# New Primes of the Form $k \cdot 2^n + 1$

## By Robert Baillie

Abstract. All primes of the form $k \cdot 2^n + 1$ for $k$ odd, $1 \leqslant k < 150$, $1 \leqslant n \leqslant 1500$, have now been computed. Those not previously published are given here. Numbers with $151 \leqslant k \leqslant 999$ and $n \leqslant 600$ were also tested. Three new factors of Fermat numbers and a large pair of twin primes were found.

This report extends those of Matthew and Williams [5], Robinson [8], and Shippee [9], who searched for primes of the form $k \cdot 2^n + 1$.

The method used to test $k \cdot 2^n + 1$ for primality was stated originally by Proth [6], and is proven in [7]. Given $N = k \cdot 2^n + 1$ with $k < 2^n$, we look for a number $D$ which makes the Jacobi symbol $(D/N) = -1$. If $3 \nmid k$, we may take $D = 3$; if $3 \mid k$, a (usually short) search is conducted for a suitable $D$. Then $N$ is prime if and only if $D^{(N-1)/2} \equiv -1 \pmod{N}$.

Each prime found was checked to see if it was a factor of a Fermat number. Three factors were found: $629 \cdot 2^{257} + 1 \mid F_{255}$, $247 \cdot 2^{302} + 1 \mid F_{298}$, and $225 \cdot 2^{547} + 1 \mid F_{544}$. For a list of all other Fermat numbers known to be composite, see [4], [5], and [9].

Also, for each prime $N = k \cdot 2^n + 1$ found here, as well as those in [5] and [8] with $n \geqslant 200$, $N - 2$ was checked for primality using Theorem 13 of [1]. The largest pair of twin primes thus found was $297 \cdot 2^{546} \pm 1$. The largest pair previously published is $156 \cdot 5^{202} \pm 1$ [11]. (Since this note was originally submitted, an even larger pair of 303-digit twin primes was published. See [2].)

The table lists the primes of the form $k \cdot 2^n + 1$ for $1 \leqslant k < 150$, $1 \leqslant n \leqslant 1500$ which were not given in [5] and [8]. $\pi_k$ denotes the number of primes for a given $k$ for $1 \leqslant n \leqslant 1500$. Note that some values of $k$, such as $k = 81$, yield many primes, while some, like $k = 47$ give very few. Usually, $47 \cdot 2^n + 1$ is divisible by a small prime. In fact, $47 \cdot 2^n + 1$ is divisible by 3, 5, 7, or 13 unless $n \equiv 7 \pmod{12}$. By contrast, $81 \cdot 2^n + 1$ is never divisible by 3 or 7; if $5 \mid 81 \cdot 2^n + 1$, then $n \equiv 2 \pmod 4$; if $13 \mid 81 \cdot 2^n + 1$, then $n \equiv 2 \pmod{12}$, so in this case, $81 \cdot 2^n + 1$ is also divisible by 5. Thus, $81 \cdot 2^n + 1$ has more of a chance to produce primes than does $47 \cdot 2^n + 1$.

If $n$ is fixed, the sequence $\{a_k\} = \{k \cdot 2^n + 1\}$ is prime for infinitely many $k$, by Dirichlet's theorem. If $k$ is fixed, does $\{b_n\} = \{k \cdot 2^n + 1\}$ likewise have infinitely many primes? The answer is, "Not necessarily." Stark [10] gives a value of

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$k$ ($k = 2935\ 36333\ 15419\ 25531$) for which $b_n$ is always composite. See [3, p. 420] for a brief discussion of this problem.

