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Negative Norm Estimates and Superconvergence in Galerkin Methods for Parabolic Problems

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Abstract. Negative norm error estimates for semidiscrete Galerkin-finite element methods for parabolic problems are derived from known such estimates for elliptic problems and applied to prove superconvergence of certain procedures for evaluating point values of the exact solution and its derivatives.

Our first purpose in this paper is to show how known negative norm error estimates for Galerkin-finite element type methods applied to the Dirichlet problem for second order elliptic equations can be carried over to initial-boundary value problems for nonhomogeneous parabolic equations. We then want to describe how such estimates may be used to prove superconvergence of a number of procedures for evaluating point values of the exact solution and its derivatives. These applications include in particular the case of one space dimension with continuous, piecewise polynomial approximating subspaces, where we analyze methods proposed by Douglas, Dupont and Wheeler [3]. Further, in higher dimensions we discuss the application of an averaging procedure by Bramble and Schatz [1] for elements which are uniform in the interior and in the nonuniform case a method employing a local Green's function considered by Louis and Natterer [4].

The error analysis of this paper takes place in the general framework introduced in Bramble, Schatz, Thomée and Wahlbin [2] allowing approximating subspaces which do not necessarily satisfy the homogeneous boundary conditions of the exact solution. These subspaces are assumed to permit approximation to order $O(h^r)$ in L_2 $(r \geqslant 2)$ and to yield $O(h^{2r-2})$ error estimates for the elliptic problem in norms of order $-(r-2)$. The superconvergent order error estimates which we aim for in the parabolic problem are then of this higher order. In [2], estimates of the type considered here were obtained for homogeneous parabolic equations by spectral representation; our basic results in this paper are derived by the energy method.

1. Preliminaries. We shall be concerned with the approximate solution of the initial-boundary value problem ($u_t = \partial u/\partial t$, $R_+ = \{t; t \geqslant 0\}$)

$$(1.1)\qquad \begin{aligned} Lu \equiv u_t + Au &= f \quad && \text{in } \Omega \times R_+, \\ u(x, t) &= 0 && \text{on } \partial\Omega \times R_+, \\ u(x, 0) &= v(x) && \text{on } \Omega. \end{aligned}$$

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Here Ω is a bounded domain in R^N with sufficiently smooth boundary $\partial\Omega$,

$$Au = -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a_{jk}\frac{\partial u}{\partial x_k}\right) + a_0 u,$$

with a_{jk} and a_0 sufficiently smooth time-independent functions, the matrix (a_{jk}) symmetric and uniformly positive definite and a_0 nonnegative in $\overline{\Omega}$.

In order to introduce some notation, we consider first the corresponding elliptic problem

$$(1.2) \qquad Au = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

and denote by $T: L_2(\Omega) \longrightarrow H_0^1(\Omega) \cap H^2(\Omega)$ its solution operator, defined by $u = Tf$. Notice that by the symmetry of A, T is selfadjoint and positive definite in $L_2(\Omega)$. Recall also the elliptic regularity estimate

$$\|Tf\|_{s+2} \leqslant C\|f\|_s \quad \text{for } s \geqslant 0,$$

where $\|\cdot\|_s$ denotes the norm in $H^s(\Omega)$.

Set now for s a nonnegative integer and $v, w \in L_2(\Omega)$, with $(\cdot\,,\cdot)$ the inner product in $L_2(\Omega)$,

$$(1.3) \qquad (v, w)_{-s} = (T^s v, w), \qquad \|v\|_{-s} = (T^s v, v)^{1/2}.$$

Since T is positive definite, $(\cdot\,,\cdot)_{-s}$ is an inner product. One can show that $\|\cdot\|_{-s}$ is equivalent to the norm

$$\sup\left\{\frac{(v, \varphi)}{\|\varphi\|_s};\ \varphi \in \dot{H}^s(\Omega)\right\},$$

where

$$\dot{H}^s(\Omega) = \{\varphi \in H^s(\Omega);\ A^j\varphi = 0 \text{ on } \partial\Omega \text{ for } j < s/2\}.$$

In fact, with $\{\lambda_j\}_1^\infty$ and $\{\varphi_j\}_1^\infty$ the eigenvalues and orthonormal eigenfunctions of A (with Dirichlet boundary conditions) an equivalent norm to $\|\cdot\|_s$ on $\dot{H}^s(\Omega)$ is

$$\|v\|_{\dot{H}^s(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^s (v, \varphi_j)^2\right)^{1/2};$$

with this notation,

$$(v, w)_{-s} = \sum_{j=1}^{\infty} \lambda_j^{-s}(v, \varphi_j)(w, \varphi_j).$$

For the purpose of approximation, let $\{S_h\}$ denote a family of finite dimensional subspaces of $L_2(\Omega)$ depending on the "small" mesh parameter h, and let $\{T_h\}$ denote a corresponding family of approximate linear solution operators $T_h: L_2(\Omega) \longrightarrow S_h$ of (1.2). Following [2], we shall assume throughout below that $\{S_h\}$ and $\{T_h\}$ are tied together by the following two properties:

(i) T_h is selfadjoint, positive semidefinite on $L_2(\Omega)$ and positive definite on S_h.

(ii) There is an integer $r \geqslant 2$ such that

$$\|(T_h - T)f\|_{-p} \leqslant Ch^{p+q+2}\|f\|_q \quad \text{for } 0 \leqslant p,\, q \leqslant r-2,\, f \in H^q(\Omega).$$

One example of a family $\{T_h\}$ with the above properties is exhibited by the standard Galerkin method, where for each h, $S_h \subset H_0^1(\Omega)$, where $\{S_h\}$ satisfies the approximation property ($\|\cdot\| = \|\cdot\|_0$)

$$\inf_{\chi \in S_h} \{\|w - \chi\| + h\|w - \chi\|_1\} \leqslant Ch^s\|w\|_s, \qquad 1 \leqslant s \leqslant r,\ w \in H_0^1(\Omega) \cap H^s(\Omega),$$

and where T_h is defined by

(1.4) $$A(T_h f, \chi) = (f, \chi) \quad \text{for } \chi \in S_h,$$

with

$$A(v, w) = \int_\Omega \left(\sum_{j,k=1}^{N} a_{jk} \frac{\partial v}{\partial x_k} \frac{\partial w}{\partial x_j} + a_0 v w \right) dx.$$

The properties (i) and (ii) hold also in other instances, including situations where the bilinear form used in the definition of T_h contains boundary terms, added to deal with the difficulty of satisfying the homogeneous boundary conditions in S_h.

Introducing the elliptic projection $P_1 = T_h A$, the property (ii) reduces to the well-known error estimate

(1.5) $$\|(I - P_1)v\|_{-p} \leqslant Ch^{p+q}\|v\|_q \quad \text{for } 0 \leqslant p \leqslant r-2,\ 2 \leqslant q \leqslant r,\quad v \in H_0^1(\Omega) \cap H^q(\Omega),$$

valid, in fact, for the standard Galerkin method for $-1 \leqslant p \leqslant r-2$, $1 \leqslant q \leqslant r$. Notice also (cf. [2]) that for the orthogonal projection $P_0\colon L_2(\Omega) \longrightarrow S_h$ we have as a result of (ii),

$$\|(I - P_0)v\|_{-p} \leqslant Ch^{p+q}\|v\|_q \quad \text{for } 2 \leqslant p,\ q \leqslant r,\ v \in H_0^1(\Omega) \cap H^q(\Omega).$$

With the aid of the operator $T = A^{-1}$, the initial-boundary value problem (1.1) may be written

(1.6) $$Tu_t + u = Tf \quad \text{for } t \geqslant 0, \text{ with } u(0) = v.$$

We shall consider the following semidiscrete analogue, namely to find $u_h\colon R_+ \longrightarrow S_h$ such that

(1.7) $$T_h u_{h,t} + u_h = T_h f \quad \text{for } t \geqslant 0, \text{ with } u_h(0) = v_h,$$

where v_h is some approximation to v. Notice that since T_h is positive definite on S_h, this defines u_h for $t \geqslant 0$. When $S_h \subset H_0^1(\Omega)$ and T_h is defined by (1.4), the problem (1.7) is equivalent to the standard Galerkin problem

