# Monotone Difference Approximations
# for Scalar Conservation Laws

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**Abstract.** A complete self-contained treatment of the stability and convergence proper-
ties of conservation-form, monotone difference approximations to scalar conservation
laws in several space variables is developed. In particular, the authors prove that gener-
al monotone difference schemes always converge and that they converge to the physical
weak solution satisfying the entropy condition. Rigorous convergence results follow
for dimensional splitting algorithms when each step is approximated by a monotone
difference scheme.

      The results are general enough to include, for instance, Godunov's scheme, the
upwind scheme (differenced through stagnation points), and the Lax-Friedrichs scheme
together with appropriate multi-dimensional generalizations.

**Introduction.** Perhaps the simplest mathematical models exhibiting behavior typ-
ical of that encountered in inviscid continuum mechanics are the initial-value problems
for a scalar conservation law. These problems are of the form

$$(0.1) \quad \begin{cases} \text{(i)} \quad u_t + \sum_{i=1}^{N} f_i(u)_{x_i} = 0, \quad \text{for } t > 0, x = (x_1, \dots, x_N) \in \mathbf{R}^N, \\[2mm] \text{(ii)} \quad u(x, 0) = u_0(x), \quad \text{for } x \in \mathbf{R}^N, \end{cases}$$

where the $f_i$ are smooth real-valued functions and $u$ is a scalar. It is well known (see
[14]) that even if the initial value $u_0$ is smooth, the solution to (0.1) typically develops
discontinuities as $t$ increases to some $t_0 > 0$ (i.e. shock waves form). Thus the differen-
tial equation must be understood in a generalized or weak sense. However, there can
be an infinite number of generalized solutions of (0.1) with the same initial data $u_0$;
and an additional principle, the entropy condition, is needed to select the unique
"physical" weak solution (see [14]).

    The main new result of this work establishes the convergence of general conserva-
tion-form, monotone difference approximations to (0.1) to the unique generalized solu-
tion which satisfies the entropy condition. For notational simplicity in the sequel we
restrict the presentation to the case $N = 2$ for the most part. The corresponding defi-
nitions and results for the general case will be clear from this. For $N = 2$
we write $(x, y)$ rather than $(x_1, x_2)$. Selecting mesh sizes $\Delta x$, $\Delta y$, $\Delta t > 0$, the value

of our numerical approximation at $(j\Delta x, k\Delta y, n\Delta t)$ will be denoted by $U_{j,k}^n$. Capital letters $U$, $V$, etc. will denote functions on the $x$, $y$ lattice $\Delta = \{(j\Delta x, k\Delta y) : j, k$ are integers$\}$; and the value of $U$ at $(j\Delta x, k\Delta y)$ will be written $U_{j,k}$. Thus $U^n$, the state of our numerical approximation at the level $n\Delta t$, is a function on $\Delta$ with values $U_{j,k}^n$.

The standard notations $\lambda^x = \Delta t/\Delta x$, $\lambda^y = \Delta t/\Delta y$, $(\Delta_+^x U)_{j,k} = U_{j+1,k} - U_{j,k}$, $(\Delta_+^y U)_{j,k} = U_{j,k+1} - U_{j,k}$, etc., will be used. The difference approximations of (0.1) of interest here are explicit marching schemes of the form

(0.2)          $$U_{j,k}^{n+1} = G(U_{j-p,k-r}^n, \dots, U_{j+q+1,k+s+1}^n),$$

where $p$, $s$, $q$, $r$ are nonnegative integers and $G$ is a function of $(p + q + 2)(r + s + 2)$ real variables. (We are ignoring $\lambda^x$, $\lambda^y$ dependence for the moment, as these quantities will typically be fixed.) To simplify notation, (0.2) will be written as

(0.3)          $$U^{n+1} = \vec{G}(U^n),$$

with the choice of $p$, $s$, $q$, $r$ dictated by the context. The difference approximation (0.2)–(0.3) is said to have *conservation form* (see [10]) if there are functions $g_1$, $g_2$ such that

(0.4)
$$\begin{aligned}
\vec{G}(U)_{j,k} &= G(U_{j-p,k-r}, \dots, U_{j+q+1,k+s+1}) \\
&= U_{j,k} - \lambda^x \Delta_+^x g_1(U_{j-p,k-r}, \dots, U_{j+q,k+s+1}) \\
&\quad - \lambda^y \Delta_+^y g_2(U_{j-p,k-r}, \dots, U_{j+q+1,k+s}).
\end{aligned}$$

In order that (0.3) be *consistent* with (0.1) when (0.4) holds we must have

(0.5)          $$g_i(u, \dots, u) = f_i(u) \quad \text{for } u \in \mathbf{R} \text{ and } i = 1, 2.$$

The functions $g_i$ are called the *numerical fluxes* of the approximation. Finally, the difference approximation is *monotone* on the interval $[a, b]$ if $G(\alpha_1, \dots, \alpha_{(p+q+2)(r+s+2)})$ is a nondecreasing function of each argument $\alpha_i$ so long as all arguments lie in $[a, b]$.

It follows from the results of [13] that for $u_0 \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ there is a unique weak solution $u \in L^\infty(\mathbf{R}^2 \times [0, \infty))$, which satisfies the entropy condition of [13]. (See Section 2 for more details.) Moreover, we can write $u(x, y, t) = (S(t)u_0) \cdot (x, y)$, where $S(t) : L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2) \to L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$ for each $t \geqslant 0$ and $t \to S(t)u_0$ is continuous into $L^1(\mathbf{R}^2)$. To compute this solution numerically we set

(0.6)          $$U_{j,k}^0 = \frac{1}{\Delta x \Delta y} \int_{R_{j,k}} u_0(x, y)\,dx\,dy,$$

where

$$R_{j,k} = [(j - \tfrac{1}{2})\Delta x, (j + \tfrac{1}{2})\Delta x) \times [(k - \tfrac{1}{2})\Delta y, (k + \tfrac{1}{2})\Delta y),$$

and define $U^{n+1}$ from $U^n$ by (0.3). Finally, put $\chi_{j,k}^n = $ characteristic function of $R_{j,k} \times [n\Delta t, (n + 1)\Delta t)$ and

(0.7)          $$u^{\Delta t} = \sum_{n=0}^{\infty} \sum_{j,k=-\infty}^{\infty} U_{j,k}^n \chi_{j,k}^n.$$

The main result is

THEOREM 1. *Suppose $u_0 \in L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$ and $a \leqslant u_0 \leqslant b$ a.e. Let (0.2) be a consistent conservation-form difference approximation to (0.1)(i) which is monotone on $[a, b]$ and which has Lipschitz continuous numerical fluxes $g_i$, $i = 1, 2$. Let $u^{\Delta t}$ be given by (0.2), (0.6), (0.7). Then as $\Delta t \to 0$ with $\lambda^x$, $\lambda^y$ fixed, $u^{\Delta t}$ converges to $S(t)u_0$ in $L^1(\mathbf{R}^2)$ uniformly for bounded $t \geqslant 0$. More precisely,*

$$(0.8) \qquad \lim_{\Delta t \to 0} \; \sup_{0 \leqslant t \leqslant T} \; \iint_{\mathbf{R}^2} |u^{\Delta t}(x, y, t) - S(t)u_0(x, y)| \, dx \, dy = 0$$

*for each $T > 0$.*

Reviewing the definitions, (0.8) can also be restated as

$$(0.9) \qquad \lim_{\Delta t \to 0} \; \sup_{\substack{0 \leqslant t \leqslant T \\ n\Delta t \leqslant t < (n+1)\Delta t}} \; \sum_{j,k} \int_{R_{j,k}} |U_{j,k}^n - S(t)u_0(x, y)| \, dx \, dy = 0.$$

It should be recalled that even if $N = 1$ and $f_1$ is convex, nonmonotone schemes such as the Lax-Wendroff scheme can converge to solutions which violate the entropy condition; see [10] and [18]. The result of Theorem 1 applies to the popular dimensional splitting algorithms; see [8], [17], [21]. This follows from simple observations. For example, consider the one-dimensional conservation laws

$$(0.10) \qquad \begin{cases} \text{(i)} \quad v_t + f_1(v)_x = 0, \\ \text{(ii)} \quad w_t + f_2(w)_y = 0. \end{cases}$$

If

$$(0.11) \qquad \begin{cases} \text{(i)} \quad V_j^{n+1} = G_1(V_{j-p}^n, \dots, V_{j+q+1}^n), \\ \text{(ii)} \quad W_k^{n+1} = G_2(W_{k-r}^n, \dots, W_{k+s+1}^n), \end{cases}$$

are conservation-form difference approximations of (0.10)(i), (ii), respectively, Lax has observed that the scheme defined by

$$(0.12) \qquad \begin{cases} U_{j,k}^{n+1} = G_1(U_{j-p,k}^{n+1/2}, \dots, U_{j+q+1,k}^{n+1/2}), \\ U_{l,m}^{n+1/2} = G_2(U_{l,m-r}^n, \dots, U_{l,m+s+1}^n), \end{cases}$$

has conservation form and is consistent with (0.1); see [2] or check the definitions. When $G_1$ and $G_2$ are also monotone on $[a, b]$ the scheme in (0.12) will be monotone on $[a, b]$. (Since $a \leqslant V_{j+l} \leqslant b$ for $-p \leqslant l \leqslant q + 1$ implies $a \leqslant G_1(V_{j-p}, \dots, V_{j+q+1}) \leqslant b$ by Proposition 3.1(b) below, this is evident.) Thus, Theorem 1 applies to the split scheme (0.12). This fact, along with other results concerning splitting algorithms is developed in [17].

