REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the indexing system printed in Volume 28, Number 128, October 1974, pages 1191–1194.

6[7.80, 7.105, 12.05.1].—H. Exton, *Handbook of Hypergeometric Integrals*, Ellis Horwood, Ltd., Chichester, U. K. and Halsted Press: a division of John Wiley & Sons, New York, 1978, 316 pp., 24 cm. Price \$37.50.

One of the simplest hypergeometric functions goes by the name of Gauss. It can be defined as

$$(1)\qquad {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}\,dt}{(1-zt)^b},$$
$$R(c) > R(a) > 0,\ |\arg(1-z)| < \pi.$$

Here $\Gamma(a)$ is the gamma function, etc. There are other integral representations. From (1) we can derive the hypergeometric series

$$(2)\qquad {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k z^k}{(c)_k k!},$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$, etc. and c is not a negative integer or zero. The series converges for $|z| < 1$ unless it terminates. a and b are called numerator parameters and c is called a denominator parameter. A natural extension is to permit, formally at least, any number of numerator and denominator parameters. We then have a generalized hypergeometric series. The latter can also be represented by an integral, and we also speak of generalized hypergeometric functions. There are further generalizations—the so-called G- and H-functions. Also, the notions are readily extended to two and more variables by use of multiple integrals, multiple summations, etc. A typical hypergeometric series of two variables is

$$(3)\qquad F_1(a, b, \beta; c, x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(\beta)_n}{(c)_{m+n}m!n!} x^m y^n.$$

In the latter, the variables are separable, and this simple feature which characterizes the F_1 and its natural generalizations allows for many properties to flow quite readily from those for hypergeometric functions of a single variable.

Hypergeometric functions of one and more variables and their generalizations have a well established literature. See, for example, the volumes by Erdélyi et al. [1], [2], Exton [3], Luke [4] and Slater [5].

In the volume under review, the author defines a hypergeometric integral as one which has a hypergeometric function in its integrand. The purpose of the tome is to give an ordered list of the main analytical formulae for these integrals along with some simple computer programs in FORTRAN for evaluation of the integrals in regions which omit singularities.

Many integrals involving hypergeometric functions of a single variable—Eulerian integrals, Laplace transforms, Mellin transforms, Mellin-Barnes integrals and many others—are again members of the hypergeometric family; and this naturally extends to generalized hypergeometric functions of several variables. Of course, many hypergeometric integrals cannot be so simply expressed, though they can be built up and expressed as series of hypergeometric integrals. The present volume is most welcome, especially for applied workers, since it is a handy reference to identify a broad class of functions, to delineate their properties and to provide a computer program for possible development of numerics.

The book is divided into two parts. Part I is composed of seven chapters giving general theory and applications, particularly in statistical distributions and in various branches of physics and engineering. Chapter 1 treats equations (1)–(3), the generalization of (3) to any number of variables and G- and H-functions. Various properties delineated include convergence of series and differential equations. Chapter 2 is devoted to integrals of Euler type of which (1) is a simple example.

"Definite Integrals and Repeated Integrals" is the title of Chapter 3. The word 'definite' should be replaced by 'indefinite,' for a typical integral studied is

$\int_0^z e^{-bn}u^{a-1}f(n)\,du$, which could also be called an "incomplete Laplace transform." Similarly,

$\int_0^z u^{a-1}(1-u)^{b-1}f(u)\,du$ could be called an "incomplete Euler transform."

Chapters 4 and 5 study contour integrals and infinite integrals, respectively, while Chapter 6 is devoted to multiple integrals. Part I is completed with Chapter 7, which treats applications, as already noted.

In Part II, about 80 pages are in the style of a handbook of tables of hypergeometric integrals; and about 40 pages are devoted to FORTRAN programs for the computation of 50 different integrals. For example, the first program is for the computation of the integral

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 u^{a-1}(a-u)^{b-1}\;{}_2F_2(c_1, c_2; d_1, d_2; ux)\,du$$

$$= {}_3F_3(c_1, c_2, a; d_1, d_2, a+b; x).$$

That the program is written for the latter series is not stated explicitly where the integral itself is posted. The introduction to all the programs states, "it must always be ascertained that the series, single or multiple, being investigated is either convergent or a suitable asymptotic series." As a practical indicator of the speed of convergence of summations, it is suggested that one compare, for example, evaluation of M terms of the above ${}_3F_3$ with $M+1$ terms, etc. The ${}_3F_3$ converges for all $|x| < \infty$, though if $|x|$ is large use of the program can be lengthy and costly. Use of an asymptotic expansion might be far more efficient, but this topic and its pertinent application to the material in the volume is not considered. Other pertinent topics such as expansion of the ${}_3F_3$ in series of Chebyshev polynomials of the first kind are omitted.