(1.8) $$(u_{h,t}, \chi) + A(u_h, \chi) = (f, \chi) \quad \text{for } \chi \in S_h,\ t \geqslant 0, \text{ with } u_h(0) = v_h.$$

In the case of a homogeneous parabolic equation ($f = 0$), error estimates for the semidiscrete problem (1.7) were derived in [2]. It was shown in particular, using spectral representation, that with $v_h = P_0 v$ the L_2-projection of v, we have for the error $e = u_h - u$, with $D_t = \partial/\partial t$,

$$(1.9)\qquad \|D_t^j e(t)\|_{-p} \leqslant Ch^{p+q+2} t^{-1-j} \|v\|_q \quad \text{for } 0 \leqslant p,\ q \leqslant r-2,\ v \in \dot{H}^q(\Omega).$$

By an iteration argument this also implied the L_2-estimate

$$\|D_t^j e(t)\| \leqslant Ch^r t^{-r/2-j} \|v\|.$$

Our first purpose here is to derive negative norm estimates for the nonhomogeneous problem, valid uniformly for small t. This will be done by the energy method. In order to do so we introduce the discrete analogues of the inner product and norm in (1.3),

$$(v, w)_{-s,h} = (T_h^s v, w), \qquad \|v\|_{-s,h} = (T_h^s v, v)^{1/2}.$$

Since T_h is semidefinite on $L_2(\Omega)$ these are a semi-inner product and a seminorm, respectively. In the following lemma we shall relate these discrete seminorms to the negative norms previously defined.

LEMMA 1. *Under the above assumptions about* $\{T_h\}$, *we have for* $0 \leqslant p \leqslant r$, $v \in L_2(\Omega)$,

$$(1.10)\qquad \|v\|_{-p,h} \leqslant C\{\|v\|_{-p} + h^p \|v\|\},$$

$$(1.11)\qquad \|v\|_{-p} \leqslant C\{\|v\|_{-p,h} + h^p \|v\|\}.$$

Proof. We first prove (1.10) by induction over p. The result is trivial for $p = 0$ and also clear for $p = 1$, since

$$\|v\|_{-1,h}^2 = (T_h v, v) = (Tv, v) + ((T_h - T)v, v) \leqslant \|v\|_{-1}^2 + Ch^2 \|v\|^2,$$

by (ii). Let now $p \geqslant 1$ and assume that (1.10) is proved up to p. We have

$$\|v\|_{-(p+1),h} = \|T_h v\|_{-(p-1),h} \leqslant \|Tv\|_{-(p-1),h} + \|(T_h - T)v\|_{-(p-1),h}.$$

By the induction assumption,

$$\|Tv\|_{-(p-1),h} \leqslant C\{\|Tv\|_{-(p-1)} + h^{p-1}\|Tv\|\} = C\{\|v\|_{-(p+1)} + h^{p-1}\|v\|_{-2}\}.$$

Using, for instance, spectral representations, we have easily

$$\|v\|_{-2} \leqslant C\{h^2 \|v\| + h^{-(p-1)} \|v\|_{-(p+1)}\},$$

so that

$$\|Tv\|_{-(p-1),h} \leqslant C\{\|v\|_{-(p+1)} + h^{p+1}\|v\|\}.$$

Further, by the induction assumption and (ii) with $q = 0$,

$$\|(T_h - T)v\|_{-(p-1),h} \leqslant C\{\|(T_h - T)v\|_{-(p-1)} + h^{p-1}\|(T_h - T)v\|\} \leqslant Ch^{p+1}\|v\|.$$

This proves (1.10).

By interchanging the roles of T and T_h, (1.11) follows analogously. This completes the proof of the lemma.

Notice that since T_h is positive definite on S_h, $\|\cdot\|_{-s,h}$ is a norm on S_h and we may then also use

$$\|\chi\|_{1,h} = (T_h^{-1}\chi, \chi)^{1/2} \quad \text{for } \chi \in S_h.$$

When T_h is defined by the standard Galerkin equation (1.4), we have

$$\|\chi\|_{1,h} = A(\chi, \chi)^{1/2} \quad \text{for } \chi \in S_h. \tag{1.12}$$

2. **Error Estimates.** Consider now the initial-value problem (1.1) or (1.6) and its semidiscrete analogue (1.7). By subtraction we find immediately that the error $e = u_h - u$ satisfies the equation

$$T_h e_t + e = \rho \equiv (T_h - T)Au = (P_1 - I)u. \tag{2.1}$$

Our basic negative norm estimate is then based on the following lemma.

LEMMA 2. *Under the above assumptions about T_h, let e satisfy* (2.1). *Then for any $s \geqslant 0$,*

$$\|e(t)\|_{-s,h} \leqslant C\left\{\|e(0)\|_{-s,h} + \|\rho(0)\|_{-s,h} + \int_0^t \|\rho_t(\tau)\|_{-s,h}\, d\tau\right\}.$$

Proof. Let temporarily $(\cdot, \cdot)$ denote any semi-inner product for which T_h is selfadjoint, nonnegative and let $\|\cdot\|$ be the corresponding seminorm. We have after multiplication of the error equation by e_t,

$$(T_h e_t, e_t) + \frac{1}{2}\frac{d}{dt}\|e\|^2 = (\rho, e_t) = \frac{d}{dt}(\rho, e) - (\rho_t, e),$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 \leqslant \frac{d}{dt}(\rho, e) + \|\rho_t\| \cdot \|e\|.$$

It follows by integration

$$\begin{aligned}\|e(t)\|^2 &\leqslant C\left\{\|e(0)\|^2 + \|\rho(t)\|\,\|e(t)\| + \|\rho(0)\|\,\|e(0)\| + \int_0^t \|\rho_t\|\,\|e\|\, d\tau\right\}\\ &\leqslant C \sup_{0\leqslant\tau\leqslant t} \|e(\tau)\| \left\{\|e(0)\| + \sup_{0\leqslant\tau\leqslant t}\|\rho(\tau)\| + \int_0^t \|\rho_t\|\, d\tau\right\}.\end{aligned}$$

Now choose $\bar{t} \in [0, t]$ such that

$$\|e(\bar{t})\| = \sup_{0\leqslant\tau\leqslant t} \|e(\tau)\|.$$

We then have

$$\|e(\bar{t})\|^2 \leqslant C\|e(\bar{t})\| \left\{\|e(0)\| + \sup_{0\leqslant\tau\leqslant \bar{t}}\|\rho(\tau)\| + \int_0^{\bar{t}} \|\rho_t\| d\tau\right\},$$

and hence

$$\|e(t)\| \leqslant \|e(\bar{t})\| \leqslant C\left\{\|e(0)\| + \sup_{0\leqslant\tau\leqslant t} \|\rho(\tau)\| + \int_0^t \|\rho_t\|\,d\tau\right\}.$$

Since

$$\sup_{0\leqslant\tau\leqslant t} \|\rho(\tau)\| \leqslant \|\rho(0)\| + \int_0^t \|\rho_t\|\,d\tau,$$

the result now follows by application to $(\cdot,\cdot) = (\cdot,\cdot)_{-s,h}$.

We are now ready to state and prove our basic negative norm error estimate. Here and below we let our regularity assumptions be implicitly defined, unless explicitly stated, by the norms appearing on the right in the error estimates, recalling always that the solution vanishes on $\partial\Omega$ so that for instance the appearance of $\|v\|_q$ for $q \geqslant 1$ means that $v \in H_0^1(\Omega) \cap H^q(\Omega)$.

THEOREM 1. *Under the above assumptions we have for* $0 \leqslant p \leqslant r-2$, $2 \leqslant q \leqslant r$,

$$\|e(t)\|_{-p} \leqslant C\left\{\|e(0)\|_{-p,h} + h^{p+q}\left[\|v\|_q + \int_0^t \|u_t\|_q\,d\tau\right]\right\}.$$

Proof. In view of Lemma 1 and (1.5) we have

$$\|\rho(0)\|_{-p,h} \leqslant C\{\|(I-P_1)v\|_{-p} + h^p\|(I-P_1)v\|\} \leqslant Ch^{p+q}\|v\|_q,$$

and similarly

$$\|\rho_t\|_{-p,h} \leqslant Ch^{p+q}\|u_t\|_q.$$

The result hence follows by Lemma 2.