The plan of this work is as follows: In Section 1 various monotone difference schemes to which Theorem 1 applies are recalled. These include the Lax-Friedrichs scheme, the upwind scheme (differenced through stagnation points) and Godunov's scheme. The construction of a wide variety of multi-dimensional schemes from the

above one-dimensional ones is discussed. In view of these examples, many of the results of Le Roux [16] for specific schemes in a single space dimension are included in our general approach. Section 2 is a review of some basic facts about solutions to (0.1) and some function spaces and estimates needed in the proof of Theorem 1. We discuss the stability of monotone conservation-form difference schemes in Section 3. In particular, we prove that such schemes define $L^1$ contractions; this was proved by Jennings [11] in the case of one space variable, but even there our proof, based on a lemma of Crandall and Tartar [5], is simpler. In Section 4 we verify that if solutions of monotone conservation-form difference approximations converge, the limit satisfies the entropy conditions. This was proved by Harten, Hyman and Lax in [10] for $N = 1$; however, we build a different discrete entropy flux which yields a simpler proof for general $N$ and requires only continuity of the numerical fluxes. This generality is useful for applications. (See Section 1 and [16].) The various results of Sections 2, 3, 4 are pieced together to prove Theorem 1 in Section 5. In Section 6 we briefly discuss the inhomogeneous equation.

In fact, our arguments yield more than Theorem 1 states, for the *existence* of the solution $S(t)u_0$ of (0.1) is established while proving convergence (see Section 5). See Conway and Smoller [3], Douglis [6], and Kojima [12] for earlier uses of the (monotone) Lax-Friedrichs difference approximation to prove existence. Oharu and Takahashi prove this scheme converges to the solution satisfying the entropy condition via nonlinear semigroup methods in [19].

*Added in Proof.* Kuznecov and Vološin [23], a paper we uncovered on the day it was necessary to return the corrected proofs, states the main result of this work together with an error estimate under stronger regularity assumptions than used here.

**1. Examples.** In this section we present a variety of difference schemes to which Theorem 1 applies. Later sections are independent of the current one.

We begin with several well-known schemes in the case $N = 1$. So long as $N = 1$ we will write $f$, $g$ in place of $f_1$, $g_1$. For a single-space variable the *Lax-Friedrichs scheme* is given by

(1.1)
$$u_j^{n+1} = u_j^n - \frac{\lambda^x}{2} \Delta_0^x f(u_j^n) + \frac{1}{2} \Delta_+^x \Delta_-^x u_j^n.$$

Equivalently,

$$u_j^{n+1} = u_j^n - \lambda^x \Delta_+^x g(u_j, u_{j-1}),$$

where the numerical flux $g$ is given by

$$g(u_j, u_{j-1}) = \frac{f(u_j) + f(u_{j-1})}{2} - \frac{1}{2\lambda^x}(u_j - u_{j-1}).$$

The conservation form and consistency are obvious. A simple analysis reveals that (1.1) is also monotone on $[a, b]$ provided that the C.F.L. condition

(1.2)
$$\lambda^x \max_{a \leqslant u \leqslant b} |f'(u)| \leqslant 1$$

holds. More generally, if $\gamma$ is nondecreasing, the scheme

$$(1.3) \qquad u_j^{n+1} = u_j^n - \frac{\lambda^x}{2} \Delta_0^x f(u_j^n) + \frac{1}{2} \Delta_+^x \Delta_-^x \gamma(u_j^n)$$

has conservation form with the flux.

$$g(u_j, u_{j-1}) = \frac{f(u_j) + f(u_{j-1})}{2} - \frac{1}{2\lambda^x} (\gamma(u_j) - \gamma(u_{j-1}))$$

and (1.3) is monotone on $[a, b]$ provided that

$$(1.4) \qquad 1 - \lambda^x \gamma'(u) \geqslant 0 \quad \text{and} \quad \gamma'(u) - \lambda^x |f'(u)| \geqslant 0 \quad \text{for } a \leqslant u \leqslant b.$$

If $f$ is nondecreasing, the *upwind difference scheme* is given by

$$(1.5) \qquad u_j^{n+1} = u_j^n - \lambda^x(f(u_j^n) - f(u_{j-1}^n)) = u_j^n - \lambda^x \Delta_+^x f(u_{j-1}^n),$$

while if $f$ is nonincreasing it has the form

$$(1.6) \qquad u_j^{n+1} = u_j^n - \lambda^x \Delta_+^x f(u_j^n).$$

Next let $\alpha \in \mathbf{R}$, and assume

$$(1.7) \qquad f'(u) \geqslant 0 \quad \text{for } u \geqslant \alpha \quad \text{and} \quad f'(u) \leqslant 0 \quad \text{for } u \leqslant \alpha.$$

Thus, $f'(\alpha) = 0$ and $\alpha$ is a *stagnation point*. Without loss of generality, we may also assume that $f(\alpha) = 0$ (since changing $f$ by a constant leaves (0.1) invariant). Set

$$(1.8) \qquad \theta(s) = \begin{cases} 1 & \text{if } s \leqslant \alpha, \\ 0 & \text{if } s > \alpha, \end{cases}$$

and consider the scheme

$$(1.9) \qquad u_j^{n+1} = u_j^n - \lambda^x \Delta_+^x (\theta(u_j^n)f(u_j^n) + (1 - \theta(u_{j-1}^n))f(u_{j-1}^n)).$$

The resulting scheme (for any choice of $\theta$) is clearly consistent, has conservation form, and when $\theta \equiv 0$ or $\theta \equiv 1$ reduces to the conventional upwind schemes (1.5), (1.6). In our case ((1.7) holds, $\theta$ given by (1.8) and $f(\alpha) = 0$) it is easy to verify that (1.10) is monotone on $[a, b]$, provided that (1.2) holds.

The last scheme we mention for $N = 1$ is *Godunov's method*. Here we consider only the case in which $f'' > 0$ (in particular, $f$ is strictly convex). Godunov's method is then given by the three-point conservation-form scheme

$$(1.10) \qquad u_j^{n+1} = u_j^n - \lambda^x \Delta_+^x g(u_{j-1}^n, u_j^n),$$

where the numerical flux $g$ is defined by the complicated recipe:

$$(1.11) \qquad g(u_{j-1}, u_j) = \begin{cases} f(u_{j-1}) & \text{if } u_{j-1} \geqslant u_j \text{ and } f(u_{j-1}) \geqslant f(u_j), \\ f(u_j) & \text{if } u_{j-1} \geqslant u_j \text{ and } f(u_{j-1}) \leqslant f(u_j), \\ f(u_{j-1}) & \text{if } u_{j-1} \leqslant u_j \text{ and } f'(u_{j-1}) \geqslant 0, \\ f(u_j) & \text{if } u_{j-1} \leqslant u_j \text{ and } f'(u_j) \leqslant 0, \\ f((f')^{-1}(0)) & \text{otherwise.} \end{cases}$$

We have written down the function $g$ explicitly because these formulas show that for Godunov's method the numerical flux is Lipschitz continuous but not everywhere differentiable. In order to verify that Godunov's method defines a monotone scheme we recall the basic idea of the method. Given bounded discrete data $u_j^n$ with $a \leqslant u_j^n \leqslant b$, let

$$u_0 = \sum_{j=-\infty} u_j^n \chi_j,$$

where $\chi_j$ is the characteristic function of $(j - \frac{1}{2})\Delta x \leqslant x < (j + \frac{1}{2})\Delta x$. Now we let $u$ be the exact solution of

(1.12)
$$\begin{cases} u_t + f(u)_x = 0, \\ u(0, x) = u_0(x), \end{cases}$$

which we write as $u(x, t) = S(t)u_0(x)$. Since solutions of (1.11) propogate at finite speed at most $c_0 = \max_{a \leqslant u \leqslant b} |f'(u)|$ (see Section 2), it follows that $S^x(\Delta t)u_0(x)$, restricted to $(j - \frac{1}{2})\Delta x \leqslant x \leqslant (j + \frac{1}{2})\Delta x$, depends only on the three values $u_{j-1}^n$, $u_j^n$, $u_{j+1}^n$ provided that, as we now assume, (1.2) holds. Using the form of the exact solution of (1.12) for $0 \leqslant t \leqslant \Delta t$ (see [14]) and the standard flux balance relation, one explicitly computes

$$u_j^{n+1} = \frac{1}{\Delta x} \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} S^x(\Delta t)u_0(x)\, dx$$

and finds (1.10), (1.11). It is well known (see our proof of Theorem 1) that $v_0 \geqslant u_0$ implies $S(t)v_0 \geqslant S(t)u_0$. Hence, $u_j^{n+1}$ is a nondecreasing function of $u_{j-1}^n$, $u_j^n$, $u_{j+1}^n$, which establishes the monotonicity of the scheme.