TABLE

*Primes of the form $k \cdot 2^n + 1$*

| $k$ | $\pi_k$ | $n$ | $k$ | $\pi_k$ | $n$ |
|---|---|---|---|---|---|
| 1 | 5 | | 55 | 13 | |
| 3 | 19 | | 57 | 21 | 1312, 1399 |
| 5 | 12 | | 59 | 6 | 1085 |
| 7 | 19 | | 61 | 5 | |
| 9 | 25 | 1305, 1411, 1494 | 63 | 33 | 1150, 1290, 1441 |
| 11 | 12 | | 65 | 22 | 1317 |
| 13 | 10 | | 67 | 22 | 1148, 1366 |
| 15 | 21 | | 69 | 12 | 1450 |
| 17 | 10 | | 71 | 13 | |
| 19 | 5 | 1246 | 73 | 12 | |
| 21 | 18 | | 75 | 27 | 1018, 1054 |
| 23 | 12 | | 77 | 10 | |
| 25 | 20 | 1280, 1328 | 79 | 7 | 1330 |
| 27 | 24 | 1076, 1090 | 81 | 38 | 1384 |
| 29 | 18 | 1053, 1175 | 83 | 6 | |
| 31 | 6 | | 85 | 14 | 1300 |
| 33 | 18 | 1420 | 87 | 15 | 1268, 1302 |
| 35 | 18 | 1423, 1443 | 89 | 8 | |
| 37 | 20 | 1240 | 91 | 4 | |
| 39 | 27 | 1057 | 93 | 22 | 1020, 1110, 1478 |
| 41 | 6 | | 95 | 22 | 1039 |
| 43 | 14 | 1076 | 97 | 9 | |
| 45 | 16 | 1374 | 99 | 31 | |
| 47 | 2 | 1483 | 101 | 21 | 1103 |
| 49 | 11 | 1202 | 103 | 10 | |
| 51 | 23 | 1485 | 105 | 27 | |
| 53 | 11 | | 107 | 9 | |

TABLE (*continued*)

| $k$ | $\pi_k$ | $n$ |
|---|---|---|
| 109 | 5 | |
| 111 | 11 | |
| 113 | 13 | 1045 |
| 115 | 11 | 1298 |
| 117 | 15 | 1416 |
| 119 | 10 | 1115 |
| 121 | 10 | |
| 123 | 17 | 1173 |
| 125 | 15 | 1205, 1279, 1411 |
| 127 | 14 | 1098, 1420 |
| 129 | 12 | 1071 |
| 131 | 21 | 1, 3, 9, 13, 19, 21, 25, 51, 55, 97, 153, 165, 199, 261, 285, 361, 373, 465, 475, 529, 765 |
| 133 | 13 | 4, 6, 10, 16, 30, 124, 174, 192, 336, 600, 720, 1092, 1138 |
| 135 | 33 | 1, 2, 4, 6, 10, 15, 18, 20, 30, 31, 35, 38, 39, 51, 85, 90, 106, 108, 202, 238, 253, 282, 330, 361, 452, 459, 646, 895, 922, 1201, 1402, 1441, 1462 |
| 137 | 8 | 3, 27, 39, 83, 203, 395, 467, 875 |
| 139 | 3 | 2, 14, 914 |
| 141 | 33 | 1, 3, 5, 7, 8, 12, 15, 20, 31, 33, 37, 41, 61, 65, 91, 93, 103, 117, 133, 137, 141, 160, 291, 303, 343, 488, 535, 555, 556, 640, 756, 897, 917 |
| 143 | 7 | 53, 77, 293, 333, 393, 809, 825 |
| 145 | 11 | 6, 16, 28, 70, 76, 250, 276, 312, 562, 636, 1366 |
| 147 | 20 | 8, 11, 15, 18, 19, 26, 44, 60, 84, 90, 91, 134, 155, 179, 258, 275, 475, 620, 824, 888 |
| 149 | 20 | 3, 7, 9, 15, 17, 27, 33, 35, 57, 125, 127, 137, 191, 513, 819, 827, 921, 931, 1047, 1147 |

*Note Added in Proof.* I subsequently tested the range $20 \leqslant n \leqslant 100$, $30001 \leqslant k < 130000$ and found another Fermat factor:

$$92341 \cdot 2^{96} + 1 \mid F_{93}.$$

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