Notice in particular, with for instance $v_h = P_0 v$ or $v_h = P_1 v$,

$$\|e(t)\|_{-p} \leqslant Ch^{r+p}\left\{\|v\|_r + \int_0^t \|u_t\|_r\,d\tau\right\}.$$

For the homogeneous equation we have then

$$\|u_t(\tau)\|_r \leqslant C\|u(\tau)\|_{r+2} \leqslant C\tau^{-(1-\epsilon/2)}\|v\|_{r+\epsilon} \quad \text{if } v \in \dot{H}^{r+\epsilon}(\Omega),$$

and hence for any $\epsilon > 0$,

$$\|e(t)\|_{-p} \leqslant Ch^{r+p}\|v\|_{r+\epsilon} \quad \text{if } v \in \dot{H}^{r+\epsilon}(\Omega).$$

Generalizing the argument in [2] for $p = 0$, one easily shows the somewhat more precise estimate

$$\|e(t)\|_{-p} \leqslant Ch^{r+p}\|v\|_r \quad \text{for } v \in \dot{H}^r(\Omega),\ 0 \leqslant p \leqslant r-2.$$

The above theorem is complemented by the following well-known H^1 error estimate for the standard Galerkin method (cf. e.g. [6]).

THEOREM 2. *Consider the standard Galerkin method* (1.8), *and let* $2 \leqslant q \leqslant r$. *Then*

$$\|e(t)\|_1 \leqslant C\left\{\|e(0)\|_1 + h^{q-1}\left[\sup_{\tau\leqslant t}\|u(\tau)\|_q + \left(\int_0^t \|u_t\|_{q-1}^2\, d\tau\right)^{1/2}\right]\right\}.$$

Proof. Setting $\theta = u_h - P_1 u$, we have now

$$(\theta_t, \chi) + A(\theta, \chi) = -(\rho_t, \chi) \quad \text{for } \chi \in S_h,$$

and hence with $\chi = \theta_t$,

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}A(\theta, \theta) = -(\rho_t, \theta_t) \leqslant \frac{1}{2}\|\theta_t\|^2 + \frac{1}{2}\|\rho_t\|^2.$$

By integration this yields

$$A(\theta, \theta)(t) \leqslant A(\theta, \theta)(0) + \int_0^t \|\rho_t\|^2\, d\tau,$$

or since $\theta = e - \rho$,

$$\|e(t)\|_1^2 \leqslant C\left\{\|e(0)\|_1^2 + \sup_{\tau\leqslant t}\|\rho(\tau)\|_1^2 + \int_0^t \|\rho_t\|^2\, d\tau\right\}.$$

Recalling that (cf. (1.5))

$$\|(I - P_1)v\|_j \leqslant Ch^{q-1}\|v\|_{q-1+j}, \qquad j = 0, 1, \text{ for } 2 \leqslant q \leqslant r,$$

the result obtains.

Our applications will require estimates for time derivatives of the error for positive time. We start with a lemma on the homogeneous semidiscrete equation.

LEMMA 3. *Let* $u_h(t) \in S_h$ *be the solution of*

$$T_h u_{h,t} + u_h = 0 \quad \textit{for } t \geqslant 0,\ u_h(0) = v_h,$$

and let $-1 \leqslant p \leqslant s$, $j \geqslant 0$. *Then*

$$\|D_t^j u_h(t)\|_{-p,h} \leqslant Ct^{-(j+\frac{1}{2}(s-p))}\|v_h\|_{-s,h} \quad \textit{for } t > 0.$$

Proof. By the definition of the discrete negative norm, we have with $T_h^{-1} = A_h$,

$$\|\chi\|_{-p,h} = (T_h^p\chi, \chi)^{1/2} = (T_h^s A_h^{\frac{1}{2}(s-p)}\chi, A_h^{\frac{1}{2}(s-p)}\chi)^{1/2} = \|A_h^{\frac{1}{2}(s-p)}\chi\|_{-s,h}.$$

Since

$$u_h(t) = e^{-A_h t}v_h,$$

we have

$$D_t^j u_h(t) = (-A_h)^j e^{-A_h t}v_h.$$

The result is, therefore, an immediate consequence of the uniform boundedness for $t \geqslant 0$ of the operator $(A_h t)^l e^{-A_h t}$ ($l = j + (s - p)/2$) with respect to the norm $\|\cdot\|_{-s,h}$ which in turn follows at once by the boundedness of $\omega^l e^{-\omega}$ for $\omega \geqslant 0$.

We are now ready to prove the negative norm error estimate for time derivatives at positive time.

THEOREM 3. *Let $j \geqslant 0$, $s \geqslant 0$ and $0 \leqslant p \leqslant r-2$. We then have for $t > \delta > 0$,*

$$\|D_t^j e(t)\|_{-p} \leqslant C\|v_h - P_0 v\|_{-s,h} + Ch^{r+p}\left\{\sum_{l=0}^{j} \|D_t^l u(t)\|_r + \int_{t-\delta}^{t} \|D_t^{j+1} u\|_r\,d\tau + \int_0^t \|u_t\|_{p+2}\,d\tau\right\}.$$

Proof. Consider a fixed $t = t_1 > \delta$. Let $\varphi \in C^\infty$ be such that $\varphi(t) = 1$ for $t > -\delta/2$, $\varphi(t) = 0$ for $t < -\delta$. Set $\varphi_1(t) = \varphi(t - t_1)$, and write $u = u_1 + u_2$ where $u_1 = u\varphi_1$, $u_2 = u(1-\varphi_1)$. Recalling that $L = D_t + A$, we have

$$Lu_1 = f_1 \equiv f\varphi_1 + u\varphi_1' \qquad \text{for } t \geqslant 0,\ u_1(0) = 0,$$

$$Lu_2 = f_2 \equiv f(1-\varphi_1) - u\varphi_1' \quad \text{for } t \geqslant 0,\ u_2(0) = v.$$

Let now $u_{1,h}$ and $u_{2,h}$ be the solutions of the corresponding semidiscrete problems with $u_{1,h}(0) = 0$, $u_{2,h}(0) = P_0 v$, and let $u_{3,h}$ be the solution of the homogeneous semidiscrete equation with $u_{3,h}(0) = v_h - P_0 v$, so that $u_h = u_{1,h} + u_{2,h} + u_{3,h}$. With $e_i = u_{i,h} - u_i$ for $i = 1, 2$ we then have $e = e_1 + e_2 + u_{3,h}$.

In order to estimate $D_t^j e_1$ we notice that $D_t^j u_1$ satisfies

$$L(D_t^j u_1) = D_t^j f_1 \quad \text{for } t \geqslant 0,\ D_t^j u_1(0) = 0,$$

and that $D_t^j u_{1,h}$ is the solution of the corresponding semidiscrete problem with $D_t^j u_{1,h}(0) = 0$. Hence, by Theorem 1

$$\|D_t^j e_1(t_1)\|_{-p} \leqslant Ch^{r+p}\int_0^{t_1} \|D_t^{j+1} u_1\|_r\,d\tau \leqslant Ch^{r+p}\left\{\sum_{l=0}^{j} \|D_t^l u(t_1)\|_r + \int_{t_1-\delta}^{t_1} \|D_t^{j+1} u\|_r\,d\tau\right\}.$$

On the other hand, since $e_2(t) = u_{2,h}(t)$ and $f_2(t) = 0$ for $t > t_1 - \delta/2$, we have by Lemmas 1 and 3,

$$\|D_t^j e_2(t_1)\|_{-p} \leqslant C\{\|D_t^j e_2(t_1)\|_{-p,h} + h^p\|D_t^j e_2(t_1)\|\} \leqslant C\|e_2(t_1 - \delta/2)\|_{-(r-2),h};$$

and hence, by Theorem 1 (with $p = r-2$, $q = p+2$) since $u_{2,h}(0) = P_0 u_2(0)$,

$$\|D_t^j e_2(t_1)\|_{-p} \leqslant Ch^{r+p}\left\{\|v\|_{p+2} + \int_0^{t_1} \|u_t\|_{p+2}\,d\tau\right\} \leqslant Ch^{r+p}\left\{\|u(t_1)\|_r + \int_0^{t_1} \|u_t\|_{p+2}\,d\tau\right\}.$$

Finally, by Lemma 3

$$\|D_t^j u_{3,h}(t_1)\|_{-p} \leqslant C\|u_{3,h}(0)\|_{-s,h} = C\|v_h - P_0 v\|_{-s,h}, \tag{2.2}$$

which completes the proof.