Multi-dimensional examples of schemes to which Theorem 1 applies can be built from the one-dimensional schemes in a variety of ways. First, there are the methods of dimensional splitting as mentioned in the introduction. To this we add one additional method. Let

(1.13)
$$\begin{cases} \text{(i)} \quad V_j^{n+1} = G_1(V_{j-p}^n, \dots, V_{j+q+1}^n) = V_j^n - \lambda^x \Delta_+^x g_1(V_{j-p}^n, \dots, V_{j+q}^n), \\ \text{(ii)} \quad W_k^{n+1} = G_2(W_{k-r}^n, \dots, W_{k+s+1}^n) = W_j^n - \lambda^y \Delta_+^y g_2(W_{k-r}^n, \dots, W_{k+s}^n), \end{cases}$$

denote two conservation-form schemes consistent with

(1.14)
$$\begin{cases} \text{(i)} \quad v_t + \left(\frac{1}{\alpha}f_1(v)\right)_x = 0, \\ \text{(ii)} \quad w_t + \left(\frac{1}{1-\alpha}f_2(w)\right)_y = 0, \end{cases}$$

respectively, where $0 < \alpha < 1$. Now form the composite scheme

(1.15)  $U_{j,k}^{n+1} = \alpha G_1(U_{j-p,k}^n, \dots, U_{j+q+1,k}^n) + (1 - \alpha)G_2(U_{j,k-r}^n, \dots, U_{j,k+s+1}^n).$

Then (1.15) has conservation form and is consistent with $u_t + f_1(u)_x + f_2(u)_y = 0$. Moreover, if (1.13)(i), (ii) are monotone on $[a, b]$ so then is (1.15). Any of the

schemes discussed above may be used for $G_1$, $G_2$. For example, if $\alpha = \frac{1}{2}$ and $G_1$, $G_2$ are chosen as Lax-Friedrich's scheme (1.1), then (1.15) reads

$$u_{j,k}^{n+1} = u_{j,k}^n - \frac{\lambda^x}{2}\,\Delta_0^x f_1(u_{j,k}^n) - \frac{\lambda^y}{2}\,\Delta_0^y f_2(u_{j,k}^n) + \frac{1}{4}(\Delta_+^x \Delta_-^x + \Delta_+^y \Delta_-^y)(u_{j,k}^n),$$

which is the Lax-Friedrich's scheme in two dimensions. The C.F.L. condition guaranteeing monotonicity on $[a, b]$ is now

$$\lambda^x \max_{a \leqslant u \leqslant b} |f_1'(u)|, \; \lambda^y \max_{a \leqslant u \leqslant b} |f_2'(u)| \leqslant \frac{1}{2},$$

which is a factor of 2 more severe than (1.2).

**2. Preliminaries.** We begin by recalling some of the basic facts concerning solutions of the problem (0.1). A *weak solution* of the conservation law (0.1)(i) on $\mathbf{R}^N \times [0, T]$ is a function $u \in L^\infty(\mathbf{R}^N \times [0, T])$ such that

$$(2.1) \quad \int_0^T \int_{\mathbf{R}^N} \left( \varphi_t u + \sum_{i=1}^N \varphi_{x_i} f_i(u) \right) dx\,dt = 0 \quad \text{for every } \varphi \in C_0^1(\mathbf{R}^N \times (0, T)),$$

where $C_0^1(\mathbf{R}^N \times (0, T))$ denotes the continuously differentiable functions on $\mathbf{R}^N \times (0, T)$ with compact support. As we remarked earlier, weak solutions are not uniquely determined by their initial data (properly interpreted); and an additional condition, the entropy condition, is needed to select the desired solution. The form of the entropy condition we will use was given by Vol'pert [22]. An *entropy solution* of the conservation law (0.1)(i) on $\mathbf{R}^N \times (0, T)$ is a function $u \in L^\infty(\mathbf{R}^N \times [0, T])$ such that

$$(2.2) \quad \begin{cases} \int_0^T \int_{\mathbf{R}^N} \varphi_t |u - c| + \sum_{i=1}^N \varphi_{x_i} \operatorname{sgn}(u - c)(f_i(u) - f_i(c))\,dx\,dt \geqslant 0 \\[2mm] \text{for every } \varphi \in C_0^1(\mathbf{R}^2 \times (0, T)) \text{ with } \varphi \geqslant 0 \text{ and every } c \in \mathbf{R}. \end{cases}$$

In (2.2), $\operatorname{sgn} r = r/|r|$ for $r \neq 0$ (and the value assigned to sgn 0 is irrelevant since $f_i(u) - f_i(c) = 0$ if $u = c$). If $a \leqslant u \leqslant b$ a.e., then choosing $c = b$ and $c = a$ in (2.2) we can deduce (2.1) for $\varphi \geqslant 0$ (and hence, in general). That is, entropy solutions are weak solutions. Moreover, if $a \leqslant u \leqslant b$ a.e. and (2.2) holds for $a \leqslant c \leqslant b$, then it clearly holds for all $c$. The existence and uniqueness of entropy solutions of (0.1) which are of (locally) bounded variation (see below), when $u_0$ is of (locally) bounded variation and the initial condition is properly interpreted, was proved by Vol'pert [22]. Subsequently, Kružkov [13] extended these results. A special case of Kružkov's uniqueness theorem adequate for our purposes is:

THEOREM 2.1 (UNIQUENESS). *Let* $u_{10}, u_{20} \in L^\infty(\mathbf{R}^N)$. *Let* $u_1$, $u_2$ *be entropy solutions of* (0.1)(i) *on* $\mathbf{R}^N \times (0, T)$ *for which*

$$(2.3) \quad \text{essential limit}_{t \downarrow 0} \int_{|x| \leqslant R} |u_i(x, t) - u_{i0}(x)|\,dx = 0 \quad \text{for } R > 0, i = 1, 2.$$

*Let* $L$ *be a Lipschitz constant for the mapping* $r \to (f_1(r), \ldots, f_N(r))$ *on* $|r| \leqslant \max\{\|u_i\|_{L^\infty(\mathbf{R}^N \times (0,T))} : i = 1, 2\}$. *Then*

(2.4)
$$\begin{cases} \int_{|x|\leqslant R} |u_1(x,\,t)-u_2(x,\,t)|\,dx \leqslant \int_{|x|\leqslant R+Lt} |u_{10}(x)-u_{20}(x)|\,dx \\ \text{for } R>0 \text{ and almost all } t\in[0,\,T]. \end{cases}$$

In particular, if $u_{10}=u_{20}$, then $u_1=u_2$ a.e. If we choose $u_{20}=u_2=0$ in (2.4) (constants are entropy solutions of (0.1)(i)), we find

$$\int_{|x|\leqslant R} |u_1(x,\,t)|\,dx \leqslant \int_{|x|\leqslant R+Lt} |u_{10}(x)|\,dx.$$

If $u_{10}\in L^1(\mathbf{R}^N)$, letting $R\rightarrow\infty$ above yields

$$\int_{\mathbf{R}^N} |u_1(x,\,t)|\,dx \leqslant \int_{\mathbf{R}^N} |u_{10}(x)|\,dx,$$

so $t\rightarrow u_1(\cdot,\,t)$ is bounded into $L^1(\mathbf{R}^N)$ (or $u_1\in L^\infty(0,\,T:L^1(\mathbf{R}^N))$). It will prove convenient for us to work with this case, that is, $u_0\in L^1(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$. It is a simple matter to pass then to the more general case $u_0\in L^\infty(\mathbf{R}^N)$ via the finite domain of dependence established in (2.4) (or the finite numerical domain of dependence for our schemes).

It follows from the results of [13], [22] that for every $u_0\in L^\infty(\mathbf{R}^N)$ (0.1)(i) has an entropy solution $u$ assuming the initial-value $u_0$ in the sense (2.3). This existence result will follow easily from our investigations, so we will not belabor it here.

