > a_{\nu+1}(\beta)$. For fixed $b_1, \ldots, b_j$ this defines an interval $E(b)$

(suppressing ν, n, j) with endpoints $p_{j+\nu}(\alpha)/q_{j+\nu}(\alpha)$ and $p_{j+\nu+1}(\alpha)/q_{j+\nu+1}(\alpha)$ of length $1/q_{j+\nu}(\alpha)q_{j+\nu+1}(\alpha)$. If P_i/Q_i denotes the convergents of $[0, b_1, \ldots, b_j]$, then the set $F(b)$ of all α of the form $[0, b_1, \ldots, b_j, \alpha_{j+1}]$ is an interval with endpoints P_j/Q_j and $(P_j + P_{j-1})/(Q_j + Q_{j-1})$ of length $1/Q_j(Q_j + Q_{j-1})$. Hence,

$$|E(b)|/|F(b)| \leqslant Q_j^2/q_{j+\nu}(\alpha)^2.$$

We use the well-known identity

$$q_{j+\nu}(\alpha) = Q_j q_\nu(\beta) + Q_{j-1}q_{\nu-1}(\beta)$$

to obtain

$$|E(b)|/|F(b)| \leqslant 1/q_\nu(\beta)^2.$$

Now, where $\bigcup_b$ means the union over all $1 \leqslant b_1, \ldots, b_j < \infty$, we have

$$\bigcup_b E(b) = E_{\nu,n,j} \quad \text{and} \quad \bigcup_b F(b) = (0, 1),$$

where both unions are disjoint. Thus, we conclude that

$$|E_{\nu,n,j}| \leqslant 1/q_\nu(\beta)^2$$

(we emphasize that this is independent of j).

Now, from the definition of $\rho_\nu(n, \alpha, \beta)$, if $\alpha \notin E_{\nu,n}$, where

$$E_{\nu,n} = \bigcup_{j=n-\nu+1}^{n-1} E_{\nu,n,j},$$

then $\rho_\nu(n, \alpha, \beta) = 0$. Moreover, $|E_{\nu,n}| \leqslant \nu/q_\nu(\beta)^2$.

Set $l(n) = [n^{1/4}]$. Set

$$E_n = \bigcup_{\nu \geqslant l(n)} E_{\nu,n}.$$

Since trivially $\rho_\nu(n, \alpha, \beta) \leqslant \nu$ for all ν, n we see that if $\alpha \notin E_n$

$$\frac{1}{n}\sum_{\nu=0}^{\infty} \rho_\nu(n, \alpha, \beta) \leqslant \frac{1}{n}\sum_{\nu=0}^{l(n)-1} \nu \leqslant \frac{l(n)^2}{n}.$$

Thus, we see that, if for any positive integer m, $\alpha \notin \bigcup_{n \geqslant m} E_n$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{\nu=0}^{\infty} \rho_\nu(n, \alpha, \beta) = 0.$$

But the measure of the excluded set can be determined as

$$\left|\bigcap_{m=1}^{\infty}\bigcup_{n\geqslant m} E_n\right| = \lim_{m\to\infty}\left|\bigcup_{n\geqslant m} E_n\right| \leqslant \lim_{m\to\infty}\sum_{n=m}^{\infty} |E_n|$$

$$\leqslant \lim_{m\to\infty}\sum_{n=m}^{\infty} \frac{l(n)}{q_{l(n)}(\beta)^2} = 0,$$

since it is well known that $q_\nu(\beta) \geqslant 2^{(\nu-1)/2}$; and so we see that $\Sigma_{n=2}^{\infty} l(n)/q_{l(n)}(\beta)^2$ is a convergent series.

4. Comparison of the RCF and NICF. We will now give a very explicit algorithm for converting the RCF into the NICF. Theorem 3 will be an easy consequence of this procedure.

First, we will set up the notation and describe the NICF. For comparison we begin by briefly describing the familiar RCF (see [2]).