In particular, for t positive, and with for instance $v_h = P_0 v$ or $P_1 v$, we have

$$\|D_t^j e(t)\|_{-p} \leqslant C(u)h^{r+p},$$

with stringent regularity assumptions on u only near t.

Using (1.9), we may further reduce the regularity assumptions away from t for maximal order negative norm error estimates, in the instance $p \geqslant 1$ at the expense of having to impose the boundary conditions of the spaces $\dot{H}^p(\Omega)$ on v and f.

THEOREM 4. *Let $j \geqslant 0$, $s \geqslant 0$ and $0 \leqslant p \leqslant r-2$. Then if $v \in \dot{H}^p(\Omega)$ and $f \in L_1(0, t; \dot{H}^p(\Omega))$, we have for $t > \delta > 0$*

$$\begin{aligned}\|D_t^j e(t)\|_{-p} &\leqslant C\|v_h - P_0 v\|_{-s,h} \\ &+ Ch^{r+p}\left\{\sum_{l=0}^{j} \|D_t^l u(t)\|_r + \int_{t-\delta}^{t} \|D_t^{j+1} u\|_r \, d\tau + \|v\|_p + \int_0^t \|f\|_p \, d\tau\right\}.\end{aligned} \tag{2.3}$$

Proof. With the notation of the proof of Theorem 3, $D_t^j e_1$ and $D_t^j u_{3,h}$ are estimated as before by the right-hand side of (2.3) and it remains only to consider $D_t^j e_2$. Let now $E(t)$ denote the solution operator of the homogeneous parabolic equation, $E_h(t)$ the solution operator of the corresponding semidiscrete problem and $F_h(t) = E_h(t)P_0 - E(t)$ the error operator corresponding to $v_h = P_0 v$. With this notation we have by (1.9), for $0 \leqslant p \leqslant r-2$,

$$\|F_h(t)v\|_{-(r-2),h} \leqslant Ch^{r+p}t^{-1}\|v\|_p \quad \text{for } v \in \dot{H}^p(\Omega), \tag{2.4}$$

and Duhamel's principle shows for $t \geqslant 0$,

$$e_2(t) = F_h(t)v + \int_0^t F_h(t-\tau) f_2(\tau)\, d\tau. \tag{2.5}$$

In the same way as above, Lemma 3 implies since $e_2(t) = u_{2,h}(t)$ and $f_2(t) = 0$ for $\tau > t_1 - \delta/4$ that

$$\|D_t^j e_2(t_1)\|_{-p} \leqslant C\|e_2(t_1 - \delta/4)\|_{-(r-2),h}.$$

Further, by (2.4) and (2.5), now since $f_2(\tau) = 0$ for $\tau > t_1 - \delta/2$,

$$\begin{aligned}\|e_2(t_1 - \delta/4)\|_{-(r-2),h} &\leqslant Ch^{r+p}\delta^{-1}\left\{\|v\|_p + \int_0^{t_1-\delta/2} \|f_2\|_p \, d\tau\right\} \\ &\leqslant Ch^{r+p}\left\{\|v\|_p + \int_0^{t_1} (\|f\|_p + \|u\|_p)\, d\tau\right\} \\ &\leqslant Ch^{r+p}\left\{\|v\|_p + \int_0^{t_1} \|f\|_p \, d\tau\right\}.\end{aligned} \tag{2.6}$$

Here the last step follows by the fact that under our assumptions on v and f, $u \in L_1(0, t; \dot{H}^p(\Omega))$ and

$$\int_0^{t_1} \|u\|_p \, d\tau \leqslant C\left\{\|v\|_p + \int_0^{t_1} \|f\|_p \, d\tau\right\}. \tag{2.7}$$

In fact, it follows immediately by eigenfunction expansions that

$$\|E(t)v\|_p \leqslant C\|v\|_p \quad \text{for } v \in \dot{H}^p(\Omega),\ t \geqslant 0,$$

which shows (2.7) by Duhamel's principle. This concludes the proof.

For the standard Galerkin method we also have the following H^1 estimate.

THEOREM 5. *Consider the standard Galerkin method and let $j \geqslant 0$ and $s \geqslant 0$. Then for $t > \delta > 0$,*

$$\|D_t^j e(t)\|_1 \leqslant C\|v_h - P_0 v\|_{-s,h}$$
$$+ Ch^{r-1}\left\{\sum_{l=0}^{j} \sup_{t-\delta\leqslant\tau\leqslant t} \|D_t^l u(\tau)\|_r + \left(\int_{t-\delta}^{t} \|D_t^{j+1}u\|_{r-1}^2\, d\tau\right)^{1/2}\right\}$$
$$+ Ch^r\left\{\|v\| + \int_0^t \|f\|\, d\tau\right\}.$$

Proof. With the notation of the proof of Theorem 3 above we now have by Theorem 2,

$$\|D_t^j e_1(t_1)\|_1 \leqslant Ch^{r-1}\left\{\sup_{\tau\leqslant t_1} \|D_t^j u_1(\tau)\|_r + \left(\int_0^{t_1} \|D_t^{j+1}u_1\|_{r-1}^2\, d\tau\right)^{1/2}\right\}$$
$$\leqslant Ch^{r-1}\left\{\sum_{l=0}^{j} \sup_{t_1-\delta\leqslant\tau\leqslant t_1} \|D_t^l u(\tau)\|_r + \left(\int_{t_1-\delta}^{t_1} \|D_t^{j+1}u\|_{r-1}^2\, d\tau\right)^{1/2}\right\}.$$

Recalling that by (1.12),

$$\|\chi\|_1^2 \leqslant CA(\chi, \chi) = C\|\chi\|_{1,h}^2 \quad \text{for } \chi \in S_h,$$

we have as above by Lemma 3,

$$\|D_t^j e_2(t_1)\|_1 \leqslant C\|e_2(t_1 - \delta/4)\|_{-(r-2),h},$$

and hence as in (2.6),

$$\|D_t^j e_2(t_1)\|_1 \leqslant Ch^r\left\{\|v\| + \int_0^{t_1} \|f\|\, d\tau\right\}. \tag{2.8}$$

Finally, as in (2.2),

$$\|D_t^j u_{3,h}(t_1)\|_1 \leqslant C\|u_{3,h}(0)\|_{-s,h} = C\|v_h - P_0 v\|_{-s,h}. \tag{2.9}$$

Together these estimates complete the proof.

In particular this shows for t positive, under the appropriate assumptions about initial data and regularity,

$$\|D_t^j e(t)\|_1 \leqslant C(u)h^{r-1}.$$

The latter estimate we shall in fact only need here in the case of one space dimension with $\Omega = I = [0, 1]$ and with S_h the space of continuous functions on I which vanish at $x = 0$ and $x = 1$ and which reduce to polynomials of degree at most $r - 1$ on each subinterval $I_j = (x_j, x_{j+1})$ of a partition $0 = x_0 < x_1 < \cdots < x_m = 1$ with $\max h_j \leqslant h$ where $h_j = x_{j+1} - x_j$. In one of our results below in this case we shall also have use for a maximum-norm estimate. We first quote the following lemma

(Theorem 1 in Wheeler [7]). Here and below we denote by $|\cdot|_{r,J}$ the norm in $W^r_\infty(J)$ with r and J omitted when $r = 0$ and $J = I$, respectively.