Next, we define some function spaces needed in the sequel. From now on we set $N=2$ in this section and assume the reader can extrapolate to the case of general $N$. $L^1_{loc}(\mathbf{R}^2)$ is the space of functions which are integrable on compact subsets of $\mathbf{R}^2$. A sequence $\{g_n\}\leqslant L^1_{loc}(\mathbf{R}^2)$ converges to $g$ in $L^1_{loc}(\mathbf{R}^2)$ provided $\|g_n-g\|_{L^1(K)}\rightarrow 0$ for every compact $K\subset\mathbf{R}^2$. $BV(\mathbf{R}^2)$ denotes the subspace of $f\in L^1_{loc}(\mathbf{R}^2)$ for which $\|f\|_{BV(\mathbf{R}^2)}<\infty$, where

(2.5)
$$\|f\|_{BV(\mathbf{R}^2)} = \sup_{h\in\mathbf{R}\setminus\{0\}} \int_{\mathbf{R}^2} \frac{|f(x+h,\,y)-f(x,\,y)|}{|h|}\,dxdy$$
$$+ \sup_{h\in\mathbf{R}\setminus\{0\}} \int_{\mathbf{R}^2} \frac{|f(x,\,y+h)-f(x,\,y)|}{|h|}\,dxdy.$$

We observe that for $f\in BV(\mathbf{R}^2)$ and $h,\,s\in\mathbf{R}$

(2.6)
$$\int_{\mathbf{R}^2} |f(x+h,\,y+s)-f(x,\,y)|\,dxdy$$
$$\leqslant \int_{\mathbf{R}^2} |f(x+h,\,y+s)-f(x,\,y+s)|\,dxdy$$
$$+ \int_{\mathbf{R}^2} |f(x,\,y+s)-f(x,\,y)|\,dxdy$$
$$\leqslant (|h|+|s|)\|f\|_{BV(\mathbf{R}^2)}.$$

The space $L^1(\mathbf{R}^2)\cap BV(\mathbf{R}^2)$ is equipped with the norm

(2.7)
$$\|f\|_{L^1(\mathbf{R}^2)\cap BV(\mathbf{R}^2)} = \|f\|_{L^1(\mathbf{R}^2)} + \|f\|_{BV(\mathbf{R}^2)}.$$

We will use the following compactness criterion (which is stated with unnecessarily restrictive hypotheses).

PROPOSITION 2.2. *Let* $H \subset L^1(\mathbf{R}^2)$ *be bounded and*

$$(2.8) \qquad \lim_{(h,s)\to(0,0)} \sup_{f\in H} \int_{\mathbf{R}^2} |f(x + h, y + s) - f(x, y)| \, dxdy = 0.$$

*Then* $H$ *is precompact in* $L^1_{\mathrm{loc}}(\mathbf{R}^2)$.

This is immediate from standard results, e.g., [6, IV.8.21]. As a corollary, we note that (2.6), (2.7) imply

$$(2.9) \qquad \text{bounded subsets of } L^1(\mathbf{R}^2) \cap BV(\mathbf{R}^2) \text{ are precompact in } L^1_{\mathrm{loc}}(\mathbf{R}^2).$$

Finally, we summarize several facts which will be used in passing between continuous and discrete estimates. Let us recall the notations. $U$, $V$, etc. denote functions on the lattice $\Delta = \{(j\Delta x, k\Delta y) = \Delta t(j/\lambda^x, b/\lambda^y): j, k \text{ integers}\}$ with values $U_{j,k}$, $V_{j,k}$, etc. We set

$$(2.10) \qquad \|U\|_{L^1(\Delta)} = \sum_{j,k} |U_{j,k}| \Delta x \Delta y, \qquad \|U\|_{L^\infty(\Delta)} = \sup_{j,k} |U_{j,k}|$$

and

$$(2.11) \qquad \|U\|_{BV(\Delta)} = \frac{1}{\Delta x} \|\Delta_+^x U\|_{L^1(\Delta)} + \frac{1}{\Delta y} \|\Delta_+^y U\|_{L^1(\Delta)},$$

where $(\Delta_+^x U)_{j,k} = U_{j+1,k} - U_{j,k}$, etc. $L^1(\Delta)$, $BV(\Delta)$ consist of those $U$ for which (2.10), (2.11) are finite.

The characteristic function of the rectangle $R_{j,k} = [(j - \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta x) \times [(k - \frac{1}{2})\Delta y, (k + \frac{1}{2})\Delta y)$ is denoted by $\chi_{j,k}$. Given a lattice function $U$ we define a piecewise constant function $U_{\mathbf{R}^2}$ on $\mathbf{R}^2$ by $U_{\mathbf{R}^2} = U_{j,k}$ on $R_{j,k}$. Equivalently,

$$(2.12) \qquad U_{\mathbf{R}^2} = \sum_{j,k} U_{j,k}\chi_{j,k}.$$

Given a function $u \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$ we define a lattice function $u_\Delta$ by

$$(2.13) \qquad (u_\Delta)_{j,k} = \frac{1}{\Delta x \Delta y} \int_{R_{j,k}} u(x, y) \, dxdy.$$

One has

$$(2.14) \qquad \|U_{\mathbf{R}^2}\|_{L^1(\mathbf{R}^2)} = \|U\|_{L^1(\Delta)}, \qquad \|U_{\mathbf{R}^2}\|_{L^\infty(\mathbf{R}^2)} = \|U\|_{L^\infty(\Delta)},$$

$$(2.15) \qquad \|u_\Delta\|_{L^1(\Delta)} \leqslant \|u\|_{L^1(\mathbf{R}^2)}, \qquad \|u_\Delta\|_{L^\infty(\Delta)} \leqslant \|u\|_{L^\infty(\mathbf{R}^2)},$$

$$(2.16) \qquad \|U_{\mathbf{R}^2}\|_{BV(\mathbf{R}^2)} \leqslant \|U\|_{BV(\Delta)},$$

and

$$(2.17) \qquad \|u_\Delta\|_{BV(\Delta)} \leqslant \|u\|_{BV(\mathbf{R}^2)}.$$

Of these relations, (2.14) and (2.15) are immediate from the definitions. Also, (2.17) is easy to see for, by the definitions,

$$\|u_\Delta\|_{BV(\Delta)} = \sum_{j,k} \left( \frac{1}{\Delta x} \left| \int_{R_{j+1,k}} u(x, y)\,dxdy - \int_{R_{j,k}} u(x, y)\,dxdy \right| \right.$$

$$\left. + \frac{1}{\Delta y} \left| \int_{R_{j,k+1}} u(x, y)\,dxdy - \int_{R_{j,k}} u(x, y)\,dxdy \right| \right)$$

$$= \sum_{j,k} \left( \frac{1}{\Delta x} \left| \int_{R_{j,k}} (u(x + \Delta x, y) - u(x, y))\,dxdy \right| \right.$$

$$\left. + \frac{1}{\Delta y} \left| \int_{R_{j,k}} u(x, y + \Delta y) - u(x, y)\,dxdy \right| \right)$$

$$\leqslant \|u\|_{BV(\mathbf{R}^2)}.$$

It remains to check (2.17). To this end, observe that

$$\int_{\mathbf{R}^2} |f(x + h_1 + h_2, y) - f(x, y)|\,dxdy$$

$$\leqslant \int_{\mathbf{R}^2} |f(x + h_1 + h_2, y) - f(x + h_2, y)|\,dxdy$$

(2.18)
$$+ \int_{\mathbf{R}^2} |f(x + h_2, y) - f(x, y)|\,dxdy$$

$$= \int_{\mathbf{R}^2} |f(x + h_1, y) - f(x, y)|\,dxdy$$

$$+ \int_{\mathbf{R}^2} |f(x + h_2, y) - f(x, y)|\,dxdy.$$

That is, $g(h_1 + h_2) \leqslant g(h_1) + g(h_2)$, where $g(h_1 + h_2)$ is the left-hand side of (2.18). Assume now that $h > 0$, and write $h = (l\Delta x + \xi \Delta x)$ where $l$ is a nonnegative integer and $0 \leqslant \xi < 1$. By the above

$$\int_{\mathbf{R}^2} |U_{\mathbf{R}^2}(x + h, y) - U_{\mathbf{R}^2}(x, y)|\,dxdy$$

(2.19)
$$\leqslant l \int_{\mathbf{R}^2} |U_{\mathbf{R}^2}(x + \Delta x, y) - U_{\mathbf{R}^2}(x, y)|\,dxdy$$

$$+ \int_{\mathbf{R}^2} |U_{\mathbf{R}^2}(x + \xi \Delta x, y) - U_{\mathbf{R}^2}(x, y)|\,dxdy.$$

By the definitions

(2.20)
$$\int_{\mathbf{R}^2} |U_{\mathbf{R}^2}(x + \Delta x, y) - U_{\mathbf{R}^2}(x, y)|\,dxdy$$

$$= \sum_{j,k} |U_{j+1,k} - U_{j,k}|\,\Delta x \Delta y = \|\Delta_+^x U\|_{L^1(\Delta)}$$

and

$$\int_{\mathbf{R}^2} |U_{\mathbf{R}^2}(x + \xi\Delta x, y) - U_{\mathbf{R}^2}(x, y)|\, dxdy$$

(2.21)

$$= \sum_{j,k} \int_{R_{j,k}} |U_{\mathbf{R}^2}(x + \xi\Delta x, y) - U_{j,k}|\, dxdy$$

$$= \sum_{j,k} \int_{(j+\frac{1}{2}-\xi)\Delta x}^{(j+\frac{1}{2})\Delta x} \int_{(k-\frac{1}{2})\Delta y}^{(k+\frac{1}{2})\Delta y} |U_{j+1,k} - U_{j,k}|\, dxdy$$

$$= \sum_{j,k} \xi |U_{j+1,k} - U_{j,k}|\, \Delta x\Delta y.$$

Combining (2.19), (2.20), (2.21) and recalling $h = l\Delta x + \xi\Delta x$, we find

$$\frac{1}{h} \int_{\mathbf{R}^2} |U_{\mathbf{R}^2}(x + h, y) - U_{\mathbf{R}^2}(x, y)|\, dxdy \leqslant \frac{1}{\Delta x} \|\Delta_+^x U\|_{L^1(\Delta)}.$$

Treating the $y$-variation in a similar way gives (2.16).