Let α be an irrational number. Let $a_0 = [\alpha]$ denote the largest integer $\leqslant \alpha$. Let $\alpha_0 = \alpha$ and, inductively,**

$$\alpha_k = \frac{1}{\alpha_{k-1} - a_{k-1}}, \qquad a_k = [\alpha_k].$$

Then

$$\alpha = \alpha_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

which we abbreviate as

$$\alpha = [a_0, a_1, a_2, \ldots].$$

Set $p_{-2} = 0$, $q_{-2} = 1$, $p_{-1} = 1$, $q_{-1} = 0$ and, inductively, for $k \geqslant 0$, $p_k = a_k p_{k-1} + p_{k-2}$, $q_k = a_k q_{k-1} + q_{k-2}$. Then the a_k are called the partial quotients of α, and the p_k/q_k are called the RCF convergents of α. Moreover,

$$p_k/q_k = [a_0, a_1, \ldots, a_k] \quad \text{and} \quad \alpha_k = [a_k, a_{k+1}, \ldots].$$

Now, for an irrational number α we will give the NICF algorithm (see [6], [7], [8]). Let $N(\alpha)$ denote the integer closest to α. Set $\theta_0 = \alpha$, $b_0 = N(\alpha)$ and, inductively, set

$$\theta_n = \frac{1}{b_{n-1} - \theta_{n-1}}, \qquad b_n = N(\theta_n).$$

Then

$$\alpha = \theta_0 = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cdots}},$$

which we abbreviate as

$$\alpha = (b_0, b_1, b_2, \ldots).$$

**The explicit dependence of $a_k = a_k(\alpha)$ on α will now be suppressed.

Set $A_{-2} = 0, B_{-2} = -1, A_{-1} = 1, B_{-1} = 0$ and, inductively, for $n \geqslant 0$,

$$A_n = b_n A_{n-1} - A_{n-2}, \qquad B_n = b_n B_{n-1} - B_{n-2}.$$

The A_n/B_n are called the NICF convergents of α. Moreover,

$$A_n/B_n = (b_0; b_1, \ldots, b_n), \qquad \theta_n = (b_n; b_{n+1}, \ldots).$$

We now describe how, given the RCF, we can obtain the NICF from it.

THEOREM 4. *For a given integer $n \geqslant -1$ there are* 16 *different possibilities for $b_n, A_n, B_n, A_{n-1}, B_{n-1}$ and θ_{n+1} given by the following table, where k is chosen so that $A_n = \pm p_k$, $B_n = \pm q_k$ (same sign).*

	I	II	III	IV	V	VI	VII	VIII
b_n	a_k	$-a_k$	$1 + a_k$	$-(1 + a_k)$	$1 + a_{k-1}$	$-(1 + a_{k-1})$	$2 + a_{k-1}$	$-(2 + a_{k-1})$
A_n	$\pm p_k$	$\pm p_k$	$\pm p_k$	$\pm p_k$	$\pm p_k$	$\pm p_k$	$\pm p_k$	$\pm p_k$
A_{n-1}	$\pm p_{k-1}$	$\mp p_{k-1}$	$\pm p_{k-1}$	$\mp p_{k-1}$	$\pm p_{k-2}$	$\mp p_{k-2}$	$\pm p_{k-2}$	$\mp p_{k-2}$
θ_{n+1}	$-\alpha_{k+1}$	α_{k+1}	$-\alpha_{k+1}$	α_{k+1}	$1 + \alpha_{k+1}$	$-(1 + \alpha_{k+1})$	$1 + \alpha_{k+1}$	$-(1 + \alpha_{k+1})$

The signs for A_n, A_{n-1} are the respective signs given there, e.g. if in case IV, $A_n = -p_k$, *then $A_{n-1} = p_{k-1}$. To obtain B_n use q_k instead of p_k with the same sign; and similarly for B_{n-1}. To begin use case* II *for $n = -1 = k$ with the top choice of signs (ignore b_{-1} and a_{-1}).*

To obtain the table entry for b_{n+1}, A_{n+1}, A_n and θ_{n+2}, there are two cases.

Case a: $a_{k+2} \geqslant 2$. *Here $A_{n+1}/B_{n+1} = p_{k+1}/q_{k+1}$ (i.e., $k \mapsto k+1$ in the table).*

Case b: $a_{k+2} = 1$. *Here $A_{n+1}/B_{n+1} = p_{k+2}/q_{k+2}$ (i.e., $k \mapsto k+2$ in the table).*

The scheme below describes how to go from case to case as $n \mapsto n+1$:

	I	II	III	IV	V	VI	VII	VIII
$a \nearrow$	II^-	I^+	II^-	I^+	III^+	IV^-	III^+	IV^-
$b \searrow$	VI^-	V^+	VI^-	V^+	VII^+	VIII^-	VII^+	VIII^-

The "+", "−" superscripts indicate that either the choice of signs stays the same (for "+") or reverses (for "−").

Note finally that in cases I, II, III, IV, $a_{k+1} \geqslant 2$ *and in cases* V, VI, VII, VIII, $a_k = 1$.

Before proving Theorem 4, note that Theorem 3 is an immediate consequence. The proof given in [8] of Theorem 3 is much more direct but I think Theorem 4 has some independent interest. To aid in the understanding of the statement of Theorem 4 an example is worked out.