LEMMA 4. *Consider the standard Galerkin method in one space dimension under the above assumptions on $\{S_h\}$ and let $v_h = P_1 v$. Then for $t \geq 0$,*

$$|e(t)| \leq Ch^r \left\{ \sup_{\tau \leq t} |u(\tau)|_r + \left(\int_0^t \|u_t\|_r^2 \, d\tau \right)^{1/2} \right\}.$$

We now have

THEOREM 6. *Consider the standard Galerkin method in one space dimension under the above assumptions on S_h and let $j \geq 0$ and $s \geq 0$. Then for $t > \delta > 0$,*

$$\begin{aligned} &|D_t^j e(t)| + \max_k (h_k |D_t^j e(t)|_{1,I_k}) \\ (2.10) \qquad &\leq C\|v_h - P_0 v\|_{-s,h} + Ch^r \left\{ \sup_{t-\delta \leq \tau \leq t} \sum_{l=0}^{j} |D_t^l u(\tau)|_r + \left(\int_{t-\delta}^t \|D_t^{j+1} u\|_r^2 \, d\tau \right)^{1/2} \right. \\ &\qquad \left. + \|v\| + \int_0^t \|f\| \, d\tau \right\}. \end{aligned}$$

Proof. Still with the notation of the proof of Theorem 3, we have by Lemma 4 for the first part of the error, recalling that $u_1 = u\varphi_1$,

$$\begin{aligned} |D_t^j e_1(t_1)| &\leq Ch^r \left\{ \sup_{\tau \leq t_1} |D_t^j u_1(\tau)|_r + \left(\int_0^{t_1} \|D_t^{j+1} u_1\|_r^2 \, d\tau \right)^{1/2} \right\} \\ &\leq Ch^r \left\{ \sup_{t_1-\delta \leq \tau \leq t_1} \sum_{l=0}^{j} |D_t^l u(\tau)|_r + \left(\int_{t_1-\delta}^{t_1} \|D_t^{j+1} u\|_r^2 \, d\tau \right)^{1/2} \right\}. \end{aligned}$$

Further, by Sobolev's inequality and (2.8),

$$|D_t^j e_2(t_1)| \leq C\|D_t^j e_2(t_1)\|_1 \leq Ch^r \left\{ \|v\| + \int_0^{t_1} \|f\| \, d\tau \right\},$$

and similarly by (2.9),

$$|D_t^j u_{3,h}(t_1)| \leq C\|v_h - P_0 v\|_{-s,h}.$$

Together these estimates show that $|D_t^j e(t)|$ is bounded by the right-hand side of (2.10).

In order to complete the proof we only need to notice that for any $t \geq 0$, using the inverse property of S_h on I_k, with χ an interpolating polynomial of $D_t^j u(t)$ on I_k,

$$\begin{aligned} h_k |D_t^j e(t)|_{1,I_k} &\leq h_k |D_t^j u_k(t) - \chi|_{1,I_k} + h_k |D_t^j u(t) - \chi|_{1,I_k} \\ &\leq C|D_k^j u_h(t) - \chi|_{I_k} + Ch_k^r |D_t^j u(t)|_{r,I_k} \\ &\leq C|D_t^j e(t)|_{I_k} + Ch_k^r |D_t^j u(t)|_{r,I_k}, \end{aligned}$$

which is bounded in the way stated above.

Thus, with the appropriate regularity assumptions and choices of initial data we have for t positive, $|D_t^j e(t)| \leq C(u)h^r$. If in addition $h_k \geq ch$ with c fixed positive (in particular for quasi-uniform partitions), we have

$$|D_t^j e(t)|_{1,I_k} \leq C(u)h^{r-1}.$$

3. Superconvergence for C^0 Piecewise Polynomial Subspaces in One Space Dimension. We shall now turn to some examples of superconvergence in the case of one space dimension. Consider thus again a partition $0 = x_0 < x_1 < \cdots < x_m = 1$ of $I = [0, 1]$, set $h = \max(x_{j+1} - x_j)$, and let S_h be the finite-dimensional space of continuous functions on I which vanish at $x = 0$ and $x = 1$, and which reduce to polynomials of degree at most $r - 1$ on each subinterval $I_j = (x_j, x_{j+1})$. We then have

$$\inf_{\chi \in S_h} \{\|w - \chi\| + h\|w - \chi\|_1\} \leq Ch^r \left(\sum_{j=0}^{m-1} \|w\|_{r,I_j}^2 \right)^{1/2} \quad \text{for } w \in H_0^1(I) \cap \left(\bigcap_{j=0}^{m-1} H^r(I_j) \right). \tag{3.1}$$

Let now $u_h(t) \in S_h$ denote the solution of the standard Galerkin parabolic problem

$$(u_{h,t}, \chi) + A(u_h, \chi) = (f, \chi) \quad \text{for } \chi \in S_h,$$

with a suitable choice of $v_h = u_h(0)$. We shall prove some error estimates for this problem which combined with our above error estimates at positive time show superconvergence of different procedures for approximating u and its derivatives.

Our first such result yields superconvergence at the knots of the partition, a fact which has previously been proved by Douglas, Dupont and Wheeler [3]. Their analysis used a so-called quasi-projection of the exact solution into the subspace as a comparison function, and required a more special choice of discrete initial data and higher regularity of the exact solution than the present result.

THEOREM 7. *Under the above assumptions about $\{S_h\}$, let $\bar{x} \in (0, 1)$ be a knot for each h considered, and let $t \geq 0$ and $n \geq 0$. Then*

$$|e(\bar{x}, t)| \leq C \left\{ h^{r-1} \sum_{j=0}^{n} \|D_t^j e(t)\|_1 + h^r \|D_t^{n+1} e(t)\| + \|D_t^{n+1} e(t)\|_{-(2n+1)} \right\}.$$

Proof. Let $g = g_{\bar{x}}$ denote the Green's function corresponding to the two-point boundary value problem

$$Au = f \quad \text{in } I,\ u(0) = u(1) = 0. \tag{3.2}$$

As is well known, g is continuous on I and smooth except at $\bar{x}$ where g' has a simple discontinuity. In particular, $g \in H_0^1(I)$ and we have

$$w(\bar{x}) = A(w, g) \quad \text{for } w \in H_0^1(I). \tag{3.3}$$

Now let $w = w(x, t)$, be such that $D_t^j w \in H_0^1(I)$, $j = 0, \ldots, n$, $D_t^{n+1} w \in L_2(I)$. Then with

$$L(w, \varphi) = (w_t, \varphi) + A(w, \varphi), \tag{3.4}$$

we have

$$w(\bar{x}, t) = A(w, g) = \sum_{j=0}^{n} (-1)^j L(D_t^j w, T^j g) + (-1)^{n+1}(D_t^{n+1} w, T^n g). \tag{3.5}$$

This follows at once by (3.3) and the fact that for $j \geqslant 1$,

$$L(D_t^j w, T^j g) = (D_t^{j+1} w, T^j g) + A(D_t^j w, T^j g) = (D_t^{j+1} w, T^j g) + (D_t^j w, T^{j-1} g).$$

In order to show the theorem we apply (3.5) to $w = e = u_h - u$, observing that

$$L(e, \chi) = L(u_h, \chi) - L(u, \chi) = 0 \quad \text{for } \chi \in S_h,$$

and hence for $j = 0, \ldots, n$,

$$L(D_t^j e, T^j g) = L(D_t^j e, T^j g - \chi) \quad \text{for } \chi \in S_h.$$

In view of the fact that for each $j \geqslant 0$, $T^j g \in H_0^1(I) \cap H^r(0, \bar{x}) \cap H^r(\bar{x}, 1)$, this yields by (3.1),

$$\begin{aligned} |L(D_t^j e, T^j g)| &\leqslant \inf_{\chi \in S_h} \{ \|D_t^{j+1} e\| \; \|T^j g - \chi\| + \|D_t^j e\|_1 \|T^j g - \chi\|_1 \} \\ &\leqslant C\{h^r \|D_t^{j+1} e\| + h^{r-1} \|D_t^j e\|_1 \}. \end{aligned}$$

Also, since $g \in H_0^1(I)$,

$$\begin{aligned} |(D_t^{n+1} e, T^n g)| &= |(T^n D_t^{n+1} e, g)| = |A(T^{n+1} D_t^{n+1} e, g)| \\ &\leqslant C\| T^{n+1} D_t^{n+1} e\|_1 \|g\|_1 \leqslant C\|D_t^{n+1} e\|_{-(2n+1)}. \end{aligned}$$

Together these estimates complete the proof.