3. **Stability Properties of Monotone Conservation-Form Schemes.** Recall that our schemes have the form

(3.1)
$$U_{j,k}^{n+1} = G(U_{j-p,k-r}^n, \ldots, U_{j+q+1,k+s+1}^n),$$

which we write as $U^{n+1} = \vec{G}(U^n)$. The first result is

PROPOSITION 3.1. *Let (3.1) be a conservation-form difference scheme which is monotone on the interval* $[a, b]$. *Let* $U, V$ *satisfy* $a \leqslant U_{j,k}, V_{j,k} \leqslant b$ *for all* $j, k$. *Then*

(a)
$$U \leqslant V \;(i.e.\, U_{j,k} \leqslant V_{j,k} \text{ for all } j, k) \quad implies \quad \vec{G}(U) \leqslant \vec{G}(V).$$

(b)
$$\min_{\substack{j-p\leqslant l\leqslant j+q+1 \\ k-r\leqslant m\leqslant k+s+1}} U_{l,m} \leqslant \vec{G}(U)_{j,k} \leqslant \max_{\substack{j-p\leqslant l\leqslant j+q+1 \\ k-r\leqslant m\leqslant k+s+1}} U_{l,m} \quad \text{for all } j, k.$$

(c)
$$\|\vec{G}(U) - \vec{G}(V)\|_{L^1(\Delta)} \leqslant \|U - V\|_{L^1(\Delta)} \quad if\ U,\ V \in L^1(\Delta).$$

(d)
$$\|\vec{G}(U)\|_{BV(\Delta)} \leqslant \|U\|_{BV(\Delta)} \quad if\ U \in L^1(\Delta).$$

*Proof.* We will show first that (c) $\Rightarrow$ (d). Indeed, if $(\tau_{\Delta x} W)_{j,k} = W_{j+1,k}$, we have $\tau_{\Delta x} G(U) = G(\tau_{\Delta x} U)$ (that is, $G$ computes with grid translations). Then, by (c),

$$\|\Delta_+^x \vec{G}(U)\|_{L^1(\Delta)} = \|\tau_{\Delta x} \vec{G}(U) - \vec{G}(U)\|_{L^1(\Delta)}$$

$$= \|\vec{G}(\tau_{\Delta x} U) - \vec{G}(U)\|_{L^1(\Delta)} \leqslant \|\tau_{\Delta x} U - U\|_{L^1(\Delta)} = \|\Delta_+^x U\|_{L^1(\Delta)}.$$

Similarly, $\|\Delta_+^y \overrightarrow{G}(U)\|_{L^1(\Delta)} \le \|\Delta_+^Y U\|_{L^1(\Delta)}$. Now (d) follows from the definition (2.11). A moment's reflection shows that (a) is merely a restatement of the assumption that (3.1) is monotone on $[a, b]$. We next verify (b). Letting $V_{j,k} = \sup_{l,m} U_{l,m} = c$ for all $j,k$, we have $\overrightarrow{G}(V) = V$ ($G$ maps constants to themselves by the conservation form). Now $V \ge U$ so $c = \overrightarrow{G}(V)_{j,k} \ge \overrightarrow{G}(U)_{j,k}$ for all $j$, $k$. Estimating below in a similar way yields $\inf_{l,m} U_{l,m} \le U_{j,k} \le \sup_{l,m} U_{l,m}$. The refinement (b) of this result follows from the fact that $G(U)_{j,k}$ depends only on $U_{l,m}$ for $l$, $m$ in the range indicated in (b).

It remains to prove (c). The following elementary lemma of M. Crandall and L. Tartar is at the heart of the matter. We use the notations $f \vee g = \max(f, g)$ and $f^+ = f \vee 0$.

LEMMA 3.2. *Let $\Omega$ be a measure space and $C \subset L^1(\Omega)$ have the property that $f \vee g \in C$ whenever $f, g \in C$. Let $T: C \to L^1(\Omega)$ satisfy*

$$(3.2) \qquad \int_\Omega T(f) = \int_\Omega f \quad \textit{for } f \in C.$$

*Then the following three properties of $T$ are equivalent:*

(a) $\qquad f, g \in C$ *and* $f \le g$ *a.e.* *implies* $T(f) \le T(g)$ *a.e.*,

(b) $\qquad \displaystyle\int_\Omega (T(f) - T(g))^+ \le \int_\Omega (f - g)^+ \quad \textit{for } f, g \in C,$

(c) $\qquad \displaystyle\int_\Omega |T(f) - T(g)| \le \int_\Omega |f - g| \quad \textit{for } f, g \in C.$

The proof of Lemma 3.2 is given at the end of this section. We apply it here with $\Omega = \Delta$ equipped with the discrete measure assigning mass $\Delta x \Delta y$ to each point. Set

$$C = \{U \in L^1(\Delta) : a \le U_{j,k} \le b \text{ for all } j, k\},$$

and $T(U) = \overrightarrow{G}(U)$. It follows from Proposition 3.1(b) that

$$\|\overrightarrow{G}(U)\|_{L^1(\Delta)} \le (p + q + 2)(s + r + 2)\|U\|_{L^1(\Delta)},$$

so $\overrightarrow{G}: C \to L^1(\Delta)$. For $U \in C$ we have

$$\int_\Delta \overrightarrow{G}(U) = \sum_{j,k} G(U)_{j,k} \Delta x \Delta y = \sum_{j,k} U_{j,k} \Delta x \Delta y = \int_\Delta U,$$

because of the conservation form of $G$. Thus, (3.2) is verified in this case. Moreover, Lemma 3.2(a) is the same as Proposition 3.1(a) in this application. Thus, Proposition 3.1(c) holds by the equivalence of Lemma 3.2(a) and (c).

*Remark* 3.3. In fact, Proposition 3.1(c) and (d) hold even if $U$, $V \notin L^1(\Delta)$, as can easily be shown.

Next we examine the continuity properties in time of the function constructed as follows:   Let $U^0 \in L^1(\Delta)$ be given with $a \leqslant U^0_{j,k} \leqslant b$.  Set

$$(3.3) \quad U^{n+1} = \vec{G}(U^n) = \vec{G}^n(U^0) \quad \text{and} \quad u^{\Delta t} = \sum_{n=0}^{\infty} U^n_{\mathbf{R}^2} \chi^n = \sum_{n=0}^{\infty} \sum_{j,k} U^n_{j,k} \chi^n_{j,k},$$

where $\chi^n$ is the characteristic function of $[n\Delta t, (n+1)\Delta t)$ and $\chi^n_{j,k}$ is the characteristic function of $R_{j,k} \times [n\Delta t, (n+1)\Delta t)$.

PROPOSITION 3.4.   *Let the assumptions of Proposition 3.1 hold.  Let* $u^{\Delta t}$ *be given by* (3.3) *and* $0 \leqslant t_1 \leqslant t_2$.  *Then*

$$\int_{\mathbf{R}^2} |u^{\Delta t}(x, y, t_2) - u^{\Delta t}(x, y, t_1)| \, dxdy$$

$$\leqslant \left( \frac{|t_2 - t_1|}{\Delta t} + 1 \right) \| \vec{G}(U^0) - U^0 \|_{L^1(\Delta)}.$$

*Proof.*  Let $n\Delta t \leqslant t_2 < (n+1)\Delta t$, $m\Delta t \leqslant t_1 < (m+1)\Delta t$ so that $u(\cdot, \cdot, t_2) = U^n_{\mathbf{R}^2}$, $u(\cdot, \cdot, t_1) = U^m_{\mathbf{R}^2}$.  We will show that

$$\| U^n_{\mathbf{R}^2} - U^m_{\mathbf{R}^2} \|_{L^1(\mathbf{R}^2)} \leqslant (n - m) \| \vec{G}(U^0) - U^0 \|_{L^1(\Delta)},$$

from which (3.12) follows since $(n - m)\Delta t \leqslant t_2 - t_1 + \Delta t$ by the choice of $n$ and $m$. Using Proposition 3.1(b), $a \leqslant U^n_{j,k} \leqslant b$ for $n = 0, 1, \ldots$ and all $j, k$.  Then (2.14) and Proposition 3.1(c) yield

$$\| U^n_{\mathbf{R}^2} - U^m_{\mathbf{R}^2} \|_{L^1(\mathbf{R}^2)} = \| \vec{G}^n(U^0) - \vec{G}^m(U^0) \|_{L^1(\Delta)}$$

$$= \left\| \sum_{i=0}^{n-m-1} (\vec{G}^{n-i}(U^0) - \vec{G}^{n-(i+1)}(U^0)) \right\|_{L^1(\Delta)}$$

$$\leqslant \sum_{i=0}^{n-m-1} \| \vec{G}^{n-i}(U^0) - \vec{G}^{n-(i+1)}(U^0) \|_{L^1(\Delta)}$$

$$\leqslant \sum_{i=0}^{n-m-1} \| \vec{G}(U^0) - U^0 \|_{L^1(\Delta)} = (n-m) \| \vec{G}(U^0) - U^0 \|_{L^1(\Delta)},$$

which gives the desired result.