Let $\alpha = \sqrt{13/2} = [2; \overline{1, 1, 4}]$. For $n = k = -1$, $a_{k+1} = a_1 = 1$. So II $\longrightarrow$ V^+, $n = 0$, $k = 1$ and $b_0 = 1 + a_{k-1} = 1 + a_0 = 3$. Now $a_{k+2} = a_3 = 4$ so V $\longrightarrow$ III^+, $n = 1$, $k = 2$ and $b_1 = 1 + a_k = 1 + a_2 = 2$. Again, $a_{k+2} = a_4 = 1$ so III $\longrightarrow$ VI^-, $n = 2$, $k = 4$ and $b_2 = -(1 + a_3) = -5$. Once again, $a_{k+2} = a_6 = 4$ so VI $\longrightarrow$ IV^-, $n = 3$, $k = 5$ and $b_3 = -(1 + a_5) = -2$. Finally, $a_{k+2} = a_7 = 1$ so $\mathrm{IV}^- \longrightarrow$ V^+, $n = 4$, $k = 7$, so $b_4 = 1 + a_6 = 5$ and the cycle is complete. Hence, $\sqrt{13/2} = (3; \overline{2, -5, -2, 5})$. (Note that the period is in fact longer here, contrary to statements made in [7], [8]. However, this in no way affects anything done in those papers.)

The proof of Theorem 4 using induction on n is simply to verify each of the cases. We will only do this for

$$\mathrm{VI} \begin{array}{l} \overset{a}{\nearrow}\ \mathrm{IV}^- \\ \underset{b}{\searrow}\ \mathrm{VIII}^- \end{array}$$

the other cases being similar. First assume that $a_{k+2} \geqslant 2$ (Case a). Then, since $\theta_{n+1} = -(1 + \alpha_{k+1})$, we see that $b_{n+1} = N(\theta_{n+1}) = -(1 + a_{k+1})$; $A_{n+1} = b_{n+1}A_n - A_{n-1} = -(1 + a_{k+1})(\pm p_k) - (\mp p_{k-2}) = \mp p_{k+1} \mp (p_k - p_{k-1} - p_{k-2}) = \mp p_{k+1}$ since $a_k = 1$; $A_n = \pm p_k$; $\theta_{n+2} = 1/(b_{n+1} - \theta_{n+1}) = 1/(-a_{k+1} + \alpha_{k+1}) = \alpha_{k+2}$; thus, we have IV^-. Now assume $a_{k+2} = 1$ (Case b). Then again $\theta_{n+1} = -(1 + \alpha_{k+1})$ implies $b_{n+1} = N(\theta_{n+1}) = -(2 + a_{k+1})$; $A_{n+1} = -(2 + a_{k+1})(\pm p_k) -(\mp p_{k-2}) = \mp (p_{k+1} + p_k) \mp (p_k - p_{k-1} - p_{k-2}) = \mp p_{k+2}$ since $a_k = a_{k+2} = 1$; $A_n = \pm p_k$; $\theta_{n+2} = 1/(-1 - a_{k+1} + \alpha_{k+1}) = 1/(-1 + \alpha_{k+2}^{-1}) = -\alpha_{k+2}/(\alpha_{k+2} - 1) = -(1 + \alpha_{k+3})$, since $a_{k+2} = 1$; thus, we have VIII^-.

5. Proof of Theorem 1. Theorem 2 will now be applied to prove Theorem 1. Recall that given $\alpha = [a_0; a_1, a_2, \ldots]$ we have for integers $n \geqslant -1$, $k(n)$ defined so that $A_n/B_n = p_{k(n)}/q_{k(n)}$. From Theorem 3, we have the following recursion for $k(n)$:

$$k(-1) = -1,$$
$$(5) \qquad k(n+1) = \begin{Bmatrix} k(n) + 1 \text{ if } a_{k(n)+2} \geqslant 2 \\ k(n) + 2 \text{ if } a_{k(n)+2} = 1 \end{Bmatrix}.$$

Define $\Phi_\nu(n, \alpha)$ for integers $\nu \geqslant 0$, $n \geqslant -1$ to be the number of integers j satisfying $0 \leqslant j \leqslant k(n) - \nu$ such that $a_{j+1} = a_{j+2} = \cdots = a_{j+\nu} = 1$. That is, $\Phi_\nu(n, \alpha)$ is the number of strings of ν consecutive 1's in the first $k(n)$ partial quotients (a_0 is excluded) Also, $\Phi_\nu(n, \alpha) = \varphi_\nu(k(n), \alpha, (\sqrt{5} - 1)/2)$. The relation between the NICF and the frequency of digits in the RCF is given by

LEMMA 5. $k(n) = n + \Sigma_{\nu=1}^{\infty} (-1)^{\nu-1}\Phi_\nu(n, \alpha)$.