Combining this result with our error estimates of Section 3, we find under the appropriate regularity assumptions for t positive and with $v_h = P_0 v$ or $P_1 v$, for instance

$$|e(\bar{x}, t)| \leqslant C(u) h^{2r-2}.$$

Notice as usual that strong regularity assumptions only have to be made near t.

In our next example we shall treat a procedure for approximating the first derivative of the solution at a knot, proposed by Douglas, Dupont and Wheeler [3]. Again we shall express the error bound in terms of such norms of the error, including negative norms, for which we have estimates at our disposal, thus allowing us to show superconvergence, under somewhat milder regularity assumptions and more general choices of discrete initial data than in [3].

The starting point of the procedure is the following identity (cf. [3]):

$$\bar{x} a u'(\bar{x}) = A_{-}(u, x) - (Au, x)_{-},$$

where with $\bar{x} \in (0, 1)$, $A_{-}(\cdot, \cdot)$ and $(\cdot, \cdot)_{-}$ denote the bilinear form and L_2 inner product taken over the interval $(0, \bar{x})$ rather than over I and where $a = a_{11}$. It follows for u a solution of the parabolic problem, with $L_-(\cdot, \cdot)$ correspondingly defined from (3.4),

$$\bar{x}a \frac{\partial u}{\partial x}(\bar{x}, t) = L_-(u, x) - (f, x)_-. \tag{3.6}$$

With the semidiscrete solution $u_h(t)$ computed, we now study the approximation to $\partial u(\bar{x}, t)/\partial x$ defined by

$$\bar{x}a\tilde{u}_{1,h}(\bar{x}, t) = L_-(u_h, x) - (f, x)_-. \tag{3.7}$$

We have

THEOREM 8. *Under the assumptions of Theorem 7 we have for $t \geqslant 0$ and $n \geqslant 0$,*

$$\left|\tilde{u}_{1,h}(\bar{x}, t) - \frac{\partial u}{\partial x}(\bar{x}, t)\right| \leqslant C\left\{h^{r-1} \sum_{j=0}^{n} \|D_t^j e(t)\|_1 + h^r \|D_t^{n+1} e(t)\| + \|D_t^{n+1} e(t)\|_{-2n}\right\}.$$

Proof. We have by (3.6) and (3.7),

$$\begin{aligned}\bar{x}a\left(\tilde{u}_{1,h}(\bar{x}, t) - \frac{\partial u}{\partial x}(\bar{x}, t)\right) &= L_-(e, x) = (e_t, x)_- + A_-(e, x) \\ &= (e_t, x)_- + (e, Ax)_- + ae(\bar{x}, t) = (e_t, \varphi_1) + (e, \varphi_0) + ae(\bar{x}, t),\end{aligned}$$

where φ_0 and φ_1 are discontinuous at $\bar{x}$ but smooth otherwise. The last term is estimated by Theorem 7 and the remaining two terms by the following lemma which completes the proof.

LEMMA 5. *Under the assumptions of Theorem 7, let $\varphi \in L_2(I) \cap H^{r-2}(0, \bar{x}) \cap H^{r-2}(\bar{x}, 1)$. Then for any $n \geqslant 0$,*

$$|(e, \varphi)| \leqslant C\left\{h^{r-1} \sum_{j=0}^{n-1} \|D_t^j e\|_1 + h^r \|D_t^n e\| + \|D_t^n e\|_{-2n}\right\}.$$

Proof. The case $n = 0$ is trivial. For n positive we have similarly to above, with $\psi = T\varphi$,

$$(e, \varphi) = A(e, \psi) = \sum_{j=0}^{n-1} (-1)^j L(D_t^j e, T^j \psi) + (-1)^n (D_t^n e, T^{n-1}\psi). \tag{3.8}$$

Here for $j = 0, \ldots, n-1$,

$$L(D_t^j e, T^j\psi) = L(D_t^j e, T^j\psi - \chi) \quad \text{for } \chi \in S_h;$$

and hence, since $T^j\psi \in H_0^1(I) \cap H^r(0, \bar{x}) \cap H^r(\bar{x}, 1)$,

$$\begin{aligned}|L(D_t^j e, T^j\psi)| &\leqslant \inf_{\chi \in S_h} \{\|D_t^{j+1} e\| \, \|T^j\psi - \chi\| + \|D_t^j e\|_1 \, \|T^j\psi - \chi\|_1\} \\ &\leqslant C\{h^r \|D_t^{j+1} e\| + h^{r-1} \|D_t^j e\|_1\}.\end{aligned}$$

Also

$$|(D_t^n e, T^{n-1}\psi)| = |(D_t^n e, T^n\varphi)| = |(T^n D_t^n e, \varphi)|$$
$$\leqslant \|T^n D_t^n e\| \, \|\varphi\| \leqslant C\|D_t^n e\|_{-2n}.$$

In view of (3.8) these estimates prove the lemma.

We shall now show that a superconvergent order approximation of the first derivative can also be based on difference quotients of the semidiscrete solution, provided these are taken over mesh intervals of length at least ch with c fixed positive. Since a derivative of a smooth function can be approximated locally to any order by a linear combination of such difference quotients, it is sufficient to show a superconvergent order error bound for a forward difference quotient, say. Thus let $\bar{x} \in (0, 1)$ be a knot for the S_h considered, let $\bar{x} + \bar{h}$ denote the knot immediately to the right of $\bar{x}$, and set

$$\partial u(\bar{x}) = \bar{h}^{-1}(u(\bar{x} + \bar{h}) - u(\bar{x})).$$

Our result is then

THEOREM 9. *Under the assumptions of Theorem 7 we have for $t \geqslant 0$ and $n \geqslant 0$,*

$$|\partial e(\bar{x}, t)| \leqslant C\left\{h^{r-1} \sum_{j=0}^{n} (\|D_t^j e(t)\|_1 + |D_t^j e(t)|_{1,\bar{I}}) + h^r(\|D_t^{n+1}e(t)\| + |D_t^{n+1}e(t)|_{\bar{I}}) + \|D_t^{n+1}e(t)\|_{-2n}\right\}.$$

Proof. With $g_{\bar{x}}$ as above the Green's function of the two-point boundary value problem (3.2), and $g^{(1)}(x) = \bar{h}^{-1}(g_{\bar{x}+\bar{h}}(x) - g_{\bar{x}}(x))$, we have

$$\partial e(\bar{x}, t) = A(e, g^{(1)}) = \sum_{j=0}^{n} (-1)^j L(D_t^j e, T^j g^{(1)}) + (-1)^{n+1}(D_t^{n+1}e, T^n g^{(1)}).$$

With $\bar{I} = (\bar{x}, \bar{x} + \bar{h})$, $I_- = (0, \bar{x})$ and $I_+ = (\bar{x} + \bar{h}, 1)$ we have now

$$|L(D_t^j e, T^j g^{(1)})| \leqslant \inf_{\chi \in S_h} \{\|D_t^{j+1}e\|_{I_- \cup I_+} \|T^j g^{(1)} - \chi\|_{I_- \cup I_+} + |D_t^{j+1}e|_{\bar{I}} \|T^j g^{(1)} - \chi\|_{L_1(\bar{I})} + \|D_t^j e\|_{1,I_- \cup I_+} \|T^j g^{(1)} - \chi\|_{1,I_- \cup I_+} + |D_t^j e|_{1,\bar{I}} \|T^j g^{(1)} - \chi\|_{W_1^1(\bar{I})}\}.$$

Since for $\psi = T^j g^{(1)} - \chi$,

$$\|\psi\|_{L_1(\bar{I})} \leqslant \bar{h}|\psi|_{\bar{I}}, \qquad \|\psi\|_{W_1^1(\bar{I})} \leqslant \bar{h}|\psi|_{1,\bar{I}},$$

it follows by (3.10) of Lemma 6 below

$$|L(D_t^j e, T^j g^{(1)})| \leqslant C\{h^{r-1}(\|D_t^j e\|_1 + |D_t^j e|_{1,\bar{I}}) + h^r(\|D_t^{j+1}e\| + |D_t^{j+1}e|_{\bar{I}})\}.$$

Using also (3.9) of Lemma 6, we have

$$|(D_t^{n+1}e, T^n g^{(1)})| = |(T^n D_t^{n+1}e, g^{(1)})| \leqslant \|D_t^{n+1}e\|_{-2n}\|g^{(1)}\| \leqslant C\|D_t^{n+1}e\|_{-2n}.$$

Together these estimates show the theorem.