We next need to estimate the quantity $\| \vec{G}(U^0) - U^0 \|_{L^1(\Delta)}$.

PROPOSITION 3.5.   *Let G be given by* (0.4) *where the fluxes* $g_1, g_2$ *are Lipschitz continuous.  Then*

$$\frac{1}{\Delta t} \| \vec{G}(U) - U \|_{L^1(\Delta)} \leqslant L(p + q + 2)(r + s + 2) \| U \|_{BV(\Delta)},$$

*where L is a Lipschitz constant for* $g_1, g_2$.

*Proof.* By (0.4)

$$\|\vec{G}(U) - U\|_{L^1(\Delta)} \leqslant \sum_{j,k} (\lambda^x |g_1(U_{j-p+1,k-r}, \ldots, U_{j+q+1,k+s+1})$$

$$- g_1(U_{j-p,k-r}, \ldots, U_{j+q,k+s+1})|$$

$$+ \lambda^y |g_2(U_{j-p,k-r+1}, \ldots, U_{j+q+1,k+s+1})$$

$$- g_2(U_{j-p,k-r}, \ldots, U_{j+q+1,k+s})|)\Delta x \Delta y$$

$$\leqslant \sum_{j,k} \left( \frac{\Delta t}{\Delta x} L \sum_{l=j-p}^{j+q} \sum_{m=k-r}^{k+s+1} |U_{l+1,m} - U_{l,m}| \right.$$

$$\left. + \frac{\Delta t}{\Delta y} L \sum_{l=j-p}^{j+q+1} \sum_{m=k-r}^{k+s} |U_{l,m+1} - U_{l,m}| \right) \Delta x \Delta y$$

$$\leqslant (\Delta t) L (p+q+2)(r+s+2) \left( \frac{1}{\Delta x} \|\Delta_+^x U\|_{L^1(\Delta)} + \frac{1}{\Delta y} \|\Delta_+^y U\|_{L^1(\Delta)} \right)$$

$$= (\Delta t) L (p+q+2)(r+s+2) \|U\|_{BV(\Delta)},$$

whence the result.

The following corollary of these estimates will be what is eventually used.

COROLLARY 3.6. *Let the assumptions of Propositions 3.4 and 3.5 hold. Let* $u_0 \in L^1(\mathbf{R}^2)$, $a \leqslant u_0 \leqslant b$ *a.e., and* $u^{\Delta t}$ *be given by* (3.3), *where* $U^0 = u_{0\Delta}$; *see* (2.13). *Then there is a constant* $C$ *independent of* $u_0$, $\Delta t$ *such that for* $0 \leqslant t_1, t_2$

$$\int_{\mathbf{R}^2} |u^{\Delta t}(x, y, t_2) - u^{\Delta t}(x, y, t_1)| \, dx \, dy \leqslant C(|t_1 - t_2| + \Delta t)\|u_0\|_{BV(\mathbf{R}^2)}.$$

*Proof.* This is immediate from Propositions 3.4, 3.5 and (2.17).

*Proof of Lemma 3.2.* First, we prove that (a) implies (b). Let $f, g \in C$. Then $f \vee g = g + (f - g)^+ \in C$ by assumption, and $T(f \vee g) - T(g) \geqslant 0$ since $T$ is order preserving. Also $T(f) - T(g) \leqslant T(f \vee g) - T(g)$, and so $(T(f) - T(g))^+ \leqslant T(f \vee g) - T(g)$. Thus

$$\int_\Omega (T(f) - T(g))^+ \leqslant \int_\Omega (T(f \vee g) - T(g)) = \int_\Omega (f \vee g - g) = \int_\Omega (f - g)^+,$$

and (b) is established. That (b) implies (c) is trivial. Indeed, if (b) holds,

$$\int_\Omega |T(f) - T(g)| = \int_\Omega (T(f) - T(g))^+ + \int_\Omega (T(g) - T(f))^+$$

$$\leqslant \int_\Omega (f - g)^+ + \int_\Omega (g - f)^+ = \int_\Omega |f - g|.$$

Finally, if $f, g \in C$, $f \geqslant g$ and (c) holds, $2s^+ = |s| + s$ implies

$$2 \int_\Omega (T(g) - T(f))^+ = \int_\Omega |T(g) - T(f)| - \int_\Omega (T(g) - T(f))$$

$$\leqslant \int_\Omega |g - f| - \int_\Omega (g - f) = 0.$$

We have proved Lemma 3.2 here for completeness. The parallel result for $L^\infty$ and extensions are discussed in [5]. It is recognized that there will be results analogous to those of this paper for, in particular, equations of the form $u_t - \Delta\varphi(u) = 0$ and $u_t + f(\mathrm{grad}\ u) = 0$; and, time permitting, these will be developed.

**4. The Entropy Condition.** In this section we establish that if $u^{\Delta t}$ is an approximation of a solution of (0.1) produced by a monotone conservation-form difference scheme via the prescription (3.3); and there is a sequence $\Delta t_l \longrightarrow 0$ for which $u^{\Delta t_l}$ converges to a limit $u$ boundedly a.e., then $u$ is an entropy solution of (0.1). In the next section we show that every sequence $\Delta t_l$ convergent to zero has a subsequence with the above property, and then deduce that $u^{\Delta t}$ converges as $\Delta t \longrightarrow 0$ from the uniqueness Theorem 2.1.

If $\Delta t_l$ is given, it determines the lattices $\{(j\Delta x, k\Delta y)\} = \{(j/\lambda^x, k/\lambda^y)\Delta t_l\}$, $\{(j\Delta x, k\Delta y, n\Delta t_l)\} = \{(j/\lambda_x, k/\lambda^y, n)\Delta t_l\}$ and the associated partitions $\{R_{j,k}\}$, $\{R_{j,k} \times [n\Delta t_l, (n+1)\Delta t_l]\}$ of $\mathbf{R}^2$ and $\mathbf{R}^2 \times [0, \infty)$. These *depend on l*, but this will not be explicitly indicated by our notation. The ratios $\lambda^x, \lambda^y$ will be held constant (so $\Delta x, \Delta y$ depend on $l$). As $l$ varies, so too will the initial data involved in computing $u^{\Delta t_l}$; but this will not be indicated either.

PROPOSITION 4.1. *Let* (3.1) *be a conservation-form difference approximation consistent with* (0.1) *which is monotone on* $[a, b]$ *and has continuous numerical fluxes* $g_1, g_2$. *Suppose a sequence* $\Delta t_l$ *of positive numbers convergent to* 0 *is given. Let* $u^{\Delta t_l}$ *be computed by* (3.3), $a \leqslant U_{j,k}^0 \leqslant b$, *and* $u^{\Delta t_l} \longrightarrow u$ *boundedly a.e. on* $\mathbf{R}^2 \times [0, T]$. *Then,* $u$ *is an entropy solution of* $u_t + f_1(u)_x + f_2(u)_y = 0$ *on* $\mathbf{R}^2 \times [0, T]$.

Lax and Wendroff [15] observed that under the above assumptions $u$ will be a weak solution of the equation (even if $G$ is not monotone). We require the following simple lemma which is at the crux of their argument:

LEMMA 4.2. *Let* $\Delta t_l \longrightarrow 0$ *and*

$$\lim_{l \to \infty} \sum_{n=0}^{\infty} \sum_{j,k} V_{j,k}^n \chi_{j,k}^n = v$$

*boundedly a.e. on* $\mathbf{R}^2 \times [0, \infty)$. *Let*

$$\varphi \in C_0^1(\mathbf{R}^2 \times (0, \infty)) \quad and \quad \varphi_{j,k}^n = \varphi(j\Delta x, k\Delta y, n\Delta t_l).$$

*Then*

$$\lim_{l \to \infty} \sum_{n=0}^{\infty} \sum_{j,k} \varphi_{j,k}^n (\Delta_+^z V)_{j,k}^n \Delta x \Delta y \Delta t_l = -\int_0^\infty \int_{\mathbf{R}^2} \left(\frac{\partial\varphi}{\partial z}\right) v\, dx\, dy\, dt$$

*for* $z = x, y, t$.

The proof is elementary and is omitted. (While this lemma is not formulated in [15], the arguments there suffice for the proof.)

