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Residue Classes of the Partition Function

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Abstract. The results of computations of $p(n) \pmod{p^k}$ for primes $p < 100$ are summarized in frequency tables of residue classes.

1. Introduction. If $p(n)$ denotes the number of unrestricted partitions of n, let

$$A(m, k, x) = \{n : p(n) \equiv k \pmod{m} \text{ and } n \leqslant x\},$$

$$A(m, k) = \{n : p(n) \equiv k \pmod{m}\},$$

$$s(m, k, x) = \operatorname{card} A(m, k, x), \text{ and let}$$

$$d(m, k, x) = s(m, k, x)/x.$$

In 1960, Morris Newman [9] conjectured that $p(n)$ fills all residue classes mod m infinitely often. In our notation, this is equivalent to the conjecutre that no $A(m, k)$ is finite.

Proofs of this conjecture have been discovered for $m = 2$ ([6]), 5 ([9]), 7 ([1]), 13 ([9]), 17, 19, 29, 31 ([4]), and 11^2 ([5]). The Ramanujan congruences imply that $A(5^a\, 7^b\, 11^c, 0)$ is infinite for all natural numbers a, b, and c.

It is natural to ask the following questions about the densities and structures of the $A(m, k)$'s:

1. Does $\lim_{x\to\infty} d(m, k, x)$ exist?
2. Is $\lim_{x\to\infty} \inf d(m, k, x) > 0$?
3. Does $A(m, k)$ contain any arithmetic progressions or any other simple infinite sequences?

2. Tables of Residue Classes. Using a PDP 11/60, $A(m, k, 19000)$ was computed for m equal to powers of 2, 3, 5, 7, and 11 less than 5^5, and for m equal to primes less than 100. A summary of the results is contained in the following tables.

TABLE 1. *Values of* $s(5^\alpha, 0, 19000)$

m	5	5^2	5^3	5^4	5^5	5^6
s(m, 0, 19000)	6907	1992	724	172	57	5

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TABLE 2. *Values of* $s(7^{\alpha}, 0, 19000)$

m	7	7^2	7^3	7^4
s(m, 0, 19000)	5244	2070	308	47

TABLE 3. *Values of* $s(5^2, 5l + j, 19000)$

ℓ \ j	0	1	2	3	4
0	1992	587	615	595	596
1	1240	578	631	609	575
2	1249	587	655	617	649
3	1220	567	663	573	610
4	1206	559	604	594	630

TABLE 4. *Values of* $s(7^2, 7l + j, 19000)$

ℓ \ j	0	1	2	3	4	5	6
0	2070	308	318	363	362	317	332
1	512	298	304	323	327	339	328
2	495	332	329	317	327	327	302
3	508	376	340	318	336	328	355
4	526	303	344	326	308	307	333
5	526	332	313	333	342	300	341
6	559	342	350	363	338	310	314

TABLE 5. *Values of* $s(11^2, 11l + j, 19000)$

l \ j	0	1	2	3	4	5	6	7	8	9	10
0	487	138	164	139	151	160	159	138	134	148	163
1	290	160	143	131	147	141	138	135	148	148	138
2	309	200	145	127	150	133	144	125	154	128	143
3	272	158	130	116	143	150	138	146	154	146	145
4	276	132	154	124	123	154	137	124	143	161	123
5	281	132	143	138	162	128	126	146	140	129	141
6	295	117	138	126	144	155	136	139	136	146	138
7	269	143	131	126	154	118	147	146	134	151	131
8	317	152	157	163	136	141	148	162	145	134	137
9	307	128	122	147	153	143	140	149	167	149	118
10	279	130	139	126	136	137	156	128	154	164	144

Tables of $s(m, k, 19000)$ for m running through the primes from 29 to 97, as well as listings of $A(m, 0, 19000)$ for $m = 5^2, 5^3, 5^4, 5^5, 7^2, 7^3, 7^4, 11^2, 11^3, 11^4$, have been deposited in the UMT file.

3. **Calculation of $p(n)$.** A table of values of $p(n)$, $n \leqslant 1435$, has been deposited in the UMT file. For $n \leqslant 600$, the values obtained agree with those of Gupta [3]. The value $p(721) = 161\ 0617\ 5575\ 0279\ 4776\ 3553\ 4762$ is equal to that obtained by D. H. Lehmer [7]. As an additional check, it was determined that $13 \mid p(n)$ for $n = 747, 890, 903, 968, 1176$, and 1241, as predicted by M. Newman [8].

4. **Conclusion.** It is hoped that examination of the tables will lead to explanations of the observed values of the residue classes of $p(n)$ and to answers of the questions posed in Section 1. Kløve [5] has shown that

$$\liminf d(5, 0, x) \geqslant .20194,$$

$$\liminf d(7, 0, x) \geqslant .150697143,$$

$$\liminf d(13, 0, x) \geqslant .00204,$$

while our calculations give $d(5, 0, 19000) = .363507184$, $d(7, 0, 19000) = .273459292$, $d(13, 0, 19000) = .079416873$, showing that there may be considerable room for improvement of the theory.

TABLE 6. Values of $s(m, k, 19600)$, $0 \leqslant k \leqslant m-1$, $2 \leqslant m \leqslant 24$

m \ k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	9580	9421																						
3	6363	6318	6320																					
4	4784	4650	4796	4771																				
5	6907	2878	3168	2988	3060																			
6	3215	3122	3169	3148	3196	3151																		
7	5196	2291	2298	2343	2340	2228	2305																	
8	2385	2355	2397	2411	2399	2295	2399	2360																
9	2136	2080	2151	2119	2081	2056	2108	2157	2113															
10	3527	1438	1607	1497	1515	3380	1440	1561	1491	1545														
11	3382	1590	1566	1463	1599	1560	1569	1538	1609	1604	1521													
12	1608	1507	1578	1590	1585	1585	1607	1615	1591	1558	1611	1566												
13	1509	1491	1408	1456	1451	1479	1453	1474	1411	1456	1453	1480	1480											
14	2626	1149	1160	1156	1189	1143	1191	2570	1142	1138	1187	1151	1085	1114										
15	2353	946	1029	1008	976	2252	951	1092	978	1004	2302	981	1047	1002	1080									
16	1197	1204	1206	1237	1221	1191	1200	1177	1188	1151	1191	1174	1178	1104	1199	1183								
17	1057	1136	1100	1069	1099	1198	1149	1149	1145	1144	1084	1159	1090	1079	1130	1112	1101							
18	1087	1012	1069	1043	1094	1001	1052	1123	1045	1049	1068	1082	1076	987	1055	1056	1034	1068						
19	1018	1000	963	1068	996	949	1022	971	1006	1004	1013	1043	1032	1007	950	1051	974	1000	934					
20	1779	710	823	723	755	1620	720	779	746	764	1748	728	784	774	760	1760	720	782	745	781				
21	1764	757	725	750	820	755	788	1731	764	792	777	779	758	737	1701	770	781	816	741	715	780			
22	1713	753	791	774	793	799	825	777	836	804	774	1669	837	775	689	806	761	744	761	773	800	747		
23	800	866	791	831	832	837	798	875	853	823	795	797	838	801	804	805	844	830	887	791	862	840	801	
24	811	762	788	796	799	778	818	795	788	786	820	795	797	745	790	794	786	807	789	820	803	772	791	771

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