It thus remains to show:

LEMMA 6. *With the above notation, we have*

$$|g^{(1)}|_I \leqslant C \tag{3.9}$$

and

$$\begin{aligned}\inf_{\chi\in S_h} \{|T^j g^{(1)} - \chi|_{I_-\cup I_+} + h|T^j g^{(1)} - \chi|_{1,I_-\cup I_+} + \bar{h}|T^j g^{(1)} - \chi|_{\bar{I}} \\ + h\bar{h}|T^j g^{(1)} - \chi|_{1,\bar{I}}\} \leqslant Ch^r.\end{aligned} \tag{3.10}$$

Proof. We first recall

$$g_{\bar{x}}(x) = \begin{cases} \omega u_1(\bar{x})u_2(x) & \text{for } x \geqslant \bar{x}, \\ \omega u_1(x)u_2(\bar{x}) & \text{for } x < \bar{x}, \end{cases}$$

where u_1 and u_2 are two solutions of $Au = 0$ satisfying $u_1(0) = u_2(1) = 0$ and where $\omega =$ constant $\neq 0$ independent of x and $\bar{x}$. It follows

$$g^{(1)}(x) = \begin{cases} \omega u_1(x)\bar{h}^{-1}(u_2(\bar{x}+\bar{h}) - u_2(\bar{x})) & \text{for } x \leqslant \bar{x}, \\ \omega u_2(x)\bar{h}^{-1}(u_1(\bar{x}+\bar{h}) - u_1(\bar{x})) & \text{for } x \geqslant \bar{x}+\bar{h}, \\ \omega\bar{h}^{-1}(u_1(x)u_2(\bar{x}+\bar{h}) - u_2(x)u_1(\bar{x})) & \text{for } \bar{x} < x < \bar{x}+\bar{h}, \end{cases}$$

which in particular yields (3.9). The expression for $g^{(1)}$ further shows that

$$|g^{(1)}|_{r,I_-\cup I_+} + \bar{h}|g^{(1)}|_{r,\bar{I}} \leqslant C, \tag{3.11}$$

which implies (3.10) for $j = 0$ by choosing χ as a suitable interpolant of $g^{(1)}$.

In order to treat the case $j > 0$ we first notice that by the maximum principle and by (3.9),

$$|T^j g^{(1)}|_I \leqslant C|T^{j-1}g^{(1)}|_I \leqslant C|g^{(1)}|_I \leqslant C. \tag{3.12}$$

Further, an easy calculation shows that for each $k \geqslant 0$ and each subinterval $J < I$,

$$|w|_{k+2,J} \leqslant C\{|w|_{\partial J} + |Aw|_{k,J}\},$$

with C independent of J as long as the length of J is bounded below. By repeated application this implies, using (3.11) and (3.12),

$$|T^j g^{(1)}|_{r,I_-\cup I_+} \leqslant C\left\{\sum_{l=0}^{j} |T^j g^{(1)}|_I + |g^{(1)}|_{r,I_-\cup I_+}\right\} \leqslant C.$$

Similarly,

$$|T^j g^{(1)}|_{r,\bar I} \leqslant \bar h^{-1}(|T^j g_{\bar x}|_{r,\bar I} + |T^j g_{\bar x+\bar h}|_{r,\bar I})$$
$$\leqslant \bar h^{-1}(|T^j g_{\bar x}|_{r,I_+\cup\bar I} + |T^j g_{\bar x+\bar h}|_{r,\bar I\cup I_-}) \leqslant C\bar h^{-1},$$

and thus

$$|T^j g^{(1)}|_{r,I_-\cup I_+} + \bar h|T^j g^{(1)}|_{r,\bar I} \leqslant C,$$

completing the proof for $j > 0$.

4. Interior Superconvergence with Uniform Elements. We shall now turn to an application of our negative norm estimates in higher dimensions in which the subspaces are based on uniform partitions in a specific sense in some interior domain Ω_0. We shall not describe this uniformity assumption in detail but content ourselves by referring to such $\{S_h\}$ as r-regular on Ω_0, following the definition of this concept in [2].

We first quote the following result (Theorem 3 in [5]) which generalizes to the case of derivatives a construction due to Bramble and Schatz [1]. Here and below the interior domain over which the L_2 and maximum norm based norms are defined are indicated in the notation. For the negative norms we denote for $\Omega_0 \subset\subset \Omega$,

$$\|v\|_{-p,\Omega_0} = \sup_{\varphi\in C_0^\infty(\Omega_0)} \frac{(v,\varphi)}{\|\varphi\|_{p,\Omega_0}}.$$

LEMMA 7. *Let ∂_h^α denote the forward difference quotient corresponding to D^α, ψ the B-spline in R^N of order $r-2$ and $N_0 = [N/2] + 1$. Then there exists a function K_h of the form*

$$K_h(x) = h^{-N}\sum_\gamma k_\gamma \psi(h^{-1}x-\gamma),$$

with $k_\gamma = 0$ when $|\gamma_j| \geqslant r-1$ such that for $\Omega_1 \subset\subset \Omega_0 \subset\subset \Omega$ and $e = u_h - u$ we have

$$|K_h * \partial_h^\alpha u_h - D^\alpha u|_{\Omega_1}$$
$$\leqslant C\left\{h^{2r-2}|u|_{2r-2+|\alpha|,\Omega_0} + \sum_{|\beta|\leqslant r-2+N_0} \|\partial_h^{\alpha+\beta}e\|_{-(r-2),\Omega_0}\right.$$
$$\left. + h^{r-2}\sum_{|\beta|\leqslant r-2} |\partial_h^{\alpha+\beta}e|_{\Omega_0}\right\}.$$

In order to apply this estimate we need to have at our disposal the appropriate estimates for $\partial_h^\beta e$. These are contained in the following lemma.

LEMMA 8. *Assume that $\{S_h\}$ is r-regular in $\Omega_0 \subset\subset \Omega$, and let $\Omega_1 \subset\subset \Omega_0$. We then have at time t,*

$$\|\partial_h^\alpha e\|_{-(r-2),\Omega_1} \leqslant Ch^{2r-2} \sum_{2l\leqslant r+|\alpha|} \|D_t^l u\|_{r+|\alpha|-2l,\Omega_0}$$

$$+ C \sum_{2l\leqslant 2+|\alpha|} (h^{r-2}\|D_t^l e\|_{\Omega_0} + \|D_t^l e\|_{-(r-2),\Omega_0}),$$

and

$$|\partial_h^\alpha e|_{\Omega_1} \leqslant Ch^r \sum_{2l\leqslant r+|\alpha|+N_0} \|D_t^l u\|_{r+|\alpha|+N_0-2l,\Omega_0} + C \sum_{2l\leqslant 2+|\alpha|+N_0} \|D_t^l e\|_{\Omega_0}.$$

Proof. For the case $f = 0$ these results were proved in Lemmas 7.3 and 6.5 of [2]. The proofs for the inhomogeneous equation are obvious modifications of those in the case $f = 0$.