*Proof of Proposition* 4.1. Our proof follows the strategy of the one given by Harten, Hyman and Lax [10] for $N = 1$. Given $c \in \mathbf{R}$, we will produce continuous numerical entropy fluxes $h_1(\alpha_{-p,-r}, \ldots, \alpha_{q,s+1})$, $h_2(\alpha_{-p,-r}, \ldots, \alpha_{q+1,s})$ such that $h_i(u, \ldots, u) = \mathrm{sgn}(u - c)(f_i(u) - f_i(c))$ and

$$\Delta_+^t \,|U_{j,k}^n - c| + \lambda^x \Delta_+^x h_1(U_{j-p,k-r}^n, \ldots, U_{j+q,k+s+1}^n)$$

(4.2)

$$+ \lambda^y \Delta_+^y h_2(U_{j-p,k-r}^n, \ldots, U_{j+q+1,k+s}^n) \leqslant 0.$$

Once this is done, Proposition 4.1 follows. Indeed, multiply (4.2) by $\Delta x \Delta y \Delta t_l \varphi_{j,k}^n \geqslant 0$ and sum. Setting, respectively, $V_{j,k}^n = |U_{j,k}^n - c|$, $h_1(U_{j-p,k-r}^n, \ldots, U_{j+q,k+s+1}^n)$, $h_2(U_{j-p,k-r}^n, \ldots, U_{j+q+1,k+s}^n)$ we can take respectively, $v = |u - c|$, $\operatorname{sgn}(u - c)(f_1(u) - c)$, $\operatorname{sgn}(u - c)(f_2(u) - c)$ in Lemma 4.1. Letting $l \rightarrow \infty$ and using Lemma 4.1 yields

$$-\int_0^\infty \int_{\mathbf{R}^2} \varphi_t |u - c| + \varphi_x \operatorname{sgn}(u - c)(f_1(u) - f_1(c))$$

$$+ \varphi_y \operatorname{sgn}(u - c)(f_2(u) - f_2(c)) \, dx \, dy \, dt \leqslant 0,$$

which is the entropy condition (2.2).

It remains to produce $h_1$, $h_2$. Here we improve on [10] in simplicity and generality. With $z \vee w = \max(z, w)$, $z \wedge w = \min(z, w)$ we set

(4.3)
$$\begin{cases} h_1(U_{j-p,k-r}, \ldots, U_{j+q,k+s+1}) = g_1(c \vee U_{j-p,k-r}, \ldots, c \vee U_{j+q,k+s+1}) \\ \quad - g_1(c \wedge U_{j-p,k-r}, \ldots, c \wedge U_{j+q,k+s+1}), \\ h_2(U_{j-p,k-r}, \ldots, U_{j+q+1,k+s}) = g_2(c \vee U_{j-p,k-r}, \ldots, c \vee U_{j+q+1,k+s}) \\ \quad - g_2(c \wedge U_{j-p,k-r}, \ldots, c \wedge U_{j+q+1,k+s}). \end{cases}$$

A direct calculation using only the definitions yields the identity

$$|U_{j,k}^n - c| - \lambda^x \Delta_+^x h_1(U_{j-p,k-r}^n, \ldots) - \lambda^y \Delta_+^y h_2(U_{j-p,k-r}^n, \ldots)$$

(4.4)

$$= G(c \vee U_{j-p,k-r}^n, \ldots, c \vee U_{j+q+1,k+s+1}^n)$$

$$- G(c \wedge U_{j-p,k-r}^n, \ldots, c \wedge U_{j+q+1,k+s+1}^n).$$

We also have the relations

(4.5)
$$\begin{cases} U_{j,k}^{n+1} = G(U_{j-p,k-r}^n, \ldots, U_{j+q+1,k+s+1}^n), \\ c = G(c, c, \ldots, c). \end{cases}$$

From (4.5) and the monotonicity of $G$ we find

(4.6)
$$\begin{cases} c \vee U_{j,k}^{n+1} \leqslant G(c \vee U_{j-p,k-r}^n, \ldots, c \vee U_{j+q+1,k+s+1}^n), \\ -(c \wedge U_{j,k}^{n+1}) \leqslant -G(c \wedge U_{j-p,k-r}^n, \ldots, c \wedge U_{j+q+1,k+s+1}^n). \end{cases}$$

Adding the inequalities (4.6) and entering the result in (4.4) yields (4.2). This completes the proof.

**5. The Proof of Theorem 1.** We will actually prove more than Theorem 1 states here, for we will not need to *assume* the existence of the solution $S(t)u_0$ of (0.1). This will follow from our proofs. To begin, we assume that $a \leq u_0 \leq b$ a.e., $u_0 \in BV(\mathbf{R}^2)$ and $u_0$ has compact support. Let $u^{\Delta t}$ be given by (3.3) with $U^0 = u_{0\Delta}$; see (2.13).

From the stability estimates of Proposition 3.1 and (2.14)–(2.17) we deduce

$$(5.1) \qquad \begin{cases} \|u^{\Delta t}(\cdot, \cdot, t)\|_Z \leq \|u_0\|_Z \quad \text{for } Z = L^1(\mathbf{R}^2), BV(\mathbf{R}^2), \text{ and} \\ a \leq u^{\Delta t} \leq b \text{ everywhere.} \end{cases}$$

Also, because $\lambda^x, \lambda^y$ are fixed, it is easy to see that there is a $c_0 > 0$ such that if $u_0$ vanishes for $|x| + |y| \geq R$, then

$$(5.2) \qquad u^{\Delta t}(x, y, t) = 0 \quad \text{for } |x| + |y| \geq R + c_0 + c_0 t, 0 \leq \Delta t \leq 1.$$

Because bounded subsets of $BV(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ are precompact in $L^1_{\text{loc}}(\mathbf{R}^2)$ (see (2.9)), (5.1) and (5.2) imply that if $T > 0$, then

$$(5.3) \qquad \{u^{\Delta t}(\cdot, \cdot, t): 0 \leq t \leq T, 0 \leq \Delta t \leq 1\} \text{ is precompact in } L^1(\mathbf{R}^2).$$

Next the estimate of Corollary 3.5 supplies us with

$$(5.4) \quad \|u^{\Delta t}(\cdot, \cdot, t_1) - u^{\Delta t}(\cdot, \cdot, t_2)\|_{L^1(\mathbf{R}^2)} \leq c(|t_1 - t_2| + 2\Delta t)\|u_0\|_{BV(\mathbf{R}^2)}.$$

By the proof of Arzela-Ascoli's theorem, (5.3) and (5.4) guarantee that if $\Delta t_l \to 0$, then there is a subsequence $\Delta t_{l(k)}$ and a function $u : [0, \infty) \to L^1(\mathbf{R}^2)$ such that

$$(5.5) \qquad \lim_{k \to \infty} \max_{0 \leq t \leq T} \|u^{\Delta t_{l(k)}}(\cdot, \cdot, t) - u(\cdot, \cdot, t)\|_{L^1(\mathbf{R}^2)} = 0 \quad \text{for } T > 0.$$

From (5.1) and (5.4) we deduce $a \leq u \leq b$ a.e. and

$$(5.6) \qquad \|u(\cdot, \cdot, t_1) - u(\cdot, \cdot, t_2)\|_{L^1(\mathbf{R}^2)} \leq c|t_1 - t_2| \|u_0\|_{BV(\mathbf{R}^2)}.$$

Passing to a further subsequence if necessary, we can have $u^{\Delta t_{l(k)}} \to u$ boundedly a.e. By Proposition 4.1, $u$ is an entropy solution of the conservation law in (0.1). By (5.6), and $u(\cdot, \cdot, 0) = u_0$, $u$ assumes the initial value $u_0$ continuously in $L^1(\mathbf{R}^2)$. The uniqueness Theorem 2.1 guarantees then the uniqueness of $u$, and we conclude

$$(5.7) \qquad \lim_{\Delta t \to 0} \|u^{\Delta t}(\cdot, \cdot, t) - u(\cdot, \cdot, t)\|_{L^1(\mathbf{R}^2)} = 0 \text{ uniformly for bounded } t.$$

This completes the analysis when $u_0$ has compact support.

For general $u_0 \in L^1(\mathbf{R}^2)$ with $a \leq u_0 \leq b$ a.e. we choose $u_{0,m} \in BV(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ with compact support such that

$$(5.8) \qquad \begin{cases} a \leq u_{0,m} \leq b \quad \text{and} \\ \|u_0 - u_{0,m}\|_{L^1(\mathbf{R}^2)} \to 0 \quad \text{as } m \to \infty. \end{cases}$$

This can be done by any standard method. Denote the difference scheme solution corresponding to the initial-value $u_{0,m}$ by $u_m^{\Delta t}$. Below, we regard all functions as functions of $t$ with values in $L^1(\mathbf{R}^2)$. We know that

(5.9)                    $\lim_{\Delta t \to 0} u_m^{\Delta t}(t) = u_m(t)$ exists uniformly for bounded $t$.