In the time one can devote to a review of this type, it is impossible to examine the volume in detail so as to uncover misprints and errors. I am reminded of the late Professor Arthur Erdélyi's observation that a handbook presents one with an idea of the type of results expected; but the actual formulas given may be incorrect, and one must always exercise caution. It therefore behooves each user to rederive or make some check on a formula to be employed. In the present instance I noticed that the result for equation (5.4.4.1) is patently false, as the integral does not exist. If the Bessel function $I_c(px^{1/2})$ is replaced by $J_c(px^{1/2})$, if p^2 is replaced by $-p^2$ in the ${}_0F_1$, and if in the final result for the integrals, 2^c is replaced by 2^{-2s}, then all is correct provided $\mathrm{Re}(s + c/2) > 0$ and $\mathrm{Re}(s) < 3/4$. Further, I find that none of the integrals on pages 118 and 119 exist. In place of (5.4.4.2) perhaps the author meant to have $\int_0^\infty x^{s-1}\,\Gamma(a, px)ax = \Gamma(s + a)/sp^s$, $\mathrm{Re}(s) > 0$. In (5.4.4.3)–(5.4.4.6), perhaps the author intended

$$I = \int_0^\infty z^{s-1} e^{-z} H_v(pz^{1/2})\,dz.$$

If so, I find

$$I = \frac{\Gamma\left(\frac{v+1}{2} + s\right) p^{v+1}}{\pi^{1/2} 2^v \Gamma(v + 3/2)}\, {}_2F_2\left(\begin{matrix} 1, \frac{v+1}{2} + s \\ 3/2, v + 3/2 \end{matrix} \,\middle|\, -p^2/4\right), \quad p > 0,\ R(2s + v) > -1.$$

I have a suggestion. When confronted with a wide variety of hypergeometric integrals, bear in mind the infinite integral ($\int_0^\infty$) of a product of G-functions is a G-function. See [4, Vol. I, p. 159]. This is a valuable master formula from which a multitude of results follow.

A special case of the remaining integrals is

$$I = \int_0^\infty y^{s-1}\, {}_2F_2(a, b; c; -\lambda y)\,dy$$

with $\lambda = -1$ which does not exist. The integral does exist if $|\arg \lambda| < \pi$, $0 < \mathrm{Re}(s) < \mathrm{Re}(a)$ and $0 < \mathrm{Re}(s) < \mathrm{Re}(b)$. In this event

$$I = \frac{\lambda^{-s}\Gamma(s)\Gamma(a - s)\Gamma(b - s)\Gamma(c)}{\Gamma(a)\Gamma(c - s)}.$$

Concerning the results as noted on pages 118 and 119, there is a chain reaction in that they are said to be special cases of results given previously in the volume. I have not tracked the ultimate source of the difficulties.

There is an excellent selected but rather lengthy bibliography. Further, there is an index of symbols and subject index to facilitate use of the volume

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1. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER & F. G. TRICOMI, *Higher Transcendental Functions,* Vols. I, II, III, McGraw-Hill, New York, 1953.

2. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER & F. G. TRICOMI, *Tables of Integral Transforms,* Vols. I, II, McGraw-Hill, New York, 1954.

3. H. EXTON, *Multiple Hypergeometric Functions and Applications,* Ellis Horwood, Ltd., Chichester, U. K., 1976.

4. Y. L. LUKE, *The Special Functions and Their Approximations,* Vols. I, II, Academic Press, New York, 1969.

5. L. J. SLATER, *Generalized Hypergeometric Functions,* Cambridge Univ. Press, London and New York, 1966.

7[7.10].—HANS A. LARSEN, *Natural Tangents and Trigonometrical Quadratic Surds to 50 Decimal Places,* Ms. of 6 pp., 8 × 11½ in., deposited in the UMT file.

As a sequel to his earlier calculation of natural sines to 50D [1] the author has calculated tan x for $x = 1^\circ(1^\circ)89^\circ$ in at least two ways to more than 60D, and the results, rounded to 50D, are tabulated in the present manuscript. Also included are closed expressions of the tangents of the integral multiples of 3° in the stated range. The eleven basic irrational numbers whose rational multiples appear in these expressions are also tabulated to 50D.

This calculation, like its predecessor, was motivated by a desire to check and extend the corresponding table of Herrmann [2].

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1. *Math. Comp.,* v. 18, 1964, p. 327, UMT 41.

2. A. HERRMANN, "Bestimmung der trigonometrischen Functionen aus den Winkeln und der Winkel aus den Functionen, bis zu einer beliebigen Grenze der Genauigkeit," *Wiener Sitzungsberichte,* v. 1, 1848, pp. 465–481.