We obtain immediately by combination of Lemmas 7 and 8:

THEOREM 10. *Assume that $\{S_h\}$ is r-regular on $\Omega_0 \subset\subset \Omega$, let D^α be an arbitrary derivative with respect to x and $\Omega_1 \subset\subset \Omega_0$. We then have for $t \geqslant 0$,*

$$|K_h * \partial_h^\alpha u_h(t) - D^\alpha u(t)|_{\Omega_1}$$

$$\leqslant Ch^{2r-2} \sum_{2l\leqslant 2r-2+|\alpha|+N_0} \|D_t^l u(t)\|_{2r-2+|\alpha|+N_0-2l,\Omega_0}$$

$$+ C \sum_{2l\leqslant r+|\alpha|+N_0} (h^{r-2}\|D_t^l e(t)\|_{\Omega_0} + \|D_t^l e(t)\|_{-(r-2),\Omega_0}).$$

As a consequence, observing that

$$\|v\|_{-p,\Omega_0} \leqslant C\|v\|_{-p} \quad \text{for } p \geqslant 0,$$

we obtain in view of our previous error estimates that for positive time and suitable discrete initial data,

$$|K_h * \partial_h^\alpha u_h(t) - D^\alpha u(t)|_{\Omega_1} \leqslant C(u)h^{2r-2},$$

with stringent regularity assumptions only near t.

5. Superconvergence Based on Local Green's Functions. In our final example, we shall show that also in the case of nonuniform partitions in arbitrary dimensions it is possible to construct superconvergent order approximations to $u(x_0, t)$ for x_0 an interior point and t positive. This approximation will depend on $u_h(x, t)$ and $f(x, t)$ in a fixed neighborhood of x_0 independent of h. Although it might not be easy to apply, our result shows that u_h carries with it through the computation the capability of reproducing u to superconvergent order.

For motivation, we consider first the elliptic problem (1.2) (cf. Louis and Natterer [4]). Let $x_0 \in \Omega_0 \subset \Omega$, and let $G_0 = G_0^{x_0}$ be the Green's function with singularity at x_0 of the Dirichlet problem

$$Aw = f \quad \text{in } \Omega_0, \qquad w = 0 \quad \text{on } \partial\Omega_0, \tag{5.1}$$

so that for w smooth and vanishing on $\partial\Omega_0$,

$$w(x_0) = \int_{\Omega_0} Aw(y)G_0(y)\,dy. \tag{5.2}$$

In particular, with $\varphi \in C_0^\infty(\Omega_0)$, $\varphi \equiv 1$ in a neighborhood of $\bar{\Omega}_1 \supset \Omega_1 \ni x_0$ we have for the solution u of (1.2),

$$u(x_0) = (A(\varphi u), G_0)_{\Omega_0},$$

where, if $G_0 \notin L_2(\Omega_0)$, $(\cdot, \cdot)_{\Omega_0}$ denotes the integral of the product. Introducing the solution operator T_0 of (5.1), we have $T_0Av = v$ for $v \in \dot{H}^2(\Omega_0)$ so that for u sufficiently smooth and any $k \geqslant 0$,

$$T_0^kA^{k+1}(\varphi u) = A(\varphi u).$$

This yields

$$u(x_0) = (T_0^kA^{k+1}(\varphi u), G_0)_{\Omega_0} = (A^{k+1}(\varphi u), T_0^kG_0)_{\Omega_0}. \tag{5.3}$$

By Leibniz' formula,

$$A^{k+1}(\varphi u) = \varphi A^{k+1}u + \sum_{\substack{|\alpha+\beta| \leqslant 2k+2 \\ \alpha \neq 0}} c_{\alpha\beta}D^\alpha\varphi D^\beta u. \tag{5.4}$$

Hence, noticing that $T_0^kG_0$ is smooth except at x_0, we conclude from (5.3), using integration by parts for the terms of the sum in (5.4),

$$u(x_0) = (\varphi A^{k+1}u,\ T_0^kG_0) + (u, \psi_k) \quad \text{with } \psi_k \in C_0^\infty(\Omega_0\backslash\bar{\Omega}_1). \tag{5.5}$$

In order to find an accurate approximation of the solution of the elliptic problem (1.2) we may utilize this representation for $k = 0$ and set

$$\tilde{u}_h(x_0) = (\varphi f,\ G_0) + (u_h, \psi_0). \tag{5.6}$$

We have at once

$$\tilde{u}_h(x_0) - u(x_0) = (e, \psi_0) = (e, T^sA^s\psi_0) = (T^se, A^s\psi_0),$$

and hence for any $s \geqslant 0$ the error estimate

$$|\tilde{u}_h(x_0) - u(x_0)| \leqslant C\|e\|_{-s}. \tag{5.7}$$

In order to avoid the computation of the singular integral in (5.6) we could alternatively have used (5.5) with $k > 0$ and set

$$\tilde{u}_h(x_0) = (\varphi A^kf,\ T_0^kG_0) + (u_h, \psi_k),$$

again concluding (5.7).

We now turn to the parabolic case and first state the following representation result.

LEMMA 9. *Let $x_0 \in \Omega_0 \subset \Omega$, let G_0 and T_0 be as above, and let $\varphi \in C_0^\infty(\Omega_0), \varphi \equiv 1$ on $\bar{\Omega}_1$ where $x_0 \in \Omega_1$. Then for any $k \geqslant 0$ there is a $\psi_k \in C_0^\infty(\Omega_0\backslash\bar{\Omega}_1)$ such that for $t \geqslant 0$,*

$$u(x_0, t) = \sum_{j=0}^{k+1} (-1)^{k+1-j}\binom{k+1}{j}(\varphi D_t^{k+1-j}L^ju,\ T_0^kG_0) + (u, \psi_k).$$

Proof. This follows at once from (5.5) upon noticing that since L and D_t commute,

$$A^{k+1} = (L - D_t)^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} (-D_t)^{k+1-j} L^j.$$

We may now use the above representation to define an approximation of $u(x_0, t)$ in terms of $u_h(x, t)$ and $f(x, t)$ on Ω_0 by

$$(5.8)\qquad \begin{aligned} \widetilde{u}_h(x_0, t) &= \sum_{j=1}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} (\varphi D_t^{k+1-j} L^{j-1} f, T_0^k G_0) \\ &\quad + (-1)^{k+1} (\varphi D_t^{k+1} u_h, T_0^k G_0) + (u_h, \psi_k). \end{aligned}$$

We have then the following error estimate. For suitable choice of k and s this shows superconvergence by our previous estimates.

THEOREM 11. *Let k_0 be such that $T_0^{k_0} G_0 \in L_2(\Omega_0)$, and let $\widetilde{u}_h$ be defined by* (5.8) *with $k \geqslant k_0$. Then for any $s \geqslant 0$,*

$$|\widetilde{u}_h(x_0, t) - u(x_0, t)| \leqslant C\{\|D_t^{k+1} e\|_{-2(k-k_0)} + \|e\|_{-s}\}.$$

Proof. We have by Lemma 9 and (5.8),

$$\widetilde{u}_h(x_0, t) - u(x_0, t) = (-1)^{k+1} (\varphi D_t^{k+1} e, T_0^k G_0) + (e, \psi_k).$$

Here the second term is estimated as in the elliptic case treated above to yield

$$|(e, \psi_k)| \leqslant C\|e\|_{-s} \quad \text{for any } s \geqslant 0.$$

For the first term we have easily

$$\begin{aligned} (\varphi D_t^{k+1} e, T_0^k G_0) &= (T_0^{k-k_0}(\varphi D_t^{k+1} e), T_0^{k_0} G_0)_{\Omega_0} \\ &\leqslant C\|T_0^{k-k_0}(\varphi D_t^{k+1} e)\|_{\Omega_0} \leqslant C\|D_t^{k+1} e\|_{-2(k-k_0)}. \end{aligned}$$

Together these estimates prove the theorem.

Starting instead of (5.2) with the identity

$$\frac{\partial u}{\partial x_l}(x_0) = \int_{\Omega_0} Au(y) \frac{\partial G_0}{\partial x_l}(y, x_0)\, dy,$$

we may similarly construct a superconvergent order approximation to $\partial u/\partial x_l(x_0)$ by replacing G_0 by $G_l = \partial G_0/\partial x_l$ everywhere above. We notice that since G_l is more singular than G_0 near x_0 we may now have to choose k_0 larger than before.

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