Moreover, from Proposition 3.1(c) it follows that

$$\sup_{t \geqslant 0} \| u_m^{\Delta t}(t) - u_l^{\Delta t}(t) \|_{L^1(\mathbf{R}^2)} \leqslant \| u_{0,m} - u_{0,l} \|_{L^1(\mathbf{R}^2)},$$

and so $\{u_m\}$ is Cauchy in $C([0, \infty) : L^1(\mathbf{R}^2))$ and hence converges uniformly to a limit $u \in C([0, \infty) : L^1(\mathbf{R}^2))$. Finally,

$$\| u^{\Delta t}(t) - u(t) \|_{L^1(\mathbf{R}^2)}$$

$$\leqslant \| u^{\Delta t}(t) - u_m^{\Delta t}(t) \|_{L^1(\mathbf{R}^2)} + \| u_m^{\Delta t}(t) - u_m(t) \|_{L^1(\mathbf{R}^2)} + \| u_m(t) - u(t) \|_{L^1(\mathbf{R}^2)}$$

$$\leqslant \| u_0 - u_{0,m} \|_{L^1(\mathbf{R}^2)} + \| u_m^{\Delta t}(t) - u_m(t) \|_{L^1(\mathbf{R}^2)} + \| u_m(t) - u(t) \|_{L^1(\mathbf{R}^2)}.$$

The first and third terms above can be made small uniformly in $t$ by taking $m$ large. Then the middle term can be made small by taking $\Delta t$ small (uniformly for $t$ bounded). Hence $\lim_{\Delta t \to 0} u^{\Delta t} = u$ locally uniformly in $L^1(\mathbf{R}^2)$ (and $u$ is continuous into $L^1(\mathbf{R}^2)$ by its construction). This completes the proof.

A reexamination of the proof of Theorem 5.1 and the various ingredients of the preceding sections shows that we have in fact proved the following results on existence and properties of solutions of (0.1) (given the uniqueness Theorem 2.1).

COROLLARY 5.1. *Let the conservation law have Lipschitz continuous fluxes $f_i$. Then for every initial-data $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ there is a unique entropy solution $u \in C([0, \infty) : L^1(\mathbf{R}^N))$ of (0.1) with $u(0) = u_0$. Denoting this solution by $S(t)u_0$ we have:*

(a) *If $u_0 \in BV(\mathbf{R}^N)$, $t \to S(t)u_0$ is Lipschitz continuous into $L^1(\mathbf{R}^N)$ and $\| S(t)u_0 \|_{BV(\mathbf{R}^N)} \leqslant \| u_0 \|_{BV(\mathbf{R}^N)}$.*
(b) *$\| S(t)u_0 - S(t)v_0 \|_{L^1(\mathbf{R}^N)} \leqslant \| u_0 - v_0 \|_{L^1(\mathbf{R}^N)}$.*
(c) *$u_0 \leqslant v_0$ a.e. implies $S(t)u_0 \leqslant S(t)v_0$ a.e.*
(d) *$a \leqslant u_0 \leqslant b$ a.e. implies $a \leqslant S(t)u_0 \leqslant b$ a.e.*

All of the above is well known, even in much greater generality; see [13], in particular. Crandall [4] and Benilan [1] also treat cases in which the $f_i$ need not be Lipschitz. In fact, it is enough that the $f_i$ are continuous and

$$\lim_{r \to 0} |f_i(r) - f_i(0)| / |r|^{(N-1)/N} = 0,$$

in order that Theorem 2.1 hold with $u_{i0} \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$, provided $R = \infty$ in (2.3) and (2.4). In order to compute solutions of (0.1) in this non-Lipschitz case, one would have to approximate $(f_1, \ldots, f_N)$ by smoother functions $(f_1^l, \ldots, f_N^l)$, solve a difference approximation to the resulting problem with $\lambda_l^x$, $\lambda_l^y$ chosen appropriately, and then let $l \to \infty$, $\Delta t \to 0$. The modulus of continuity in time must be treated appropriately, but we will not consider this here.

**6. The Inhomogeneous Equation.** We briefly remark on how the analysis given above can be carried out for the more general problem

$$(6.1) \qquad \begin{cases} u_t + \displaystyle\sum_{i=1}^{N} f_i(u)_{x_i} = F(x,\, t), \\[2mm] u(x,\, 0) = u_0(x). \end{cases}$$

The corresponding difference schemes have the form (for $N = 2$)

$$(6.2) \qquad U_{j,k}^{n+1} = G(U_{j-p,\,k-r}^n,\, \ldots,\, U_{j+q+1,\,k+s+1}^n) + \Delta t F_{j,k}^n,$$

or

$$(6.3) \qquad U^{n+1} = \vec{G}(U^n) + \Delta t F^n,$$

where $G$ satisfies the conditions of the preceding sections.

THEOREM 6.1. *Let G define a conservation-form difference scheme consistent with (0.1), which is monotone on $[a,\, b]$ and has Lipschitz continuous fluxes. Let $T > 0$, $u_0 \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ and $F \in L^\infty(\mathbf{R}^2 \times (0,\, T)) \cap L^1(\mathbf{R}^2 \times (0,\, T))$ satisfy $a + T\|F\|_{L^\infty} \leqslant u_0 \leqslant b - \|F\|_{L^\infty}T$ a.e. Let $U^0 = u_{0\Delta}$ and*

$$F_{j,k}^n = \frac{1}{\Delta t \Delta x \Delta y} \int_{n\Delta t}^{(n+1)\Delta t} \int_{R_{j,k}} F(x,\, y,\, t)\, dx\, dy\, dt.$$

*Then $U^n$, defined by (6.3) satisfies $a \leqslant U_{j,k}^n \leqslant b$ for all $n$, $j$, $k$ such that $n\Delta t \leqslant T$ and*

$$u^{\Delta t} = \sum_{n=0}^{\infty} \sum_{j,k} U_{j,k}^n \chi_{j,k}^n$$

*converges in $L^1(\mathbf{R}^2)$ uniformly on compact subsets of $[0,\, T)$ to the unique entropy solution of (6.1).*

*Sketch of Proof.* In addition to the arguments in preceding sections we require a uniqueness theorem (see [13] for this) and a few estimates given now. From Proposition 3.1 we deduce immediately

$$(6.4) \qquad \begin{cases} \displaystyle\sup_{j,k} U_{j,k}^{n+1} \leqslant \sup_{j,k} U_{j,k}^n + \Delta t \sup_{j,k} F_{j,k}^n, \\[3mm] \displaystyle\inf_{j,k} U_{j,k}^{n+1} \geqslant \inf_{j,k} U_{j,k}^n + \Delta t \inf_{j,k} F_{j,k}^n \end{cases}$$

and

$$(6.5) \qquad \|U^{n+1}\|_X \leqslant \|U^n\|_X + \Delta t \|F^n\|_X$$

for $X = L^1(\Delta)$ and $X = BV(\Delta)$. Moreover, if $\hat{U}^{n+1} = G(\hat{U}^n) + \Delta t \hat{F}^n$, we find

$$(6.6) \qquad \|U^n - \hat{U}^n\|_{L^1(\Delta)} \leqslant \|U^0 - \hat{U}^0\|_{L^1(\Delta)} + \sum_{j=0}^{n-1} \Delta t \|F^j - \hat{F}^j\|_{L^1(\Delta)}.$$

Finally, the estimate of Proposition 3.4 is replaced (with $n\Delta t \leqslant t_2 < (n+1)\Delta t$, $m\Delta t \leqslant t_1 < (m+1)\Delta t$ and $m \leqslant n$ as before) by

$$\int_{\mathbf{R}^2} |u^{\Delta t}(x, y, t_2) - u^{\Delta t}(x, y, t_1)| dt$$

$$= \| U^n - U^m \|_{L^1(\Delta)} \leqslant \sum_{j=m}^{n-1} \| U^{j+1} - U^j \|_{L^1(\Delta)}$$

$$= \sum_{j=m}^{n-1} \| \vec{G}(U^j) + \Delta t F^j - U^j \|_{L^1(\Delta)}$$

$$\leqslant \sum_{j=m}^{n-1} \| \vec{G}(U^j) - U^j \|_{L^1(\Delta)} + \sum_{j=m}^{n-1} \Delta t \| F_j \|_{L^1(\Delta)}.$$

Now we use Proposition 3.5 and (6.5) to estimate the right-hand side above by

$$\Delta t(n - m) \text{ cons.} \left( \| U^0 \|_{BV(\Delta)} + \sum_{j=0}^{n-1} \Delta t \| F^j \|_{BV(\Delta)} \right) + \sum_{j=m}^{n-1} \Delta t \| F_j \|_{L^1(\Delta)}.$$

Thus, if

$$\int_0^T \| F(\cdot, \cdot, t) \|_{BV(\mathbf{R}^2)} dt < \infty,$$

we have the type of equicontinuity of $u^{\Delta t}$ as $\Delta t \to 0$ used in the proof of Theorem 1; and the proof may be completed much as before. (One begins, say, with $u_0 \in C_0^\infty(\mathbf{R}^2)$, $F \in C_0^\infty(\mathbf{R}^2 \times [0, T))$ and passes to the general case via (6.6).)

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