Proof. Denoting the right-hand side by $h(n)$, it suffices to show that $h(n)$ satisfies the same recursion (5) that $k(n)$ does. First, it is clear that $\Phi_\nu(-1, \alpha) = 0$ for all $\nu \geqslant 1$ and so $h(-1) = -1$ as desired. Set

$$\epsilon_n = \begin{cases} 0 & \text{if } a_{k(n)+2} \geqslant 2, \\ 1 & \text{if } a_{k(n)+2} = 1, \end{cases}$$

so that $k(n+1) = k(n) + 1 + \epsilon_n$. Then, from (5) with $h(n)$ we see that we need to prove that for all $n \geqslant -1$

$$n + 1 + \sum_{\nu=1}^{\infty} (-1)^{\nu-1}\Phi_\nu(n+1, \alpha) = n + \sum_{\nu=1}^{\infty} (-1)^{\nu-1}\Phi_\nu(n, \alpha) + 1 + \epsilon_n$$

or

$$\sum_{\nu=1}^{\infty} (-1)^{\nu-1}\Phi_\nu(n+1, \alpha) = \sum_{\nu=1}^{\infty} (-1)^{\nu-1}\Phi_\nu(n, \alpha) + \epsilon_n. \tag{6}$$

There are two cases.

Case a: $k(n) = k(n-1) + 1$. That is, $a_{k(n-1)+2} \geqslant 2$ or $a_{k(n)+1} \geqslant 2$. We see then that $\Phi_1(n+1, \alpha) = \Phi_1(n, \alpha) + \epsilon_n$ and $\Phi_\nu(n+1, \alpha) = \Phi_\nu(n, \alpha)$ for all $\nu \geqslant 2$, and it is clear that (6) holds in this case.

Case b: $k(n) = k(n-1) + 2$. That is, $a_{k(n)} = a_{k(n-1)+2} = 1$. Now if $a_{k(n)+1} \geqslant 2$, then as before $\Phi_1(n+1, \alpha) = \Phi_1(n, \alpha) + \epsilon_n$ and $\Phi_\nu(n+1, \alpha) = \Phi_\nu(n, \alpha)$ for all $\nu \geqslant 2$; and we are done. So assume $a_{k(n)+1} = 1$. Choose $r \geqslant 0$ so that $a_r \neq 1$, $a_{r+1} = a_{r+2} = \cdots = a_{k(n)+1} = 1$ (if $a_1 = \cdots = a_{k(n)+1} = 1$ set $r = 0$). Then, we see easily that

$$\Phi_\nu(n+1, \alpha) = \begin{cases} \Phi_\nu(n, \alpha) + 1 + \epsilon_n & \text{if } 1 \leqslant \nu \leqslant k(n) + 1 - r, \\ \Phi_\nu(n, \alpha) + \epsilon_n & \text{if } \nu = k(n) + 2 - r, \\ \Phi_\nu(n, \alpha) & \text{if } \nu > k(n) + 2 - r. \end{cases}$$

Now we observe that $k(n) + 1 - r$ is even. Since $a_r \geqslant 2$ (or $r = 0$) and $a_{r+1} = 1$, we see that there is an m such that $r + 1 = k(m)$. Then $a_{k(m)} = a_{k(m)+1} = \cdots = a_{k(n)} = 1$ and so we see $k(n) - k(m)$ is even. Thus, $k(n) + 1 - r = k(n) + 2 - k(m)$ is even, also. From this and (7) we see that the first $k(n) + 1 - r$ terms of $1 + \epsilon_n$ in (6) cancel out, and the ϵ_n in the term for the odd number $k(n) + 2 - r$ is exactly what is needed. This completes the proof of Lemma 5.

Theorem 1 now follows, since for almost all α

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{k(n)} &= \lim_{n\to\infty} \frac{1}{k(n)} \left(k(n) + \sum_{\nu=1}^{\infty} (-1)^\nu \Phi_\nu(n, \alpha) \right) \\ &= \lim_{n\to\infty} \frac{1}{k(n)} \sum_{\nu=0}^{\infty} (-1)^\nu \varphi_\nu\left(k(n), \alpha, \frac{\sqrt{5}-1}{2} \right) = \frac{\log\left(\frac{1+\sqrt{5}}{2}\right)}{\log 2} \end{aligned}$$

from Lemma 5 and Theorem 2(b).

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